Riemann-Roch formulae in Arakelov geometry and applications

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Contents

Preface

The following is a set of very informal notes on the subject of my minicourse at the CRM in Barcelona, given during the last week of February 2006. Few proofs are given and the aim of the text is to show the computational power of the Riemann-Roch theorem. Bibliographical references to the sources of the results presented here are sketchy and by no means exhaustive. The ideal prerequisites for the course are the first three chapters of R. Hartshorne's textbook [H].

Chapter 1

Riemann-Roch formulae in algebraic geometry

Convention. A scheme will be short for a noetherian scheme separated over Spec $\mathbb{Z}$.

1.1 The Grothendieck-Riemann-Roch formula

Let C be a smooth projective curve over $\mathbb{C}$. Let $D := \sum_i n_i D_i$ be a divisor on C. The simplest instance of the Grothendieck-Riemann-Roch formula is probably the well-known equality

$$\chi(\mathcal{O}(D)) := \dim_{\mathbb{C}} H^0(C, \mathcal{O}(D)) - \dim_{\mathbb{C}} H^1(C, \mathcal{O}(D)) = \deg D + 1 - g \qquad (1.1)$$

where $\deg D := \sum_i n_i$ is the degree of D and $g := \dim_{\mathbb{C}} H^0(C, \Omega_C)$ is the genus of C. One can show that

$$\deg D = \int_C c^1(\mathcal{O}(D)),$$

where $c^1(\mathcal{O}(D))$ is the first Chern class of D, so (1.1) is a formula for the Euler characteristic $\chi(\mathcal{O}(D))$ in terms of integrals of cohomology classes.

The Grothendieck-Riemann-Roch formula aims at giving such a formula for the Euler characteristic of any vector bundle, on any (regular, quasi-projective) scheme and in a relative setting. Furthermore, the Grothendieck-Riemann-Roch formula is universal in the sense that it is independent of the cohomology theory. This chapter is dedicated to the formulation of this theorem.

We first define

Definition 1.1.1. Let X be a scheme. The group $K_0(X)$ (resp. $K_0'(X)$) is the free abelian group generated by the isomorphism classes of locally free sheaves (resp.

coherent sheaves) on X, with relations $E = E' + E''$ if there is a short exact sequence

$$0 \to E' \to E \to E'' \to 0$$

If $f : X \to Y$ is a proper morphism of schemes, we define the map of abelian groups $Rf_* : K_0'(X) \to K_0'(Y)$ by the formula

$$Rf_*(E) := \sum_{k \geqslant 0} (-1)^k R^k f_*(E).$$

This is well-defined, for $Rf_*E = Rf_*E' + Rf_*E''$ in $K_0'(Y)$ by the long exact sequence in cohomology. The group $K_0(X)$ is a commutative ring under the tensor product $\otimes$ and $K_0'(X)$ is a $K_0(X)$-module under the natural map of abelian groups $K_0(X) \to K_0'(X)$. This map is an isomorphism if X is regular. Via this isomorphism, we obtain a map $Rf_* : K_0(X) \to K_0(Y)$, if both X and Y are regular. For any morphism $f : X \to Y$ of schemes, there is a pull-back map $Lf^* : K_0(Y) \to K_0(X)$, defined in the obvious way, which is a map of rings. Similarly, this gives a pull-back map $Lf^* : K_0'(Y) \to K_0'(X)$, if X and Y are regular

A theory kindred to K_0-theory is Chow theory:

Definition 1.1.2. The group $CH^{\cdot}(X)$ is the free abelian group on all integral closed subschemes of X, with relations div $f = 0$, $f \in k(Z)^*$ a rational function on a closed integral subscheme Z of X.

A (p-)cycle in X is a formal $\mathbb{Z}$-linear combination of integral closed subschemes (of codimension p) of X. If V is a closed subscheme of X, we write $[V]$ for the element $\sum_{C \in \mathrm{Irr}(V)} \mathrm{length}(\kappa(C)) C \in \mathrm{CH}^{\cdot}(X)$, if the lengths $\mathrm{length}(\kappa(C))$ are finite.

By work of Gillet and Soulé, if X is regular, the group $\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}$ can be made into a commutative ring such that $[W] \cdot [Z] = [Y \cap Z]$, if W, Z are closed integral subschemes of X intersecting transversally. If $f : X \to Y$ is a proper map, there is a push-forward map $f_* : \mathrm{CH}(X) \to \mathrm{CH}(Y)$ such that $f_*([Z]) = [k(Z) : k(f_*(Z))] \cdot [f_*Z]$ if dim $f_*Z =$ dim Z and such that $f_*([Z]) = 0$ otherwise. If X and Y are of finite type over a regular scheme S and f is a flat S-morphism, there is a pull-back map such that $f^*[Z] = [f^*Z]$. We denote by $\mathrm{CH}^k(X)$ the subgroup generated by the closed integral subschemes of codimension k. If X is regular, the group $\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}$ is naturally a graded ring.

If X is projective and smooth over $\mathbb{C}$ (say) then we may consider the singular cohomology group $H^{\mathrm{ev}}(X(\mathbb{C}), \mathbb{C})$. If $j : Z \hookrightarrow X$ is the inclusion morphism of a codimension p integral closed subscheme, we may define a $\mathbb{C}$-linear functional on $H^{2\dim_{\mathbb{C}}(X)-2p}(X(\mathbb{C}), \mathbb{C})$ by the formula

$$\eta \mapsto \int_{Z_{\mathrm{ns}}} j^*(\eta)$$

where Z_{ns} is th non-singular locus of Z. Poincaré duality then gives an element $\mathrm{cl}(Z) \in H^{2p}(X(\mathbb{C}), \mathbb{C})$, the *cycle class* of Z. The cycle class commutes with push-forward.

Suppose now that X is regular, flat and quasi-projective over a Dedekind domain. There is a unique ring homomorphism

$$\mathrm{ch} : K_0(X) \to \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}$$

called the *Chern character*, with the following properties:
- it is functorial with respect to (flat) pull-back;
- if Z is a 1-cycle in X, then $\mathrm{ch}(\mathcal{O}(Z)) = \exp([Z])$.

There is also a unique map

$$\mathrm{Td} : K_0(X) \to \mathrm{CH}^{\cdot}(X)^{*}_{\mathbb{Q}}$$

called the *Todd class*, with the properties:
- Td is functorial with respect to flat pull-back;
- $\mathrm{Td}(x + x') = \mathrm{Td}(x)\mathrm{Td}(x')$;
- if Z is a 1-cycle in X, then

$$\mathrm{Td}(\mathcal{O}(Z)) = \frac{[Z]}{1 - \exp(-[Z])}.$$

Finally, for each $k \geqslant 0$, there is a unique map $c : K_0(X) \to \mathrm{CH}^{\cdot}(X)^{*}_{\mathbb{Q}}$, called the total Chern class, such that
- it is functorial as above;
- $c(x + x') = c(x)c(x')$;
- if Z is a 1-cycle in X, then $c(\mathcal{O}(Z)) = 1 + [Z]$.

The element $c^k(x) := c(x)^{[k]}(x)$ ($[k]$ takes the k-th graded part) is called the k-th Chern class of $x \in K_0(X)$. For a vector bundle E/X, the following identities hold

$$\mathrm{ch}(E) = 1 + c^1(E) + \frac{1}{2}(c^1(E)^2 - 2c^2(E)) + \frac{1}{6}(c^1(E)^3 - 3c^1(E)c^2(E) + 3c^3(E)) + \dots$$

and

$$\mathrm{Td}(E) = 1 + \frac{1}{2}c^1(E) + \frac{1}{12}(c^1(E)^2 + c^2(E)) + \frac{1}{24}c^1(E)c^2(E)$$

If X is smooth and projective over $\mathbb{C}$, then $\mathrm{cl}(c^k(E)) = c^k(E) \in H^{\mathrm{ev}}(X(\mathbb{C}), \mathbb{C})$. We can now formulate the Grothendieck-Riemann-Roch theorem:

Theorem 1.1.3. *Let X, Y be regular schemes, which are quasi-projective over the spectrum S of a Dedekind ring. Let $f : X \to Y$ be a smooth S-morphism. Then*

$$\mathrm{ch}(Rf_*(x)) = f_*(\mathrm{Td}(Tf)\mathrm{ch}(x))$$

for any $x \in K_0(X)$. Here $Tf := \Omega_f^{\vee}$.

