Deligne's functorial Riemann-Roch theorem

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The Adams-Riemann-Roch theorem

The following theorem was discussed in the last lecture:

Theorem (Adams-Riemann-Roch theorem)

Suppose that $f : X \to Y$ is a projective and smooth morphism of schemes, which are quasi-projective over an affine noetherian scheme. Then

- *The element $\theta^k(\Omega_f)$ is invertible in $\mathrm{K}(X)[\frac{1}{k}]$.*
- *For all $x \in \mathrm{K}(X)$, the equality*

$$\psi^k(\mathrm{R}^\bullet f_*(x)) = \mathrm{R}^\bullet f_*(\theta^k(\Omega_f)^{-1} \otimes \psi^k(x))$$

  holds in $\mathrm{K}(Y)[\frac{1}{k}]$.

A formal consequence

The following equality is a consequence of the Adams-Riemann-Roch theorem and of some general properties of the K-theory of schemes. Suppose that $\dim(X) = \dim(Y) + 1$. Then

$$18\cdot \mathrm{c}_1(\mathrm{R}^\bullet f_* L) = 18\cdot \mathrm{c}_1(\mathrm{R}^\bullet f_*\mathcal{O}) + 6\cdot \mathrm{c}_1(\mathrm{R}^\bullet f_*(L^{\otimes 2}\otimes\omega^\vee)) - 6\cdot \mathrm{c}_1(\mathrm{R}^\bullet f_*(L\otimes\omega^\vee))$$

in $\mathrm{CH}^1(Y)_{\mathbb{Q}}$ for any line bundle L on X.

Deligne's functorial Riemann-Roch theorem

Deligne's functorial form of the Grothendieck-Riemann-Roch theorem for fibrations of curves in particular asserts the following:

Theorem

There is a canonical isomorphism of line bundles

$$\det(\mathrm{R}^\bullet f_* L)^{\otimes 18} \simeq$$
$$\det(\mathrm{R}^\bullet f_* \mathcal{O})^{\otimes 18} \otimes \det(\mathrm{R}^\bullet f_*(L^{\otimes 2} \otimes \omega^\vee))^{\otimes 6} \otimes \det(\mathrm{R}^\bullet f_*(L \otimes \omega^\vee))^{\vee,\otimes 6}$$

on Y. This isomorphism is compatible with arbitrary base-change.

Applications I

Let us specialise the above isomorphism to the case $g = 1$ and L a line bundle such that on every geometric fiber X_y, $L|_{X_y}$ is a line bundle of degree 0 and not of order $\leqslant 2$.

Since $\mathrm{R}^j f_* L = \mathrm{R}^j f_*(L^{\otimes 2} \otimes \omega^\vee) = \mathrm{R}^j f_*(L \otimes \omega^\vee) = 0$ for all $j \geqslant 0$, we obtain a canonical trivialisation

$$t_L : \mathcal{O} \simeq \det(\mathrm{R}^\bullet f_* \mathcal{O})^{\otimes 18}$$

which depends on L.

Applications II

On the other hand, there is a canonical trivialisation

$$\Delta : \mathcal{O} \simeq \det(\mathrm{R}^\bullet f_* \mathcal{O})^{\otimes 12}$$

given by the discriminant modular form.

Hence

$$u_L := t_L^{\otimes 2} \circ (\Delta^{\otimes 3})^{-1}$$

is an element of $H^0(Y, \mathcal{O}_Y^*)$.

Modular units

Let $N \geqslant 3$ be an odd number. Let $\mathcal{A}_{1,N}$ be the moduli space over $\mathbb{Z}[\frac{1}{N}]$ of elliptic curves with N-level structure.

For every $t \in \mathbb{Z}/N\mathbb{Z}^2 \backslash 0$, the universal family over $\mathcal{A}_{1,N}$ gives rise to a *modular unit*

$$u_t := u_{\mathcal{O}(t-O)} \in \mathrm{H}^0(\mathcal{A}_{1,N}, \mathcal{O}^*).$$

Question: can u_t be computed explicitly as a function on the upper-half plane ?

Quillen metrics I

To tackle this problem, we need to consider supplementary data on L.

We consider again the isomorphism in Deligne's theorem and we suppose that Y is smooth and defined over $\mathbb{C}$.

Fix a hermitian metric on L and a Kähler metric on $X(\mathbb{C})$. The bundles

$$\mathrm{R}^\bullet f_* L,\ \mathrm{R}^\bullet f_* \mathcal{O},\ \mathrm{R}^\bullet f_*(L^{\otimes 2} \otimes \omega^\vee),\ \mathrm{R}^\bullet f_*(L \otimes \omega^\vee)$$

can then all be endowed with a canonical metric, the *Quillen metric*.

