Perfect points on abelian varieties in positive characteristic

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Set up

$\diamond$ K the function field of a smooth and proper curve C over a perfect field k of characteristic $p > 0$

$\diamond$ A an abelian variety over K

$\diamond$ $\mathcal{A}$ a semiabelian model of A over C with zero section $\epsilon : C \to \mathcal{A}$

$\diamond$ $\omega = \omega_{\mathcal{A}} := \epsilon^*(\Omega_{\mathcal{A}/C})$

$\diamond$ $F_{A/K} : A \to A^{(p)}$ and $F_{\mathcal{A}/C} : \mathcal{A} \to \mathcal{A}^{(p)}$ the relative Frobenii

$\diamond$ $V_{A^{(p)}/K} : A^{(p)} \to A$ the relative Verschiebung morphism

$\diamond$ $K^{\mathrm{perf}} := K^{p^{-\infty}} = \cup_{i \geq 0} K^{p^{-i}} \subseteq \bar{K}$.

Perfect points

Object of interest: $A(K^{\mathrm{perf}})$

Also of interest:

$A(K^{\mathrm{perf}})^{\#} := \bigcap_{i\geq 0} p^i \cdot A(K^{\mathrm{perf}})$

and

$\mathrm{IVD}(A(K^{\mathrm{perf}})) := \bigcup_{l\geq 0} \bigcap_{i\geq 0} V^{\circ i}_{A^{(p^i)}/K}(A^{(p^i)}(K^{p^{-l}}))$

Note that $\mathrm{IVD}(A(K^{\mathrm{perf}})) \subseteq A(K^{\mathrm{perf}})^{\#} \subseteq A(K^{\mathrm{perf}})$.

Also, if A is ordinary then $\mathrm{IVD}(A(K^{\mathrm{perf}})) = A(K^{\mathrm{perf}})^{\#}$ and if A has p-rank 0 then $A(K^{\mathrm{perf}})^{\#} = A(K^{\mathrm{perf}})$.

Some conjectures

From now on, suppose for simplicity that $\mathrm{Tr}_{K|k}(A) = 0$.
Recall that in this situation, $A(K)$ is finitely generated by the Lang-Néron theorem.

Question: is $A(K^{\mathrm{perf}})$ also finitely generated?

Answer: no! Abelian varieties of p-rank 0 provide counterexamples.

One might nevertheless make the following **conjectures**.

- $A(K^{\mathrm{perf}})$ is finitely generated if A is ordinary (Scanlon, Ziegler, Ghioca and Moosa).

- $\mathrm{IVD}(A(K^{\mathrm{perf}}))$ is finite and of prime to p order (Esnault and Langer).

- $\mathrm{Tor}(A(K^{\mathrm{perf}}))$ is of finite order (Esnault).

Some results

A related conjecture is the following. Suppose $X \subseteq A$ is a subvariety of general type.

Conjecture: $X(K^{\mathrm{perf}})$ not Zariski dense in X.

Note that this conjecture is verified if $A(K^{\mathrm{perf}})$ is finitely generated by work of Hrushovski.

Here is what is known about the above conjectures.

- $A(K^{\mathrm{perf}})$ is finitely generated if $k = \bar{\mathbb{F}}_p$ and A is ordinary, geometrically simple and $\mathcal{A}$ is not an abelian scheme (R.; Ghioca when $\dim(A) = 1$).

- $\mathrm{IVD}(A(K^{\mathrm{perf}}))$ is torsion if $\mathcal{A}$ is an abelian scheme (Esnault, Langer) and is finite and of prime to p order if $k = \bar{\mathbb{F}}_p$ (R.).

- $\mathrm{Tor}(A(K^{\mathrm{perf}}))$ is finite if $k = \bar{\mathbb{F}}_p$ (d'Addezio, Ambrosi).

- $X(K^{\mathrm{perf}})$ is not Zariski dense in X if X is a curve and k is a finite field (Kim).

The Artin-Milne map

To study $A(K^{\mathrm{perf}})$ a basic tool is the following. Let $E \subseteq C$ be the divisor of bad reduction of $\mathcal{A}$ and $U := C \backslash E$.