Example. Let $X := C$ be a smooth and projective curve over $\mathbb{C}$ as at the beginning. Let $S := \text{Spec } \mathbb{C} = Y$. Notice that $CH^{\cdot}(S) = \text{CH}^0(S) = \mathbb{Z}$ and that the Chern character of a vector bundle is in this case simply its rank. If we apply the Theorem 1.1.3 to $E := \mathcal{O}(D)$, we obtain,

$$\begin{aligned}\text{ch}(Rf_*(\mathcal{O}(D))) &= \chi(\mathcal{O}(D)) = f_*((1 + \frac{1}{2}c^1(TC))(1 + c^1(\mathcal{O}(D)))) \\ &= f_*(c^1(\mathcal{O}(D)) - \frac{1}{2}c^1(\Omega_C)) = \deg(D) - \frac{1}{2}\deg(K_C) = \deg(D) - \frac{1}{2}(2g-2) \\ &= \deg(D) + 1 - g\end{aligned}$$

which is the formula (1.1).

The Theorem 1.1.3 can be extended to hold for any projective morphism. If $f : X \to Y$ has a factorisation

$$f : X \xrightarrow{j} \mathbf{P}^r \times_S Y \xrightarrow{\pi} Y,$$

where j is a closed immersion and π is the natural projection, then the theorem still holds if one replaces $\text{Td}(Tf)$ by $j^*\text{Td}(T\pi)\text{Td}(N)^{-1}$. Here N is the normal bundle of the closed immersion j. The expression $j^*\text{Td}(T\pi)\text{Td}(N)^{-1}$ can be shown to be independent of the factorisation into j and π.

Bibliographical and historical notes. The Riemann-Roch theorem for curves was discovered by B. Riemann at the end of the nineteenth century. The generalisation of the theorem to higher dimensional manifolds (but still not in a relative situation) is due to F. Hirzebruch, who described the latter theorem in his book [Hi]. The general relative case was treated in the seminar [SGA6]. The presentation of the Grothendieck-Riemann-Roch theorem given here follows W. Fulton's book [F].

1.2 Thomasson's fixed point formula

In this section, we shall review a relative fixed point formula which is formally similar to the Theorem 1.1.3, but whose mathematical content is quite different. In the next subsection, this formula will be joined to the Grothendieck-Riemann-Roch theorem to obtain the equivariant Grothendieck-Riemann-Roch theorem.

Let S be a noetherian affine scheme. Let X be a regular scheme which is quasi-projective over S. Let μ_n be the diagonalisable group scheme over S which corresponds to $\mathbb{Z}/n\mathbb{Z}$. Suppose that X carries a μ_n-action over S; furthermore, suppose that there is an ample line bundle on X, which carries a μ_n-equivariant structure compatible with the μ_n-equivariant structure of X. We shall write $K_0^{\mu_n}(X)$ for the Grothendieck group of locally free sheaves on X which carry a compatible μ_n-equivariant structure. This group is defined exactly as in Definition 1.1.1. Replacing locally free sheaves by coherent sheaves in the definition of $K_0^{\mu_n}(X)$ leads to the group $K_0^{'\mu_n}(X)$, which is naturally isomorphic, as before. If the μ_n-equivariant structure of X is trivial, then the datum of a (compatible) μ_n-equivariant structure on a locally free sheaf E on X is equivalent to the datum of a $\mathbb{Z}/n\mathbb{Z}$-grading of E. For any μ_n-equivariant locally free sheaf E on X, we write $\Lambda_{-1}(E)$ for $\sum_{k=0}^{\mathrm{rk}(E)}(-1)^k\Lambda^k(E) \in K_0^{\mu_n}(X)$, where $\Lambda^k(E)$ is the k-th exterior power of E. There is a unique isomorphism of rings $K_0^{\mu_n}(S) \simeq K_0(S)[T]/(1-T^n)$ with the following property: it maps the structure sheaf of S endowed with a homogenous $\mathbb{Z}/n\mathbb{Z}$-grading of weight one to T and it maps any locally free sheaf carrying a trivial equivariant structure to the corresponding element of $K_0(S)$ $(= K_0'^{\mu_1}(S))$.

The functor of fixed points associated to X is by definition the functor

$$\textbf{Schemes}/S \to \textbf{Sets}$$

described by the rule

$$T \;\mapsto\; X(T)_{\mu_n(T)}.$$

Here $X(T)_{\mu_n(T)}$ is the set of elements of $X(T)$ which are fixed under each element of $\mu_n(T)$. The functor of fixed points is representable by a scheme X_{μ_n} and the canonical morphism $X_{\mu_n} \to X$ is a closed immersion. Furthermore, the scheme X_{μ_n} is regular . We shall denote the immersion $X_{\mu_n} \hookrightarrow X$ by i. Write $N^\vee$ for the dual of the normal sheaf of the closed immersion $X_{\mu_n} \hookrightarrow X$. It is locally free on X_{μ_n} and carries a natural μ_n-equivariant structure. This structure corresponds to a μ_n-grading, since X_{μ_n} carries the trivial μ_n-equivariant structure and it can be shown that the weight 0 term of this grading vanishes.

Let Y be a regular scheme which is quasi-projective over S and suppose that Y carries a μ_n-action over S. Let $f : X \to Y$ be a projective S-morphism which respects the μ_n-actions and write f^{μ_n} for the induced morphism $X_{\mu_n} \to Y$. The morphism f induces a direct image map $Rf_* : K_0'^{\mu_n}(X) \to K_0'^{\mu_n}(Y)$, which is a homomorphism of groups described by the formula $Rf_*(E) := \sum_{k\geqslant 0}(-1)^k R^k f_*(E)$ for a μ_n-equivariant coherent sheaf E on X. Here $R^k f_*(E)$ refers to the k-th higher direct image sheave of E under h; the sheaves $R^k f_*(E)$ are coherent and carry a natural μ_n-equivariant structure. The morphism h also induces a pull-back map $Lf^* : K_0^{\mu_n}(Y) \to K_0^{\mu_n}(X)$; this is a ring morphism which sends a μ_n-equivariant locally free sheaf E on Y on the locally free sheaf $f^*(E)$ on X, endowed with its natural μ_n-equivariant structure. For any elements $z \in K_0^{\mu_n}(X)$ and $w \in K_0^{\mu_n}(Y)$, the projection formula $Rf_*(z \cdot Lf^*(w)) = w \cdot Rf_*(z)$ holds. This implies that the group homomorphism Rf_* is a morphism of $K_0^{\mu_n}(S)$-modules, if the group $K_0^{\mu_n}(X)$ (resp. $K_0^{\mu_n}(Y)$) is endowed with the $K_0^{\mu_n}(S)$-module structure induced by the pull-back map $K_0^{\mu_n}(S) \to K_0^{\mu_n}(X)$ (resp. $K_0^{\mu_n}(S) \to K_0^{\mu_n}(Y)$).

Let $\mathcal{R}$ be a $K_0^{\mu_n}(S)$-algebra such that $1-T^k$ is a unit in R for all k such that $1 \leqslant k < n$.

Theorem 1.2.1. (1) *The element $\lambda_{-1}(N^\vee)$ is a unit in the ring $K_0^{\mu_n}(X_{\mu_n}) \otimes_{K_0{}^{\mu_n}(S)} \mathcal{R}$.*

(2) *If the μ_n-equivariant structure on Y is trivial, then for any element $x \in K_0^{\mu_n}(X)$, the equality*

$$Rf_*(x) = Rf^{\mu_n,*}(\Lambda_{-1}(N^\vee)^{-1} \cdot Li^*(x))$$

holds in $K_0^{\mu_n}(Y)\otimes_{K_0^{\mu_n}(S)}\mathcal{R}$.

For $\mathcal{R}$ one may for example choose $\mathbb{C}$ or $\mathbb{Q}(\mu_n)$ (sending T on a primitive root of unity). Notice that the formal analogy between Theorem 1.2.1 and Theorem 1.1.3: Li^* takes the place of the Chern character and $\Lambda_{-1}(N^\vee)^{-1}$ takes the place of the Todd class.

Bibliographical and historical notes. In the formulation given above, the Theorem 1.2.1 is contained in the article [LRR1], provided the base scheme S is a Dedekind ring which is embeddable in $\mathbb{C}$ and X and Y are flat over S, although these assumptions are not necessary. If $\mathcal{R}$ is taken to be a field, then the Theorem 1.2.1 is a consequence of the main result of [T3].

