On the generalized Mordell-Lang conjecture over function fields of positive characteristic

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The generalized Mordell-Lang conjecture in char. 0

The following theorem is the result of the work of Faltings, Serre, Hindry and McQuillan:

Theorem (generalized Mordell-Lang conjecture)

Let A be an abelian variety over $\mathbb{C}$. Let $X \hookrightarrow A$ be a closed subvariety, which is not the translate of an abelian subvariety of A. Let $\Gamma \subseteq A(\mathbb{C})$ be a group such that $\Gamma \otimes \mathbb{Q}$ is a finite-dimensional $\mathbb{Q}$-vector space. Then $X \cap \Gamma$ is not Zariski dense in X.

The special case of the generalized Mordell-Lang conjecture where Γ is a group of torsion points (resp. is a finitely generated group) is referred to as the Manin-Mumford conjecture (resp. the Mordell-Lang conjecture).

Special cases

The following are consequences of GML.

Let C be a curve of genus $\geqslant 2$ defined over a number field K.

Theorem (Mordell conjecture; Faltings [1984])

$C(K)$ is finite.

Theorem (Manin-Mumford conjecture for C; Raynaud [1983])

Let $P \in C(\overline{K})$. Then the set

$$\{Q \in C(\overline{K}) | \exists n \in \mathbb{N}^* \text{ such that } nQ - nP \text{ is principal}\}$$

is finite.

Positive characteristic

It is natural to wonder if the analog of GML is true in positive characteristic. The following is known.

Let K be an algebraically closed field of char. $p > 0$. Let A be an abelian variety over K. Suppose that A has no factors, which are defined over a finite field.

Let $X \hookrightarrow A$ be a closed subvariety of A. Suppose that X is not the translate of an abelian subvariety of A.

Finally, let $\Gamma \subseteq A(K)$ be a group st $\Gamma \otimes \mathbb{Z}_p$ is a $\mathbb{Z}_p$-module of finite rank.

Theorem (Hrushovski [1996])

$X \cap \Gamma$ is not Zariski dense in X.

Around GML in positive characteristic

E. Hrushovski's proof of GML in positive characteristic relies heavily on model-theoretic methods. More precisely, his proof relies on a trichotomy theorem for the theory of separably closed fields with a finite p-basis. To establish this trichotomy, he needs his work with B. Zilber on Zariski geometries.

Here is what was already proven some years ago by algebraic methods:

- the case where A is ordinary (Abramovich-Voloch [1996], Pillay-Ziegler [2003]);
- the case where X is a curve and K is of transcendence degree 1 over $\mathbb{F}_p$ (Grauert-Samuel [1966], Abramovich-Voloch [1996], Buium-Voloch [1996])

From Manin-Mumford to generalized Mordell-Lang

The aim of this talk is to sketch an algebraic proof of the following statement :

In positive characteristic, the Manin-Mumford conjecture implies

the generalized Mordell-Lang conjecture. (*)

Combining this proof with the existing algebraic proof of the Manin-Mumford conjecture (see Pink-R. [2004]), one obtains an algebraic proof of the Mordell-Lang conjecture in positive characteristic.

The idea that the reduction (*) may be tractable was suggested by A. Pillay in a talk he gave in Paris in December 2010. He also outlined a model-theoretic approach to (*).

Jet schemes

Let U be a smooth curve over $\bar{\mathbb{F}}_p$.

Let $\mathcal{X}$ be a scheme of over U.

We write $J^n(\mathcal{X}/U)$ for the n-th jet scheme of $\mathcal{X}$ over U.

$J^n(\mathcal{X}/U)$ is a U-scheme, which depends covariantly on $\mathcal{X}$.

For our purpose, the most interesting property of jet schemes is the following :

- For any closed point $u \in U$, there is a natural identification

$$J^n(\mathcal{X}/U)(\kappa(u)) \simeq \mathcal{X}(u_n),$$

where $u_n \hookrightarrow U$ is the n-th infinitesimal neighborhood of u in U.