Theorem (Artin-Milne)

There is a canonical injective group homomorphism

$$H^1_{\mathrm{fppf}}(K, F_{A/K}) \hookrightarrow \mathrm{Hom}_K(F_K^*(\omega_K), \Omega_{K/k}).$$

This can be refined as follows:

Theorem (R.)

There is a canonical injective group homomorphism

$$\mathrm{Sel}(K, F_{A/K}) \hookrightarrow \mathrm{Hom}_C(F_C^*(\omega), \Omega_{C/k}(E)).$$

Recall that

$$A^{(p)}(K)/F_{A/K}(A(K)) \subseteq \mathrm{Sel}(K, F_{A/K}) \subseteq H^1_{\mathrm{fppf}}(K, F_{\mathcal{A}_K/K}).$$

This theorem can be proven by using a semistable compactification of $\mathcal{A}$ (Faltings-Chai, Mumford) or by a direct analysis of Raynaud uniformisations at the points of E (ongoing work of my student Zhenhua Wu).

The interest of the theorem is that the theory of semistability of vector bundles can be brought to bear on the problem.

The Harder-Narasimhan filtration of ω turns out to be related to the structure of the group scheme $\ker F_{\mathcal{A}/C}$.

The HN-filtration of ω

Recall that ω has a filtration

$$0 \subsetneq \omega_1 \subsetneq \omega_2 \subsetneq \cdots \subsetneq \omega_\rho \subsetneq \omega$$

whose quotients are semistable, with strictly decreasing slopes.
We shall write

$$\mu_{\max}(\omega) := \mu(\omega_1) = \text{slope of } \omega_1 := \frac{\deg(\omega_1)}{\mathrm{rk}(\omega_1)}$$

and

$$\mu_{\min}(\omega) := \mu(\omega/\omega_\rho) = \text{slope of } \omega/\omega_\rho := \frac{\deg(\omega/\omega_\rho)}{\mathrm{rk}(\omega/\omega_\rho)}.$$

By work of Langer and others, the HN filtration of $F_C^{\circ n,*}(\omega)$ stabilises for large $n \geq 0$.

Hence it makes sense to define

$$\bar{\mu}_{\max}(\omega) := \lim_{n\to\infty} \frac{\mu_{\max}(F_C^{\circ n,*}(\omega))}{p^n}$$

and

$$\bar{\mu}_{\min}(\omega) := \lim_{n\to\infty} \frac{\mu_{\min}(F_C^{\circ n,*}(\omega))}{p^n}$$

A basic fact of the theory is the following. If V and W are vector bundles on C and

$$\bar{\mu}_{\min}(V) > \bar{\mu}_{\max}(W)$$

then

$$\mathrm{Hom}(V, W) = 0.$$

In particular, if $\bar{\mu}_{\min}(V) > 0$ then for $n >> 0$,

$$\bar{\mu}_{\min}(F_C^{\circ n,*}(V)) = p^n \cdot \bar{\mu}_{\min}(V) > \bar{\mu}_{\max}(W)$$

and so $\mathrm{Hom}(F_C^{\circ n,*}(V), W) = 0$.

The HN filtration of $F_C^{\circ n,*}(\omega)$ and the structure of ker $F_{\mathcal{A}/C}$

Let $G :=$ ker $F_{\mathcal{A}/C}$.

Note that the coLie algebra of G is simply ω. The map $\omega \to F_C^*(\omega)$ given by "pull-back by Verschiebung" gives the p-coLie algebra structure of G.

Recall that there is an antiequivalence of categories between finite flat group schemes of height one over C and p-coLie algebras (Demazure-Gabriel).

Proposition (R. - inspired by a result of Bost in char. 0)

Suppose that the HN filtration of ω is Frobenius stable. Then the quotients of non-positive maximal slope in the HN filtration of ω are p-coLie algebras. The quotients of strictly negative slope correspond to biinfinitesimal group schemes.

Proposition (R.)

There is a (necessarily unique) closed subgroup scheme $G_\mu \subseteq G$, such if H is a multiplicative group scheme of height one over C and $\phi : H \to G$ a group homomorphism, then ϕ factors through G_μ. The group scheme G_μ is compatible with Frobenius twists.

The proof relies on the previous proposition, together with a basic descent lemma, which allows to untwist multiplicative subgroup schemes of twists of finite flat groups schemes.