1.3 An equivariant Grothendieck-Riemann-Roch theorem

If we combine the Grothendieck-Riemann-Roch theorem and the fixed point theorem of Thomasson and Nori, we obtain the following theorem. We set $\zeta_n := \exp(2i\pi/n)$.

Theorem 1.3.1. *Let X, Y be regular schemes, which are quasi-projective over the spectrum S of a Dedekind ring. Suppose that they are both equipped with a μ_n-action over S and that they both carry μ_n-equivariant ample line bundles. Suppose also that the μ_n-structure of Y is trivial. Let $f : X \to Y$ be a μ_n-equivariant projective morphism. Then for any $x \in K_0^{\mu_n}(X)$, the formula*

$$\mathrm{ch}_{\mu_n}(Rf_*(x)) = f_*(\mathrm{ch}_{\mu_n}(\Lambda_{-1}(N^\vee))^{-1}\mathrm{Td}(Tf^{\mu_n})\mathrm{ch}_{\mu_n}(x))$$

holds in $\mathrm{CH}^{\cdot}(Y)_{\mathbb{C}}$.

Here again, N refers to the normal bundle of the immersion $X_{\mu_n} \to X$. Here, if E is a μ_n-equivariant vector bundle on X, we write E_k for the k-th graded piece of the restriction of E to X_{μ_n} and

$$\mathrm{ch}_{\mu_n}(E) := \sum_{k\in\mathbf{Z}/n} \zeta_n^k \mathrm{ch}(E_k) \in \mathrm{CH}(X_{\mu_n})_{\mathbb{C}}.$$

We could also have replaced $\mathbb{C}$ by $\mathbb{Q}(\mu_n)$ in the Theorem 1.3.1.

Example. Let $S = \mathrm{Spec}\ \mathbb{C}$ and let $Y := S$ and X be a a projective complex manifold of dimension d over S. Suppose that X is endowed with a $\mu_{n\mathbb{C}}$-action. This is equivalent to specifying an action of the group $\mu_n(\mathbb{C})$. We shall apply Theorem 1.3.1 to the de Rham complex

$$x := 1 - \Omega_X + \Lambda^2\Omega_X - \Lambda^3\Omega_X + \cdots + (-1)^d\Lambda^d\Omega_X,$$

which consists of naturally equivariant vector bundles on X. Thanks to the exact sequence on X_{μ_n}

$$0 \to N^\vee \to \Omega_X|_{X_{\mu_n}} \to \Omega_{X_{\mu_n}} \to 0$$

we have the equality

$$\Lambda_{-1}(N^{\vee})\Lambda_{-1}(\Omega_{X_{\mu_n}}) = \Lambda_{-1}(\Omega_X|_{X_{\mu_n}})$$

in $K_0^{\mu_n}(X_{\mu_n})$. We can thus compute the localised side of the formula of Theorem 1.3.1 as

$$\mathrm{ch}_{\mu_n}(\mathrm{ch}_{\mu_n}(\Lambda_{-1}(N^{\vee}))^{-1}\mathrm{Td}(X_{\mu_n})\Lambda_{-1}(\Omega_X|_{X_{\mu_n}})) = \mathrm{ch}(\Lambda_{-1}(\Omega_{X_{\mu_n}})\mathrm{Td}(\Omega^{\vee}_{X_{\mu_n}})) = c^{\mathrm{top}}(TX_{\mu_n})$$

whereas the global side can be computed as

$$\begin{aligned}\mathrm{ch}_{\mu_n}(Rf_*(1-\Omega_X+\Omega_X^2-\Omega^3X+\cdots+(-1)^d\Omega_X^d)) &= \sum_{i,j}(-1)^{i+j}\mathrm{Tr}\,_{\zeta_n}(H^i(X,\Omega_X^j)) \\ &= \sum_k(-1)^k\mathrm{Tr}\,_{\zeta_n}(H^k(X(\mathbb{C}),\mathbb{C}))\end{aligned}$$

the last equality being justified by the Hodge decomposition theorem and the fact that analytic and algebraic cohomology coincide on smooth projective manifolds over $\mathbb{C}$. We thus obtain, after application of the cycle class

$$\sum_k(-1)^k\mathrm{Tr}\,_{\zeta_n}(H^k(X(\mathbb{C}),\mathbb{C})) = \int_{X_{\mu_n}} c^{\mathrm{top}}(TX_{\mu_n})$$

and in particular, if X_{μ_n} consists of a finite set of points

$$\sum_k(-1)^k\mathrm{Tr}\,_{\zeta_n}(H^k(X(\mathbb{C}),\mathbb{C})) = \#X_{\mu_n}(\mathbb{C}).$$

This last formula is just the classical topological Lefschetz fixed point formula for $X(\mathbb{C})$ and the endomorphism given by ζ_n.

Chapter 2

Riemann-Roch formulae in Arakelov geometry

2.1 Arakelov geometry

Arakelov geometry is an extension of scheme-theoretic algebraic geometry, where one tries to treat the places at infinity (corresponding to the archimedean valuations) on the same footing as the finite ones. To be more precise, consider a scheme S which is proper over Spec $\mathbb{Z}$ and generically smooth. For each prime $p \in$ Spec $\mathbb{Z}$, we then obtain by base-change a scheme $S_{\mathbb{Z}_p}$ on the spectrum of the ring of p-adic integers $\mathbb{Z}_p$. The set $S(\mathbb{Q}_p)$ is then endowed with the following natural notion of distance. Let $P, R \in S(\mathbb{Q}_p)$; by the valuative criterion of properness, we can uniquely extend P and Q to elements $\widetilde{P}, \widetilde{R}$ of $S(\mathbb{Z}_p)$. We can then define the distance $d(P, R)$ by the formula

$$d(P,R) := p^{-\max\{k\in\mathbb{Z}|\widetilde{P} \bmod p^k = \widetilde{Q} \bmod p^k\}}$$

This distance arises naturally from the scheme structure. No such distance arises for the set $S(\mathbb{C})$ and the strategy of Arakelov geometry is to equip $S(\mathbb{C})$, as well as the vector bundles thereon with a metric in order to make up for that lack. The scheme S together with a metric on $S(\mathbb{C})$ is then understood as a "compactification" of S, in the sense that it is supposed to live on the "compactification" of Spec $\mathbb{Z}$ obtained by adding the archimedean valuation. The introduction of hermitian metrics, which are purely analytic data, implies that Arakelov will rely on a lot of analysis to define direct images, intersection numbers etc. Here is the beginning of a list of extensions of classical scheme-theoretic objects that have been worked out:

S	S with a hermitian metric on $S(\mathbb{C})$
E a vector bundle on S	E a vector bundle on S with a hermitian metric on $E(\mathbb{C})$
cycle Z on S	a cycle Z on S with a Green current for $Z(\mathbb{C})$
the degree of a variety	the height of a variety
the determinant of cohomology	the determinant of cohomology equipped with its Quillen metric
the Todd class of Tf	the arithmetic Todd class multiplied by (1-$R(Tf)$)
$\vdots$	$\vdots$

Many theorems of classical algebraic geometry have been extended to Arakelov theory. In particular, there are analogs of the Hilbert-Samuel theorem (see [GS8] and [Abbes]), of the Nakai-Moishezon of ampleness (see [Zhang]), of the Grothendieck-Riemann-Roch theorem (see [GS8]) and finally there is an analog of the equivariant Grothendieck-Riemann-Roch, whose description is the main aim of this series of lectures.

Bibliographical and historical notes. Arakelov geometry started officially in S. Arakelov's paper [Ara]. It was then further developped by G. Faltings, who extended the Riemann-Roch theorem for surfaces in [Fal] and by L. Szpiro and his students. The theory was then vastly generalised by H. Gillet and C. Soulé, who defined compactified Chow rings, Grothendieck groups and characteristic classes in all dimensions (see [GS2] and [GS3]). For an introduction to Arakelov geometry, see the book [SABK].

2.2 An arithmetic equivariant Grothendieck-Riemann-Roch theorem

The aim of this section is to formulate the analog in Arakelov geometry of the Theorem 1.3.1.

Let D be a regular arithmetic ring. By this we mean a regular, excellent, Noetherian integral ring, together with a finite set $\mathcal{S}$ of injective ring homomorphisms of $D \to \mathbf{C}$, which is invariant under complex conjugation. We fix as before $\zeta_n := \exp(2i\pi/n)$.

