Nori's approach to the Riemann-Roch theorem

PIMS, Sept. 7th 2007

The geometric fixed point formula I

We shall work with the following setup:

- X a projective smooth variety over $\mathbb{C}$
- g an automorphism of finite order of X
- $\mathrm{K}_{\mathrm{eq}}(\bullet)$ the Grothendieck group of g-equivariant vector bundles
- $\rho : X_g \hookrightarrow X$ the fixed point variety of X

The geometric fixed point formula II

The group $\mathrm{K}_{\mathrm{eq}}(\bullet)$ has a natural *covariant* and *contravariant* structure. A natural question is thus: does the following diagram commute ?

$$\begin{array}{ccc} \mathrm{K}_{\mathrm{eq}}(X) & \xrightarrow{\rho^*} & \mathrm{K}_{\mathrm{eq}}(X_g) \\ \downarrow {\scriptstyle \mathrm{R}^\bullet\Gamma} & & \downarrow {\scriptstyle \mathrm{R}^\bullet\Gamma} \\ \mathrm{K}_{\mathrm{eq}}(\mathbb{C}) & \xrightarrow{=} & \mathrm{K}_{\mathrm{eq}}(\mathbb{C}) \end{array}$$

The geometric fixed point formula III

The answer is NO.

Nevertheless, there is the following formula.

Theorem (geometric fixed point formula)

Let N be the normal bundle of X_g in X.

- *The element $\Lambda_{-1}(N^\vee) := \sum_{k\geqslant 0}(-1)^k\Lambda^k(N^\vee)$ is invertible in $\mathrm{K}_{\mathrm{eq}}(X_g)\otimes_{\mathrm{K}_{\mathrm{eq}}(\mathbb{C})}\mathbb{C}$.*
- *For every g-equivariant vector bundle E on X, the equality*

$$\mathrm{Trace}[\ \mathrm{R}^\bullet\Gamma((\Lambda_{-1}(N^\vee))^{-1}\otimes\rho^*(E))\ ] = \mathrm{Trace}[\ \mathrm{R}^\bullet\Gamma(E)\ ]$$

  holds in $\mathbb{C}$.

The Lefschetz fixed point formula

Suppose momentarily that X_g is discrete.
The *Lefschetz fixed point formula* is then the following equality

$$\mathrm{Trace}[\ \mathrm{H}^{\bullet}_{\mathrm{sing}}(X)\ ] = \# X_g .$$

Using the Hodge decomposition of singular cohomology, it can be deduced from the geometric fixed point formula, applied to the element

$$\Lambda_{-1}(\Omega_X) = \sum_{k \geqslant 0} (-1)^k \Lambda^k(\Omega_X).$$

The Riemann-Roch formula for curves I

Let us now turn to an apparently unrelated theorem.

Suppose that C is a curve.

Let D be a divisor on C.

Theorem (Riemann-Roch theorem for curves)

The equality

$$\chi(\mathcal{O}(D)) = \deg(D) - \frac{1}{2}\deg(\Omega_C)$$

holds.

The Riemann-Roch theorem for curves II

By induction on $\deg(D)$, this Theorem can be reduced to the equality

$$\chi(\mathcal{O}_C) = -\frac{1}{2}\deg(\Omega_C).$$

Again by induction on $\deg(D)$, one can show that

$$\deg(L) = \chi(L) - \chi(\mathcal{O}_C)$$

for any line bundle L on C. Hence the Theorem can be further reduced to the equality

$$\boxed{\chi(\mathcal{O}_C) = -\chi(\Omega_C).}$$

The black magic of the diagonal I

We shall now apply the geometric fixed point formula to the following situation:

- $X = C \times C$, where C is a curve;
- g is the automorphism of X swapping the two factors;
- $E := \mathcal{O}_X$.

The black magic of the diagonal II

In this case

- X_g is the diagonal in $C \times C$;
- $N^\vee = \Omega_C$;
- $\rho^*(E) = \mathcal{O}_C$ where $\mathcal{O}_C$ has a trivial g-equivariant structure.