Quillen metrics II

Deligne then proves that the isomorphism

$$\det(\mathrm{R}^\bullet f_* L)^{\otimes 18} \simeq$$
$$\det(\mathrm{R}^\bullet f_* \mathcal{O})^{\otimes 18} \otimes \det(\mathrm{R}^\bullet f_*(L^{\otimes 2} \otimes \omega^\vee))^{\otimes 6} \otimes \det(\mathrm{R}^\bullet f_*(L \otimes \omega^\vee))^{\vee, \otimes 6}$$

of the Theorem *is an isometry* up to a factor depending only on X_y.

He also proves:

- the factor vanishes if $g = 1$;
- the discriminant Δ has Quillen norm 1;
- the factor depends only on the genus of the general fiber of $X \to Y$ if $g \geqslant 3$.

Explicit computations of the modular units I

We shall use the above refinement of Deligne's theorem to compute the unit u_t.

The Quillen metric is the product of the L^2-metric with (the exponential of) Ray and Singer's analytic torsion. We shall denote the latter by $\tau(\bullet)$.

Recall that the line bundles

$$L, \; L \otimes \omega^\vee, \; L^{\otimes 2} \otimes \omega^\vee$$

have no cohomology. The determinant of the cohomology of such a bundle is canonically trivialised by a section whose squared Quillen norm is Ray and Singer's analytic torsion.

Since the discriminant has Quillen norm 1, we get

$$\begin{aligned} |u_t| &= \tau(L)^{-18} \cdot \tau(L^{\otimes 2} \otimes \omega^\vee)^6 \cdot \tau(L \otimes \omega^\vee)^{-6} \\ &= \tau(L)^{-24} \cdot \tau(L^{\otimes 2})^6. \end{aligned}$$

Explicit computation of the modular units II

The analytic torsion is explicitly given as a regularised determinant of the eigenvalues of the Kodaira-Laplace operator. In the situation above, it was computed explicitly by Ray and Singer.

Let $E \simeq \mathbb{C}/[\tau, 1]$ be an elliptic curve over $\mathbb{C}$.

Let $z \in \mathbb{C}$, $\Im(z) > 0$ and let $P := z(\mod [\tau, 1]\ ) \in E(\mathbb{C})$ be the associated point. Let $M := \mathcal{O}(P - O)$.

Endow M with a flat hermitian metric.

The analytic torsion of M is

$$\left| e^{-z \cdot \mathrm{quasiperiod}(z)/2} \sigma(z, \tau) \Delta(\tau)^{\frac{1}{12}} \right|.$$

Explicit computation of modular units III

Let $(r_1, r_2) \in \mathbb{Z}^2$, $0 \leqslant r_1, r_2 \leqslant N-1$ be such that $t = (r_1, r_2) \pmod{N}$. Consider the function of τ

$$|H_{r_1,r_2}| := \left| e^{-\frac{1}{2}(r_1\tau N + r_2/N)\cdot(\eta(\tau)r_1/N + \eta(1)r_2/N)} \sigma(r_1\tau/N + r_2/N, \tau)\Delta(\tau)^{\frac{1}{12}} \right|$$

The function H_{r_1,r_2} is called the *Siegel function* and is holomorphic in τ.

From the above, there exists a constant $K \in \mathbb{C}$, $|K| = 1$, such that

$$u_t = K \cdot H_{r_1,r_2}^{-24} \cdot H_{2r_1,2r_2}^{6}$$

is a unit on $\mathcal{A}_{1,N}$.

Arithmetic application

Let E be an elliptic curve with complex multiplication.

Example: $E(\mathbb{C}) \simeq \mathbb{C}/(i\mathbb{Z} + \mathbb{Z})$, which has complex multiplication by the Gaussian integers.

Suppose that N has at least two prime factors.

Let $P \in E(\mathbb{C})$ be an N-torsion point.

Then u_P is an algebraic unit.

This follows from the following facts:

- E has a model over a number field with good reduction everywhere;
- P never reduces to 0.

Interpretation of the Fourier development of u_t

Let $z := r_1\tau/N + r_2/N$. Let $u := \exp(2\pi i z)$ and let $q := \exp(2\pi i\tau)$. The Fourier development of H_{r_1,r_2} is described by the product

$$q^{\frac{1}{2}(r_1^2 - r_1 + 1/6)}(1-u)\prod_{n\geqslant 1}(1-q^n u)(1-q^n u^{-1})$$

The polynomial $\frac{1}{2}(X^2 - X + 1/6)$ is the second Bernoulli polynomial.

The term of lowest degree in this development can be interpreted geometrically as the degeneracy of Deligne's canonical isomorphism when approaching a semi-stable fibre.