We shall call **equivariant arithmetic variety** an integral scheme which is regular and quasi-projective over Spec D, endowed with a μ_n-equivariant structure over D and such that there is an ample μ_n-equivariant line bundle on X. We write $X(\mathbf{C})$ for the set of complex points of the variety $\coprod_{e\in\mathcal{S}} X \times_D \mathbf{C}$, which naturally carries the structure of a complex manifold. The groups $\mu(\mathbb{C})$ acts on $X(\mathbf{C})$ by holomorphic automorphisms and we shall write g for the automorphism corresponding to ζ_n. As we have seen, the fixed point scheme X_{μ_n} is regular and there are natural isomorphisms of complex manifolds $X_{\mu_n}(\mathbf{C}) \simeq (X(\mathbf{C}))_g$, where $(X(\mathbf{C}))_g$ is the set of fixed points of X under the action of $\mu(\mathbb{C})$. Complex conjugation induces an antiholomorphic automorphism of $X(\mathbf{C})$ and $X_{\mu_n}(\mathbf{C})$, both of which we denote by F_∞.

We write $\widetilde{\mathfrak{A}}(X_{\mu_n})$ for

$$\widetilde{\mathfrak{A}}(X(\mathbf{C})_g) := \bigoplus_{p\geqslant 0}(\mathfrak{A}^{p,p}(X(\mathbf{C})_g)/(\operatorname{Im}\partial + \operatorname{Im}\overline{\partial})),$$

where $\mathfrak{A}^{p,p}(\cdot)$ denotes the set of smooth complex differential forms ω of type (p,p), such that $F_\infty^*\omega = (-1)^p\omega$.

A hermitian equivariant sheaf (resp. vector bundle) on X is a coherent sheaf (resp. a vector bundle) E on X, assumed locally free on $X(\mathbf{C})$, equipped with a μ_n-action which lifts the action of μ_n on X and a hermitian metric h on $E(\mathbf{C})$, the bundle associated to E on the complex points, which is invariant under F_∞ and μ_n. We shall write (E,h) or $\overline{E}$ for an hermitian equivariant sheaf (resp. vector bundle). There is a natural $\mathbf{Z}/(n)$-grading $E|_{X_{\mu_n}} \simeq \oplus_{k\in\mathbf{Z}/(n)}E_k$ on the restriction of E to X_{μ_n}, whose terms are orthogonal, because of the invariance of the metric. We write $\overline{E}_k$ for the k-th term ($k \in \mathbf{Z}/(n)$), endowed with the induced metric. We also often write $\overline{E}_{\mu_n}$ for $\overline{E}_0$.

If $\overline{V} = (V, h_V)$ is a hermitian vector bundle on X_{μ_n} we write $\operatorname{ch}(\overline{V})$ for the differential form $\operatorname{Tr}(\exp(\Omega))$, where Ω is the curvature form associated to the connection on $V(\mathbb{C})$ whose matrix is given locally by $\partial H \cdot H^{-1}$. This differential form represents the Chern character in de Rham cohomology. We write $\operatorname{ch}_g(\overline{E})$ for the equivariant Chern character form

$$\operatorname{ch}_g((E_{\mathbf{C}}, h)) := \sum_{k\in\mathbb{Z}/(n)} \zeta_n^k \operatorname{ch}(\overline{E}_k).$$

The symbol $\operatorname{Td}_g(\overline{E})$ refers to the differential form

$$\operatorname{Td}(\overline{E}_{\mu_n})\Big(\sum_{i\geqslant 0}(-1)^i\operatorname{ch}_g(\Lambda^i(\overline{E}))\Big)^{-1}.$$

If $\mathcal{E} : 0 \to E' \to E \to E'' \to 0$ is an exact sequence of equivariant sheaves (resp. vector bundles), we shall write $\overline{\mathcal{E}}$ for the sequence $\mathcal{E}$ together with $\mu(\mathbb{C})$- and F_∞- invariant hermitian metrics on $E'(\mathbf{C})$, $E(\mathbf{C})$ and $E''(\mathbf{C})$. To $\overline{\mathcal{E}}$ and ch_g is associated an equivariant Bott-Chern secondary class $\widetilde{\operatorname{ch}}_g(\overline{\mathcal{E}}) \in \widetilde{\mathfrak{A}}(X_{\mu_n})$, which satisfies the equation $\frac{i}{2\pi}\partial\overline{\partial}\widetilde{\operatorname{ch}}_g(\overline{\mathcal{E}}) = \operatorname{ch}_g(\overline{E}') + \operatorname{ch}_g(\overline{E}'') - \operatorname{ch}_g(\overline{E})$. This class is functorial for any morphism of arithmetic varieties and vanishes if the sequence $\overline{\mathcal{E}}$ splits isometrically.

Definition 2.2.1. The arithmetic equivariant Grothendieck group $\widehat{K}_0'^{\mu_n}(X)$ (resp. $\widehat{K}_0^{\mu_n}(X)$) of X is the free abelian group generated by the elements of $\widetilde{\mathfrak{A}}(X_{\mu_n})$ and by the equivariant isometry classes of hermitian equivariant sheaves (resp. vector bundles), together with the relations

(a) for every exact sequence $\overline{\mathcal{E}}$ as above, $\widetilde{\operatorname{ch}}_g(\overline{\mathcal{E}}) = \overline{E}' - \overline{E} + \overline{E}''$;

(b) if $\eta \in \widetilde{\mathfrak{A}}(X_{\mu_n})$ is the sum in $\widetilde{\mathfrak{A}}(X_{\mu_n})$ of two elements η' and η'', then $\eta = \eta' + \eta''$ in $\widehat{K}_0^{'\mu_n}(X)$ (resp. $\widehat{K}_0^{\mu_n}(X)$).

We shall now define a ring structure on $\widehat{K}_0^{\mu_n}(X)$ (resp. $\widehat{K}_0^{\mu_n}(X)$-module structure on $\widehat{K}_0^{'\mu_n}(X)$). Let $\overline{V}$ be a hermitian equivariant vector bundle and let $\overline{V}'$ be a hermitian equivariant sheaf. Let η, η' be elements of $\widetilde{\mathfrak{A}}(X_{\mu_n})$. We define a product $\cdot$ by the rules $\overline{V} \cdot \overline{V}' := \overline{V} \otimes \overline{V}'$, $\overline{V} \cdot \eta = \eta \cdot \overline{V} := \mathrm{ch}_g(\overline{V}) \wedge \eta$ and $\eta \cdot \eta' := \frac{i}{2\pi}\partial\overline{\partial}\eta \wedge \eta'$ and we extend it by linearity. We omit the proof that it is well-defined. Notice that the definition of $\widehat{K}_0^{'\mu_n}(X)$ (resp. $\widehat{K}_0^{\mu_n}(X)$) implies that there is an exact sequence of abelian groups

$$\widetilde{\mathfrak{A}}(X_{\mu_n}) \to \widehat{K}_0^{\mu_n\,'}(X) \to K_0^{'\mu_n}(X) \to 0 \tag{2.1}$$

(resp.

$$\widetilde{\mathfrak{A}}(X_{\mu_n}) \to \widehat{K}_0^{\mu_n}(X) \to K_0^{\mu_n}(X) \to 0 \quad),$$

where $K_0^{'\mu_n}(X)$ (resp. $K_0^{\mu_n}(X)$) is the ordinary Grothendieck group of μ_n-equivariant coherent sheaves (resp. locally free sheaves). Notice finally that there is a map from $\widehat{K}_0^{\mu_n'}(X)$ to the space of complex closed differential forms, which is defined by the formula $\mathrm{ch}_g(\overline{E} + \kappa) := \mathrm{ch}_g(\overline{E}) + \frac{i}{2\pi}\partial\overline{\partial}\kappa$ ($\overline{E}$ an hermitian equivariant sheaf, $\kappa \in \widetilde{\mathfrak{A}}(X_{\mu_n})$). This map is well-defined and we shall denote it by $\mathrm{ch}_g(\cdot)$ as well. We have as before: if X is regular then the natural morphism $\widehat{K}_0^{\mu_n}(X) \to \widehat{K}_0^{'\mu_n}(X)$ is an isomorphism.