Properties of jet schemes

Here are further properties of jet schemes that we shall need :

- There are affine smooth "forgetful" morphisms

$$\cdots \to J^n(\mathcal{X}/U) \to J^{n-1}(\mathcal{X}/U) \to \cdots \to J^1(\mathcal{X}/U) = \mathcal{X}.$$

- There is a splitting map

$$\sigma_n : \mathcal{X}(U) \to J^n(\mathcal{X}/U),$$

which is natural in $\mathcal{X}$ but does not arise from a scheme morphism.

Properties of jet schemes II

- If $\mathcal{X} = \mathcal{A}$ is a an abelian scheme over U then the forgetful morphism $J^n(\mathcal{A}/U) \to J^{n-1}(\mathcal{A}/U)$ is a morphism of group schemes and its kernel is a vector bundle.
- There is a U-morphism

$$"p^{n-1}" : \mathcal{A} \to J^n(\mathcal{A}/U)$$

  such that the composition of $"p^{n-1}"$ with the forgetful morphism $J^n(\mathcal{A}/U) \to \mathcal{A}$ is the morphism p^{n-1}.
- $p^{n-1} \circ \sigma_n = \sigma_n \circ p^{n-1} = "p^{n-1}"$ on $\mathcal{A}(U)$.

The critical and exceptional schemes of a subscheme of an abelian scheme

Let now $\mathcal{A}/U$ be an abelian scheme and $\mathcal{X} \hookrightarrow \mathcal{A}$ be a closed integral subscheme. We let

$$\mathrm{Crit}^n(\mathcal{X}) := J^n(\mathcal{X}/U) \cap \text{"}p^{n-1}\text{"}(\mathcal{A})$$

The schemes $\mathrm{Crit}^n(\mathcal{X})$ are proper over U and there are natural finite morphisms

$$\cdots \to \mathrm{Crit}^n(\mathcal{X}) \to \mathrm{Crit}^{n-1}(\mathcal{X}) \to \cdots \to \mathrm{Crit}^1(\mathcal{X}) = \mathcal{X}$$

Let

$$\mathrm{Exc}(\mathcal{X}) := \bigcap_{n \geqslant 1} \mathrm{Im}(\mathrm{Crit}^n(\mathcal{X}) \to \mathcal{X})$$

The non-density of the exceptional scheme

Let L be the function field of U.

Suppose that $\mathcal{A}_{\bar{L}}$ has no factors, which are defined over a finite field. Suppose that $\mathcal{X}_{\bar{L}}$ is not a union of translates of abelian subvarieties of $\mathcal{A}_{\bar{L}}$.

Proposition (non-density of Exc)

If the Manin-Mumford conjecture holds for $\mathcal{X}_{\bar{L}}$, then the set $\mathrm{Exc}(\mathcal{X})$ is not Zariski dense in $\mathcal{X}$.

We shall sketch the proof later.

The exceptional properties of $\mathcal{X}(U)$

We contend that for $n >> 1$, we have an inclusion

$$\boxed{\mathcal{X} \cap p^n \mathcal{A}(U) \subseteq \mathrm{Exc}(\mathcal{X})}$$

Indeed, using the properties of jet schemes listed above, we get

$$\mathcal{X} \cap p^{n-1} \mathcal{A}(U) \subseteq \mathrm{Im}(\mathrm{Crit}^n(\mathcal{X}) := J^n(\mathcal{X}/U) \cap \text{"} p^{n-1} \text{"} (\mathcal{A}) \stackrel{\text{forgetful}}{\longrightarrow} \mathcal{X})$$

for any $n \geqslant 1$.

Application to the Mordell-Lang conjecture

Granting the proposition on the non-density of Exc, we deduce that there is $n_0 = n_0(\mathcal{A}, \mathcal{X} \hookrightarrow \mathcal{A})$ such that the set

$$\mathcal{X} \cap p^{n_0}\mathcal{A}(U)$$

is not dense in $\mathcal{X}$.