With this input, the geometric fixed point formula says that

$$\boxed{\mathrm{Trace}[\ \mathrm{R}^\bullet\Gamma(\mathcal{O}_X)\ ] = \mathrm{Trace}[\ \mathrm{R}^\bullet\Gamma\big((1+\Omega_C)^{-1}\big)\ ] \qquad (*)}$$

where Ω_C has a *trivial* equivariant structure.

Computations I

We shall need the following lemma from linear algebra:

Lemma

Let V be a vector space over $\mathbb{C}$ and let ι be the automorphism of $V \otimes_{\mathbb{C}} V$ swapping the factors. Then

$$\mathrm{Trace}(\iota) = \dim(V).$$

For the proof, choose any basis (v_i) of V and write the matrix of ι in the basis $(v_i \otimes v_j)$ of $V \otimes_{\mathbb{C}} V$.

Computations II

Before resuming the computation, let us quote the following Proposition.

Proposition (Grothendieck group of a curve)

The map

$$\mathrm{K}(C) \to \mathbb{Z} \oplus \mathrm{Pic}(C)$$

given by

$$E \mapsto \mathrm{rk}(E) \oplus \det(E)$$

is an isomorphism of groups.

Computations III

Let us now resume the computation. Recall the equation (*):

$$\mathrm{Trace}[\ \mathrm{R}^{\bullet}\Gamma(\mathcal{O}_X)\ ] = \mathrm{Trace}[\ \mathrm{R}^{\bullet}\Gamma(\big(1+\Omega_C\big)^{-1}\ ].$$

We shall first be concerned with the computation of

$$\big(1+\Omega_C\big)^{-1}$$

and of

$$\mathrm{Trace}[\ \cdot\ ].$$

Computations IV

We start with the former. Let $k \geqslant 0$. We first compute that

$$(\Omega_C - 1)^{\otimes k} = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \Omega_C^{\otimes j}$$

and thus, if $k \geqslant 1$, that

$$\mathrm{rk}((\Omega_C - 1)^{\otimes k}) = (1-1)^k = 0.$$

Similarly, if $k \geqslant 2$,

$$\det((\Omega_C - 1)^{\otimes k}) = \Omega^{\frac{\mathrm{d}}{\mathrm{d}x}(x-1)^k|_{x=1}} = \Omega_C^{k(x-1)^{k-1}|_{x=1}} = \mathcal{O}.$$

Computations V

Using the Proposition, we thus obtain that

$$(1-\Omega_X)^{\otimes k}=0$$

in $\mathrm{K}(C)$ if $k\geqslant 2$.

Using this, we can finally compute that

$$\begin{aligned}(1+\Omega_C)^{-1} &= \frac{1}{2-(1-\Omega_C)}=\frac{1/2}{1-\frac{1}{2}(1-\Omega_C)}\\ &= \frac{1}{2}+\frac{1}{4}(1-\Omega_C)\end{aligned}$$

in $\mathrm{K}(C)$.

Computations VI

We can thus rewrite (*) as

$$\mathrm{Trace}[\ \mathrm{R}^\bullet\Gamma(\mathcal{O}_X)\ ] = \mathrm{Trace}[\ \mathrm{R}^\bullet\Gamma(\frac{1}{2} + \frac{1}{4}(1 - \Omega_C))\ ].$$

Using the Lemma from linear algebra, we see that this can be rewritten as

$$\chi(\mathcal{O}_C) = \frac{1}{2}\chi(\mathcal{O}_C) + \frac{1}{4}\chi(\mathcal{O}_C) - \frac{1}{4}\chi(\Omega_C)$$

or

$$\boxed{\chi(\mathcal{O}_C) = -\chi(\Omega_C)}$$

Generalisation

The computation above can be generalised to any dimension. The corresponding formula is known as the *Adams-Riemann-Roch formula*.

The *Grothendieck-Riemann-Roch formula* can be deduced from it.

The link between the geometric fixed point formula and the Grothendieck-Riemann-Roch theorem was first seen by M. Nori.

We shall study the implications of the Adams-Riemann-Roch formula in the next talk of this series.