Now let $f : X \to Y$ be an equivariant projective morphism of relative dimension d over D of equivariant arithmetic varieties. We suppose that f is smooth over the generic point of D. We endow X with a Kähler fibration structure; this is a family of Kähler metrics on the fibers of $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$, satisfying a supplementary condition that we do not have the room to detail here. It is encoded in a $(1,1)$-form ω_f on $X(\mathbb{C})$. We shall see an example of such a structure in the applications. We suppose that ω_f is g-equivariant. Suppose that the action of μ_n on Y is trivial. Suppose as well that there is a μ_n-equivariant line bundle over X, which is very ample relatively to f. Let $\overline{E} := (E, h)$ be an equivariant hermitian sheaf on X. Suppose that $R^k f_*(E)_{\mathbb{C}}$ is locally free for all $k \geqslant 0$. We let $R^{\cdot} f_* \overline{E} := \sum_{k \geqslant 0} (-1)^k R^k f_* \overline{E}$ be the alternating sum of the higher direct image sheaves, endowed with their natural equivariant structures and L_2-metrics. For each $y \in Y(\mathbb{C})$, the L^2-metric on $R^i f_* E(\mathbb{C})_y \simeq H^i_{\overline{\partial}}(X(\mathbb{C})_y, E(\mathbb{C})|_{X(\mathbb{C})_y})$ is defined by the formula

$$\frac{1}{(2\pi)^d} \int_{Y(\mathbb{C})_y} h(s,t)\omega_X^d \tag{2.2}$$

where s and t are harmonic (i.e. in the kernel of the Kodaira Laplacian $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$) sections of $\Lambda^i(T^{*(0,1)}X(\mathbb{C})_y) \otimes E(\mathbb{C})|_{X(\mathbb{C})_y}$. This definition is meaningful because by Hodge theory there is exactly one harmonic representative in each cohomology class.

Let $\eta \in \widetilde{\mathfrak{A}}(X_{\mu_n})$. Consider the rule which associates the element $Rf_*\overline{E} - T_g(X, \overline{E})$ of $\widehat{K}_0^{'\mu_n}(Y)$ to $\overline{E}$ and the element $\int_{X(\mathbf{C})_g} \mathrm{Td}_g(\overline{TX})\eta \in \widehat{K}_0^{'\mu_n}(Y)$ to η. Here $T_g(\overline{E}) \in \widetilde{\mathfrak{A}}(Y)$ is the equivariant analytic torsion form. Its definition is too involved to be given in its entirety here but we shall define its component of degree 0.

Let $\Box_q^E$ be the differential operator $(\overline{\partial}+\overline{\partial}^*)^2$ acting on the C^∞-sections of the bundle $\Lambda^q T^{*(0,1)}X(\mathbb{C})_y \otimes E(\mathbb{C})|_{X(\mathbb{C})_y}$. This space of sections is equipped with the L^2-metric as above and the operator $\Box_q^{E(\mathbb{C})|_{X(\mathbb{C})_y}}$ is symmetric for that metric; we let $\mathrm{Sp}(\Box_q^{E(\mathbb{C})|_{X(\mathbb{C})_y}}) \subseteq \mathbb{R}$ be the set of eigenvalues of $\Box_q^{E(\mathbb{C})|_{X(\mathbb{C})_y}}$ (which is discrete and bounded from below) and we let $\mathrm{Eig}_q^{E(\mathbb{C})|_{X(\mathbb{C})_y}}(\lambda)$ be the eigenspace associated to an eigenvalue λ (which is finite-dimensional). Define

$$Z(\overline{E}|_{X(\mathbb{C})_y}, g, s) := \sum_{q \geqslant 1} (-1)^{q+1} q \sum_{\lambda \in \mathrm{Sp}(\Box_q^{E(\mathbb{C})|_{X(\mathbb{C})_y}}) \setminus \{0\}} \mathrm{Tr}(g^*|_{\mathrm{Eig}_q^{E(\mathbb{C})|_{X(\mathbb{C})_y}}(\lambda)}) \lambda^{-s}$$

for $\Re(s)$ sufficiently large. As a function of s, the function $Z(\overline{E}|_{X(\mathbb{C})_y}, g, s)$ has a meromorphic continuation to the whole plane, which is holomorphic around 0. By definition, the equivariant analytic torsion of $\overline{E}|_{X(\mathbb{C})_y}$ is given by $T_g(\overline{E}|_{X(\mathbb{C})_y}) := Z'(\overline{E}|_{X(\mathbb{C})_y}, g, 0)$. If E is f-acyclic (which is our assumption) then $T_g(\overline{E}|_{X(\mathbb{C})_y})$ is a C^∞-function of y and it is the degree 0-part of the equivariant analytic torsion form $T_g(\overline{E})$.

Proposition 2.2.2. *The above rule extends to a well defined group homomorphism* $f_* : \widehat{K}_0^{'\mu_n}(X) \to \widehat{K}_0^{'\mu_n}(Y)$.

We shall need the definition (due to Gillet and Soulé) of "compactified" Chow theory. Let $p \geqslant 0$. We shall write $D^{p,p}(X(\mathbb{C}))$ for the space of complex currents of type p, p on $X(\mathbb{C})$ on which F_∞^* acts by multiplication by $(-1)^p$. Now let A be a subring of $\mathbb{C}$. If Z is a p-cycle with coefficients in A on X, a Green current g_Z for Z is an element of $D^{p,p}(X(\mathbb{C}))$ which satisfies the equation

$$\frac{i}{2\pi}\partial\overline{\partial} g_Z + \delta_{Z(\mathbb{C})} = \omega_Z$$

where ω_Z is a differential form.

Definition 2.2.3. Let $p \geqslant 0$. The arithmetic Chow group $\widehat{\mathrm{CH}}_A^p(X)$ is the A-vector space generated by the ordered pairs (Z, g_Z), where Z is a p-cycle with coefficients in A on X and g_Z is a Green current for $Z(\mathbb{C})$, with the relations

(i) $\lambda \cdot (Z, g_Z) + (Z', g_{Z'}) = (\lambda \cdot Z + Z', \lambda \cdot g_Z + g_{Z'})$;

(ii) $(\mathrm{div}(f), -\log|f|^2 + \partial u + \overline{\partial} v) = 0$;

where f is a non-zero rational function defined on a closed integral subscheme of X and u (resp. v) is a complex current of type $(p-2,p-1)$ (resp. $(p-1,p-2)$) such that $F_\infty^*(\partial u+\overline{\partial}v)=(-1)^{p-1}(\partial u+\overline{\partial}v)$.

We shall write $\widehat{\mathrm{CH}}^{\cdot}(X):=\oplus_{p\geqslant 0}\widehat{\mathrm{CH}}^p(X)$. The group $\widehat{\mathrm{CH}}(X)$ is equipped with a natural A-algebra structure, such that $(Z,g_Z)\cdot(Z',g_{Z'})=(Z\cap Z',g_Z*g_{Z'})$ if Z,Z' are integral, intersect transversally. Here the symbol * refers to the star product, whose definition is too involved to be given here. The group $\widehat{\mathrm{CH}}^*(X)$ has pull-back maps (given by the obvious formula) with respect to flat and generically smooth morphisms. If $f:X\to Y$ is as above, there is a push-forward map $\widehat{\mathrm{CH}}^{\cdot}(X)\to\widehat{\mathrm{CH}}^{\cdot}(Y)$, such that $f_*(Z,g_Z)=(\deg(Z/f_*Z)f_*Z,f_*g_Z)$ for every integral closed subscheme Z of X and Green current g_Z of Z. Here we set $\deg(Z/f_*Z)=[\kappa(Z):\kappa(f_*(Z))]$ if $\dim(f_*(Z))=\dim(Z)$ and $\deg(Z/f_*Z)=0$ otherwise. It is an easy exercise to show that the map of A-modules $\mathbb{C}\to\widehat{\mathrm{CH}}^1_A(\mathbb{Z})$, defined by the recipe $z\mapsto(0,z)$ is an isomorphism.

There is a ring morphism

$$\widehat{\mathrm{ch}}:\widehat{K}_0(X)\to\widehat{\mathrm{CH}}^{\cdot}_{\mathbb{C}}(X)$$

called the arithmetic Chern character, such that

- it is functorial;
- $\widehat{\mathrm{ch}}(\eta)=(0,\eta)$;
- if $\overline{L}=(L,h)$ is a hermitian line bundle on X and s a rational section of L then $\widehat{\mathrm{ch}}(\overline{\mathcal{O}(Z)})=\exp((\mathrm{div}\ \ s,-\log\ \ h(s,s)))$.