Now suppose that $\mathcal{X}(U)$ is dense. Then for some $Q \in \mathcal{A}(U)$, the set

$$(\mathcal{X} + Q)(U) \cap p^{n_0}\mathcal{A}(U)$$

is dense in $\mathcal{X} + Q$, because $\mathcal{A}(U)$ is a finitely generated group (Mordell-Weil theorem).

We can show that $n_0(\mathcal{A}, \mathcal{X} + Q \hookrightarrow \mathcal{A})$ can be bounded independently of Q. This proves the Mordell-Lang conjecture.

On the non-density of $\mathrm{Exc}(\mathcal{X})$

We now turn to the scheme $\mathrm{Exc}(\mathcal{X})$. In order to show that $\mathrm{Exc}(\mathcal{X})$ is not dense in $\mathcal{X}$, it is sufficient to show that $\mathrm{Exc}(\mathcal{X})_u$ is not Zariski dense in $\mathcal{X}_u$.

Looking at the definitions, we see that this last statement is equivalent to the

Proposition ("generic non-liftability")

There is a natural number n_0 such that the set

$$\{P \in \mathcal{X}(\kappa(u)) \mid P \text{ lifts to } \mathcal{X}(u_{n_0}) \cap p^{n_0-1}\mathcal{A}(u_{n_0})\}$$

is not Zariski dense in $\mathcal{X}_u$.

On the non-density of $\mathrm{Exc}(\mathcal{X})$, II

The proposition on generic non-liftability was already proven by Buium-Voloch when $\mathcal{X}$ is a relative curve of genus $\geqslant 2$ with non-zero Kodaira-Spencer class and $\mathcal{A} = \mathrm{Jac}(\mathcal{X}/U)$. They showed that $n_0 = 2$ works in this situation.

The proof of the general form of the proposition is based on a galois-theoretic argument.

Let Frob be some power of the Frobenius automorphism of $\kappa(\bar{u})$, which is the identity on $\kappa(u)$.

Let g be the relative dimension of $\mathcal{A}$ over U. On $\mathcal{A}(\kappa(u))$, the automorphism Frob satisfies a Galois equation $R(\mathrm{Frob}) = 0$, where

$$R(T) = T^{2g} - (a_{2g-1}T^{2g-1} + \cdots + a_1 T + a_0)$$

is a polynomial with integer coefficients.

On the non-density of $\mathrm{Exc}(\mathcal{X})$, III

Proposition

There is a closed subscheme $\mathcal{Y} \hookrightarrow \mathcal{X}$, such that:

1. *the torsion points are dense in the generic fibre of $\mathcal{Y}$;*
2. *if $W \in \mathcal{X}(u_n)$ satisfies $R(W) = 0$ then $W \in \mathcal{Y}(u_n)$.*

Note that $\mathcal{X}_u \hookrightarrow \mathcal{Y}$. The scheme $\mathcal{Y}$ will be highly nilpotent in general.

Corollary

If Manin-Mumford holds for $\mathcal{X}$, then there is an $n_0 \in \mathbb{N}$ st the set

$$\{P \in \mathcal{X}(\kappa(u)) \mid P \text{ lifts to a } W \in \mathcal{X}(u_{n_0}) \text{ satisfying } \mathrm{R(W)=0}\}$$

is not Zariski dense.

On the non-density of $\mathrm{Exc}(\mathcal{X})$, IV

To conclude the proof of the non-density of $\mathrm{Exc}(\mathcal{X})$, notice that the set

$$\{P \in \mathcal{X}(\kappa(u)) \mid P \text{ lifts to a } W \in \mathcal{X}(u_{n_0}) \text{ satisfying } R(W) = 0\}$$

contains the set

$$\{P \in \mathcal{X}(\kappa(u)) \mid P \text{ lifts to } \mathcal{X}(u_{n_0}) \cap p^{n_0 - 1}\mathcal{A}(u_{n_0})\}$$

because the morphism "p^{n_0-1}" commutes with Frob.