Proposition (R.)

Suppose that ω has a Frobenius stable HN filtration and that $\mu_{\min}(\omega) = 0$. Then G_μ is given by the quotient of minimal slope ω/ω_ρ. If $\mu_{\min}(\omega) > 0$ then $G_\mu = 0$.

Corollary

If $\bar{\mu}_{\min}(\omega) > 0$ then $G_\mu = 0$.

Proposition (R.)

If A is ordinary then $\bar{\mu}_{\min}(\omega) \geq 0$.

Note that this proposition is a characteristic p analogue of a result of Griffiths, which asserts that the corresponding statements holds when $\mathrm{char}(k) = 0$.

Corollary

Suppose that A is ordinary. Let $n >> 0$. *Let* $x \in A^{(p^n)}(K)$. *Then the map*

$$F_C^{\circ n,*}(\omega) \to \Omega_{C/k}(E)$$

corresponding to x factors through

$$F_C^{\circ n,*}(\text{coLie algebra}(G_\mu)).$$

Proposition (R.)

Let $n >> 0$. *If* $x \in \mathrm{IVD}(A^{(p^n)}(K)) \cap A(K)$, *then the map*

$$F_C^{\circ n,*}(\omega) \to \Omega_{C/k}(E)$$

corresponding to x factors through

$$F_C^{\circ n,*}(\text{coLie algebra}(G_\mu)).$$

G_μ and the structure of $A(K^{\mathrm{perf}})$

Let us now go back to our original problem.

By the last two results, to show that $A(K^{\mathrm{perf}})$ is finitely generated when A is ordinary it would be sufficient to show that $G_\mu = 0$.

Similarly, if $G_\mu = 0$ then $\mathrm{IVD}(A(K^{\mathrm{perf}})) \cap A(K)$ would be p-divisible and hence finite of order prime to p, as conjectured.

We thus seek to replace A by an isogenous variety, such that the corresponding group G_μ vanishes.

The quotient method

We proceed as follows. Consider the sequence of isogenies

$$A \to A_1 \to A_2 \to \dots \quad (*)$$

where $A_1 = A/G_\mu$, A_2 is the quotient of A_1 by the analogue of G_μ for A_1, continuing this way. The sequence stops if the corresponding G_μ vanishes.

The sequence $(*)$ was also recently considered by Xinyi Yuan in the situation where $C = \mathbb{P}^1_k$.

Question

Is the sequence $(*)$ *always finite? Does the sequence contain only finitely isomorphism classes?*

Determining geometrically the abelian varieties A such that the sequence $(*)$ is finite is a basic concern.

The case of $k = \bar{\mathbb{F}}_p$

Proposition (R.)

If $k = \bar{\mathbb{F}}_p$ *then the sequence* $(*)$ *contains only finitely many isomorphism classes.*

For the proof, note that the modular height of the A_i can be computed and is constant. Furthermore, the whole sequence has a model over a curve defined over a finite field. The proposition follows from this.

Proposition (R.)

Suppose that $k = \bar{\mathbb{F}}_p$ and that A is geometrically simple. If the sequence $(*)$ is infinite then $\mathcal{A}$ is an abelian scheme (ie A has everywhere good reduction).

For the proof, note that by the previous proposition, the dual of some A_i has an étale endomorphism of degree a power of p. Such an abelian variety can be shown to have good reduction everywhere.

Application to perfect points

Corollary

If A is ordinary, geometrically simple and $\mathcal{A}$ is not an abelian scheme then $A(K^{\mathrm{perf}})$ is finitely generated.

Using the fact that Esnault-Langer showed that $\mathrm{IVD}(A(K^{\mathrm{perf}}))$ is a torsion group if A has everywhere good reduction and the fact that d'Addezio and Ambrosi showed that $\mathrm{Tor}(A(K^{\mathrm{perf}}))$ is finite if $k = \bar{\mathbb{F}}_p$, we also get

Corollary

If $k = \bar{\mathbb{F}}_p$ then $\mathrm{IVD}(A(K^{\mathrm{perf}}))$ is a finite torsion group.

On can show that $\mathrm{IVD}(A(K^{\mathrm{perf}}))$ is in fact prime to p torsion by a different method, which bypasses the above two results.