Example. Suppose in this example that X is of relative dimension 1 and proper over $D=\mathbb{Z}$. Suppose also that Z and Z' are two integral closed subschemes of codimension 1 of X, which intersect transversally, are flat over Spec $\mathbb{Z}$ and do not intersect on the generic fiber. As $Z(\mathbb{C})$ (resp. $Z'(\mathbb{C})$) consists of one point P (resp. P'), this last condition just says that $Z(\mathbb{C})\neq Z'(\mathbb{C})$. Now equip $\mathcal{O}(Z)$ (resp. $\mathcal{O}(Z')$) with a (conjugation invariant) hermitian metric h (resp. h') and let s be a rational section of $\mathcal{O}(Z)$ (resp. s' be a rational section of $\mathcal{O}(Z')$) vanishing exactly on Z (resp. Z'). In this case, we have

$$(Z,-\log\ h(s,s))\cdot(Z',-\log\ h'(s',s'))=(Z\cap Z',-\log\ h(s(P',P'))\delta_Z-c^1(\overline{\mathcal{O}(Z)})\log\ h'(s',s'))$$

and hence, if f is the morphism $X\to$ Spec $\mathbb{Z}$,

$$f_*(\widehat{c}^1(\overline{\mathcal{O}(Z)})\cdot\widehat{c}^1(\overline{\mathcal{O}(Z')}))=2\sum_{p\in f_*(Z\cap Z')}\log\ p-\log\ h(s(P',P'))-\int_{X(\mathbb{C})}c^1(\overline{\mathcal{O}(Z)})\log\ h'(s',s').$$

Using the arithmetic Chern character, we may also define an arithmetic Todd class $\widehat{\mathrm{Td}}:\widehat{K}_0(X)\to\widehat{\mathrm{CH}}^{\cdot}_{\mathbb{C}}(X)$ and an arithmetic total Chern class.

If $\overline{E}$ is an equivariant hermitian vector bundle on X, we write

$$\widehat{\mathrm{ch}}_{\mu_n}(\overline{E}):=\sum_{k\in\mathbf{Z}/n}\zeta_n^k\widehat{\mathrm{ch}}(\overline{E}_k)\in\widehat{\mathrm{CH}}^{\cdot}_{\mathbb{Q}(\mu_n)}(X_{\mu_n})$$

for the equivariant arithmetic Chern character. We write as before $\Lambda_{-1}(\overline{E}) := \sum_{k=0}^{\mathrm{rk}(E)}(-1)^k\Lambda^k(\overline{E}) \in \widehat{K}_0^{\mu_n}(X)$, where $\Lambda^k(\overline{E})$ is the k-th exterior power of $\overline{E}$, endowed with its natural hermitian and equivariant structure.

For any $z \in \mathbb{C}$, $|z| = 1$, define the Lerch zeta function

$$\zeta_L(z,s) := \sum_{k\geqslant 1}\frac{z^k}{k^s}$$

which is naturally defined for $Re(s) > 1$ and can be meromorphically continued to the whole plane. Define the formal complex power series

$$\widetilde{R}(z,x) := \sum_{k\geqslant 0}\Big(2\zeta_L'(z,-k) + (1+\frac{1}{2}+\cdots+\frac{1}{k})\zeta_L(z,-k)\Big)\frac{x^k}{k!}.$$

and

$$R(z,x) := \frac{1}{2}(\widetilde{R}(z,x) - \widetilde{R}(\overline{z},-x))$$

We identify $R(z,x)$ with the unique additive cohomology class it defines. For a $\mu_n(\mathbb{C})$-equivariant vector bundle on $X(\mathbb{C})$, define the cohomology class on $X(\mathbb{C})_g$ by the formula

$$R_g(E) := \sum_{k\in\mu_n(\mathbb{C})} R(\zeta_n^k, E_k).$$

Choose any μ_n-invariant (conjugation invariant) hermitian metric on $V_{\mathbb{C}}$; this hermitian metric induces a connection of type $(1,0)$ on each $V_{\mathbb{C},k}$; using this connection, we may compute a differential form representative of $R(\arg(\zeta(g)^k), E_k)$ in complex de Rham cohomology; this representative is a sum of differential forms of type (p,p) $(p \geq 0)$, which is both $\partial-$ and $\overline{\partial}-$closed. In the next theorem, we may thus consider that the values of $R_g(\cdot)$ lie in $\widetilde{\mathfrak{A}}(X_{\mu_n})$.

Let $\overline{N} = \overline{N}_{X/X_{\mu_n}}$ be the normal bundle of X_{μ_n} in X, endowed with its quotient equivariant structure and quotient metric structure (which is F_∞-invariant).

Theorem 2.2.4. *The equality*

$$\widehat{\mathrm{ch}}_{\mu_n}(f_*(x)) = f_*(\widehat{\mathrm{ch}}_{\mu_n}(\Lambda_{-1}(\overline{N}^\vee))^{-1}\mathrm{Td}(\overline{Tf}^{\mu_n})(1 - R_g(Tf))\widehat{\mathrm{ch}}_{\mu_n}(x))$$

holds in $\widehat{CH}^{\cdot}_{\mathbb{Q}(\mu_n)}(Y)$, *for any* $x \in \widehat{K}_0^{\mu_n}(X)$.

Here $\widehat{\mathrm{Td}}(\overline{Tf}^{\mu_n})$ is the arithmetic Todd class of $\overline{\Omega}^\vee(f^{\mu_n})$ if f^{μ_n} is smooth.

Bibliographical and historical notes. A complete proof of the Theorem 2.2.4 yet has to be published but a proof of the degree 1 part of the equality in 2.2.4 follows immediately from [LRR1] and [GS8]. It is important to underline that the most difficult part of the proof is analytic in nature and is contained in J.-M. Bismut's article [B3].

2.3 First applications

2.3.1 The key formula on abelian varieties

Suppose S is the spectrum of the ring of integers of a number field and let $\pi : \mathcal{A} \to S$ be an abelian scheme of relative dimension d over S. We shall apply the Theorem 2.2.4 to the morphism π and the trivial equivariant structure. Choose a line bundle L on $\mathcal{A}$ which is symmetric and ample on the generic fiber of $\mathcal{A}$. Endow it with a positive hermitian metric whose curvature form is translation invariant and endow $\mathcal{A}(\mathbb{C})$ with the Kähler metric whose Kähler form is $c^1(\overline{L})$. We also suppose that $R^k\pi_*(L) = 0$ for $k > 1$ and that the restriction of $\overline{L}$ via the zero-section is isometrically isomorphic to the trivial line bundle with the trivial metric. It can be shown that $\tau(\overline{L}) = \frac{1}{2}\chi(L)\log\frac{1}{(2\pi)^d}$ and that $\pi_*(\widehat{c}^1(\overline{L})^{d+1}) = 0$. We can thus compute

$$\begin{aligned}\widehat{c}^1(R\pi_*(\overline{L})) - \frac{1}{2}\chi(L)\log\frac{1}{(2\pi)^d} &= \widehat{c}^1(\pi_*(\overline{L})) - \frac{1}{2}\chi(L)\log\frac{1}{(2\pi)^d} = \widehat{\mathrm{Td}}(\overline{\Omega}^\vee)\pi_*(\widehat{\mathrm{ch}}(\overline{L})) \\ &= -\frac{1}{2}\chi(L)\widehat{c}^1(\overline{\Omega}).\end{aligned}$$

We now takes the direct image of both ends of the last equation under the morphism $S \to \operatorname{Spec}\mathbb{Z}$ and we divide by $\chi(L)$. We get

$$\frac{1}{\chi(L)[\kappa(S):\mathbb{Q}]}\widehat{c}^1(\pi_*(\overline{L})) = -\frac{1}{2}\frac{1}{[\kappa(S):\mathbb{Q}]}\widehat{c}^1(\overline{\Omega}) + \frac{1}{2}\log\frac{1}{(2\pi)^d} \qquad (2.3)$$

(we have sometimes identified $\overline{\Omega}$ with its restriction to C via the zero-section in the computation). The equation (2.3) is usually called the key formula. It was first proved by L. Moret-Bailly, who even proved a more general version allowing semi-abelian singularities on the finite fibers. It turns out that the quantity $\widehat{c}^1(\overline{\Omega})$ (which can be identified with a real number) is independent of the choices of the hermitian metrics. When $d = 1$ and the generic fiber has complex multiplication, it can be computed and is given (via the Chowla-Selberg formula) by linear combinations of logarithms of special values of the Γ function. For instance, in the case of the elliptic curve $y^2 = x^3 + 1$, the equality can be made completly explicit. Let Λ be the lattice $\mathbb{Z} + j\mathbb{Z}$ in $\mathbb{C}$, where $j := -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Let $\mathfrak{p}(\cdot,\Lambda)$ (resp. $\sigma(\cdot,\Lambda)$) be the associated $\mathfrak{p}$ function (resp. σ-function). We choose a number field where the curve $y^2 = x^3 + 1$ has good reduction everywhere and we take the associated S. The resulting equality is

$$\begin{aligned}&\log\int_0^1\int_0^1 |e^{-(\alpha+j\beta)(\mathfrak{p}(\frac{1}{2},\Lambda)\alpha+\mathfrak{p}(\frac{j}{2},\Lambda)\beta)}\sigma(\alpha+j\beta,\Lambda)|^2\,\mathrm{d}\alpha\,\mathrm{d}\beta \\ =\ &-\frac{3}{2}\log(\frac{1}{\sqrt{3}}(\frac{\Gamma(1/3)}{\Gamma(2/3)})^3) - \log(\frac{2}{\sqrt{3}}) + \frac{1}{2}\log\ 2 - \frac{1}{2}\log\ \pi.\end{aligned}$$

This identity seems rather mysterious and I do not know of a direct analytic proof for it.

Bibliographical and historical notes. The key formula was first proved in [MB]. A proof of the formula via the arithmetic Riemann-Roch theorem is given in [Bost].

2.3.2 Fibrations of abelian varieties

Let S be an open subscheme of the spectrum of the ring of integers of a number field. Consider an arithmetic variety C over S. Consider furthermore a principally polarised abelian scheme $\pi : \mathcal{A} \to C$ of relative dimension d. The principal polarisation induces a Kähler fibration structure on $\mathcal{A}(\mathbb{C}) \to C(\mathbb{C})$ and shall apply the Theorem 2.2.4 to the morphism π, to the trivial μ_n-structure on $\mathcal{A}$ and to the trivial bundle endowed with the trivial μ_n-structure and metric. We also normalise the Kähler fibration in such a way that the volume of the fibers is 1.

We compute in $\widehat{\mathrm{CH}}^{\cdot}_{\mathbb{Q}(\mu_n)}(C)$:

$$\begin{aligned}
\widehat{\mathrm{ch}}(R\pi_*(\overline{O})) - \tau(\overline{O}) &= \widehat{\mathrm{ch}}(\Lambda_{-1}(\overline{R^1\pi_*\mathcal{O}})) - \tau(\overline{O}) \\
&= \widehat{\mathrm{ch}}(\Lambda_{-1}(\overline{\Omega}^{\vee})) - \tau(\overline{O}) \\
&= \widehat{c}^{\mathrm{top}}(\overline{\Omega}) + \text{terms of higher degree} - \tau(\overline{O}) \\
&= \pi_*(\widehat{\mathrm{Td}}(\Omega^{\vee})(1 - R(\Omega^{\vee})) = 0
\end{aligned}$$

(again we have sometimes identified $\overline{\Omega}$ with its restriction to C via the zero-section). We have used the equality

$$\widehat{c}^{\mathrm{top}}(\overline{E})\widehat{\mathrm{Td}}(\overline{E}^{\vee}) = c^{\mathrm{top}}(\overline{E}^{\vee})$$

valid for any hermitian vector bundle $\overline{E}$. We thus obtain the equality

$$(-1)^d \widehat{c}^{\mathrm{top}}(\overline{\Omega}) = \tau(\mathcal{O})^{[d-1]}$$

We now consider the same situation again but we suppose that S does not contain a prime dividing 2 and we consider the action of μ_2 given by the inversion in the group scheme. We compute

$$\widehat{\mathrm{ch}}_{\mu_2}(R\pi_*(\overline{O})) - \tau_{-1}(\overline{O}) = 2^{2d}(1 - R_{-1}(\Omega^{\vee}))\widehat{\mathrm{ch}}_{\mu_2}(\Lambda_{-1}(\overline{\Omega}))^{-1}.$$

One can show that $\tau_{-1}(\overline{O}) = 0$. We multiply both sides by $\widehat{\mathrm{ch}}_{\mu_2}(\Lambda_{-1}(\overline{\Omega}))$ to obtain

$$\widehat{\mathrm{ch}}_{\mu_2}(\overline{H}) = 2^{2d}(1 - R_{-1}(\Omega^{\vee}))$$

where we have written $\overline{H} := \overline{\Omega} \oplus \overline{\Omega}^{\vee}$. We now need a lemma:

Lemma 2.3.1. *The equality*

$$\log^{\leqslant l}\left(\frac{1+\exp(x)}{2}\right) = -\sum_{j=1}^{\infty} \zeta_L(-1, 1-j)x^j/j!$$

holds in $\mathbb{C}[[x]]$.

Now, by definition, we have

$$\zeta_L(-1,s) = \sum_{k\geqslant 1}\frac{(-1)^k}{k^s}$$

and

$$\zeta_{\mathbb{Q}}(s) = \sum_{k\geqslant 1}\frac{1}{k^s}$$

where $\zeta_{\mathbb{Q}}$ is Riemann's ζ-function. From these equalities, we deduce that $\zeta_L(-1,s) = \zeta_{\mathbb{Q}}(s)(2^{1-s}-1)$. Resuming our computations, we get

$$\begin{aligned}
R_{-1}(\Omega^\vee) &= \frac{1}{2}\Big(\sum_{k\geqslant 0}\mathrm{ch}^{[k]}(\Omega^\vee)(2\zeta_L'(-1,-k)+(1+\cdots+\frac{1}{k})\zeta_L(-1,-k)) - \\
&- \sum_{k\geqslant 0}(-1)^k\mathrm{ch}^{[k]}(\Omega^\vee)(2\zeta_L'(-1,-k)+(1+\cdots+\frac{1}{k})\zeta_L(-1,-k))\Big) \\
&= \sum_{k\geqslant 0}\mathrm{ch}^{[2k+1]}(\Omega^\vee)(2\zeta_L'(-1,-2k-1)+(1+\cdots+\frac{1}{2k+1})\zeta_L(-1,-2k-1)) \\
&= \sum_{k\geqslant 0}\mathrm{ch}^{[2k+1]}(\Omega^\vee)\Big((\zeta_{\mathbb{Q}}'(-2k-1)(2^{3+2k}-2)-\log(2)\zeta_{\mathbb{Q}}(-2k-1)2^{2+2k}) \\
&+ (1+\cdots+\frac{1}{2k+1})\zeta_{\mathbb{Q}}(-1,-2k-1)(2^{2+2k}-1)\Big)
\end{aligned}$$

Using the lemma and applying the log map, we obtain

$$\begin{aligned}
-\sum_{k\geqslant 1} & \zeta_L(-1,1-k)\widehat{\mathrm{ch}}^k(\overline{\mathcal{H}}) = -\sum_{k\geqslant 1}\zeta_{\mathbb{Q}}(1-k)(2^k-1)\widehat{\mathrm{ch}}^k(\overline{\mathcal{H}}) \\
&= \sum_{k\geqslant 0}\mathrm{ch}^{[2k+1]}(\Omega^\vee)\Big((\zeta_{\mathbb{Q}}'(-2k-1)-\log(2)\zeta_{\mathbb{Q}}(-2k-1))(2^{3+2k}-2) \\
&+ (1+\cdots+\frac{1}{2k+1})\zeta_{\mathbb{Q}}(-1,-2k-1)(2^{2+2k}-1)\Big)
\end{aligned}$$

and in particular

$$\begin{aligned}
-\sum_{2k\geqslant 1} & \zeta_{\mathbb{Q}}(1-2k)(2^{2k}-1)\widehat{\mathrm{ch}}^{2k}(\overline{\mathcal{H}}) \\
&= \sum_{k\geqslant 0}\mathrm{ch}^{[2k+1]}(\Omega^\vee)\Big((\zeta_{\mathbb{Q}}'(-2k-1)(2^{3+2k}-2)-\log(2)\zeta_{\mathbb{Q}}(-2k-1)2^{2+2k}) \\
&+ (1+\cdots+\frac{1}{2k+1})\zeta_{\mathbb{Q}}(-1,-2k-1)(2^{2+2k}-1)\Big)
\end{aligned}$$

which implies that

$$\widehat{\mathrm{ch}}^{2k}(\overline{\Omega}) = \mathrm{ch}^{[2k-1]}(\Omega)\Big(\frac{\zeta_{\mathbb{Q}}'(-2k+1)}{\zeta_{\mathbb{Q}}(-2k+1)} - \frac{1}{1-2^{-2k}}\log(2) + \frac{1}{2}(1+\cdots+\frac{1}{2k-1})\Big)$$

Computations by J.-B. Bost and U. Kühn (see [Kuehn]) suggest that the last identity should hold also in the case of semi-abelian fibrations, in which case the bundle $\overline{\Omega}$ carries a metric with mild (logarithmic) singularities and the theory $\widehat{\mathrm{CH}}^{\cdot}(\cdot)$ has to be extended (see [BKK] and [BKK2]).

2.3.3 The Chowla-Selberg formula: a special case

In this subsection, we shall compute explicitly the Faltings height of the elliptic curve $y^2 = x^3 + 6$ using the Theorem 2.2.4. We omit notes, as we are going to follow very closely the Appendix of C. Soulé's Bourbaki talk [Bourbaki], where this computation is performed. The text of the talk is freely available at the address http://www.institut.math.jussieu.fr/Arakelov/0029.

2.3.4 The height of Grassmannians

Another application of the Theorem 2.2.4 is the computation of heights of arithmetic varieties that carry the action of a diagonalisable torus $\mathcal{T} :=$ Spec $\mathbb{Z}[X, X^{-1}]$. For every $n \geqslant 1$, there is a natural closed immersion $\mu_n \hookrightarrow \mathcal{T}$ over $\mathbb{Z}$. Suppose that we are given a smooth and projective arithmetic variety $f : X \to \operatorname{Spec}\mathbb{Z}$ (say) of relative dimension d, endowed with a $\mathcal{T}$-action. Suppose also that X is endowed with a $\mathcal{T}$-equivariant ample line bundle L. The arithmetic Hilbert -Samuel theorem implies that the height $f_*(\widehat{c}^1(\overline{L})^{d+1})$ of X with respect tot $\overline{L}$ is given by the limit

$$\lim_{k\to\infty} \frac{(d+1)!\widehat{c}^1(Rf_*(\overline{L}^{\otimes k}))}{k^{d+1}}.$$

Recall that $\zeta_n := \exp(2i\pi/n)$. It is easy to see that

$$\lim_{n\to\infty} \widehat{c}^1_{\mu_n}(Rf_*(\overline{L}^{\otimes k})) = \widehat{c}^1(Rf_*(\overline{L}^{\otimes k}))$$

and furthermore $X_{\mathcal{T}} = X_{\mu_n}$ for $n >> 0$. Hence, via the Theorem 2.2.4, we get an expression for the height of the type

$$\lim_{k\to\infty} \frac{(d+1)!}{k^{d+1}} \Big(\lim_{n\to\infty} \tau_{\mu_n}(\overline{L}^{\otimes k}) + (\text{something localised on } X_{\mathcal{T}})\Big).$$

This method was initially used by K. Köhler and the author to compute the height of the Grassmannians $G(d, n)$ of d-places in n-space, for the ample line bundle coming from the Plücker embedding. It is based on the explicit computation of the equivariant analytic torsion of Grassmannians carried out by K. Köhler in [K2]. In this case, the local contribution vanishes. The answer is

$$\text{height of G(d, n)} = \frac{1!2!\dots(d-1)!(d(n-d))!d(n-d)(d(n-d)+1)}{2(n-d)!\dots(n-1)!} +$$

$$\sum_{1\leqslant i_1<\dots i_d\leqslant n}\sum_{l=1}^{d(n-d)}\frac{(d(n-d)+1)!(-1)^l}{2l!(l+1)^2(d(n-d)-l)!}\big(d(n+1)/2-(i_1+\dots i_d)\big)^{d(n-d)-l}\Big(\sum_{k=1}^{d}\sum_{\substack{1\leqslant s\leqslant n\\ s\neq i_1,\dots i_d}}(s-i_k)^l\Big)\Big(\prod_{k=1}^{d}\prod_{\substack{1\leqslant s\leqslant n\\ s\neq i_1,\dots i_d}}\frac{1}{s-i_k}\Big)$$

This method to compute the height was greatly generalised by K. Köhler and C. Kaiser in [KK], where they give closed formulae for the height of any flag variety. H. Tamvakis, buiding on work of V. Maillot, gave in [Ta] a different approach to the computation of this height, based on an extension of the classical Schubert calculus to Arakelov geometry. In particular, he obtains the formula:

$$\text{height of } \mathrm{G(2,n)}=\Big(1+\dots+\frac{1}{n+2}\big)-\frac{2n+1}{2n+2}\Big)\binom{2n+1}{n}-\frac{4^n}{n+1}$$

The comparison of the first and second formula seems to be a hard combinatorial problem.

Bibliography

[Abbes] Abbes, A.; Bouche, T.:*Théorème de Hilbert-Samuel "arithmétique".* Ann. Inst. Fourier **45**, no. 2, 375-401 (1995).

[Ara] Arakelov, S. Ju.: *An intersection theory for divisors on an arithmetic surface.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **38**, 1179-1192 (1974).

[B3] Bismut, J.-M.: *Equivariant immersions and Quillen metrics*. J. Differential Geom. **41**, 53-157 (1995).

[Bost] Bost, J.-B.: *Intrinsic heights of stable varieties and abelian varieties.* Duke Math. J. **82**, no. 1, 21-70 (1996).

[BKK] Burgos, J.; Kramer, J.; Kühn, U.: *Cohomological Arithmetic Chow groups.* Preprint http://www.arxiv.org/abs/math.AG/0404122.

[BKK2] Burgos, J.; Kramer, J.; Kühn, U.: *Arithmetic characteristic classes of automorphic vector bundles.* Preprint http://www.arxiv.org/abs/math.AG/0502085.

[Fal] Faltings, G.: *Calculus on arithmetic surfaces.* Ann. of Math. (2) **119**, no. 2, 387-424 (1984).

[F] Fulton, W.: Intersection theory: Springer 1984.

[GS2] Gillet, H., Soulé, C.: *Arithmetic intersection theory.* Inst. Hautes Etudes Sci. Publ. Math. **72** (1990).

[GS3] Gillet, H., Soulé, C.: *Characteristic classes for algebraic vector bundles with hermitian metrics I, II.* Ann. of Math. **131**, 163-203, 205-238 (1990).

[GS8] Gillet, H., Soulé, C.: *An arithmetic Riemann-Roch theorem.* Invent. Math. **110**, 473-543 (1992).

[SGA6] Grothendieck, A., Berthelot, P., Illusie, L.: Théorie des intersections et théorème de Riemann-Roch (Lecture Notes 225): Springer 1971.

[H] Hartshorne, R.: Algebraic geometry: Springer 1977.

[Hi] Hirzebruch, F.: Topological Methods in Algebraic Geometry: Springer 1978.

[KK] Kaiser, Ch., Köhler, K.: *A fixed point formula of Lefschetz type in Arakelov geometry III: representations of Chevalley schemes and heights of flag varieties.* Invent. Math. **147**, no. 3, 633-669 (2002).

[K2] Köhler, K.: *Holomorphic torsion on Hermitian symmetric spaces.* J. Reine Angew. Math. **460**, 93-116 (1995).

[LRR1] Köhler, K.; Roessler, D.: *A fixed point formula of Lefschetz type in Arakelov geometry. I. Statement and proof.* Invent. Math. **145**, no. 2, 333–396 (2001).

[Kuehn] Kühn, U.: *Generalized arithmetic intersection numbers.* J. Reine Angew. Math. **534**, 209-236 (2001).

[M] Maillot, V.: *Un calcul de Schubert arithmétique.* Duke Math. J. **80**, No.1, 195-221 (1995).

[MB] Moret-Bailly, L. Pinceaux de variétés abéliennes. Astérisque No. **129** (1985).

[Bourbaki] Soulé, C.: *Genres de Todd et valeurs aux entiers des dérivées de fonctions L.* Séminaire Bourbaki, nov. 2005.

[SABK] Soulé, C., Abramovich, D., Burnol, J.F. and Kramer, J.: Lectures on Arakelov Geometry (Cambridge studies in math. 33): Cambridge university press 1992.

[Ta] Tamvakis, H.: *Schubert calculus on the arithmetic Grassmannian.* Duke Math. J. **98**, No. 3, 421-443 (1999).

[T3] Thomason, R.: *Une formule de Lefschetz en K-théorie équivariante algébrique.* Duke Math. J. **68**, 447-462 (1992).

[Zhang] Zhang, S.: *Positive line bundles on arithmetic varieties.* J. Amer. Math. Soc. **8**, no. 1, 187-221 (1995).