INVARIANCE PROPERTIES OF RESIDUE MAPS IN MOTIVIC COHOMOLOGY

In this text, we shall prove that residue maps (see below for the definition of this term) in motivic cohomology satisfy certain elementary invariance properties. These results are most likely known to many people but we could not find any adequate references in the literature.

We shall rely on Bloch's presentation of motivic cohomology. We first recall its main properties. For more details, see [Blo86] and [MVW06]. We shall keep working over the base field k but the results in this paragraph do not depend on the fact that k is embeddable into $\mathbb{C}$.

If $l : X \hookrightarrow Y$ is a closed k-immersion of smooth schemes over k and C is a closed subvariety Y, which meets X properly, then we can define a cycle l^*C on X by the formula

$$l^*C := \sum_{D \subseteq X \cap C, \text{ irred. comp.}} i(D, X, Y) \cdot D$$

where the numbers $i(D, X, Y) \in \mathbb{Z}$ are Serre's intersection multiplicities (see [Ful75, 2.1] for a definition in our context and further references). We extend this definition linearly to cycles on Y whose components meet X properly.

We shall write Δ^m for the m-dimensional algebraic simplex over k. By definition

$$\Delta^m := \text{Spec } k[x_0, \dots, x_m]/((\sum_i x_i) - 1)$$

For any $j \in \{0, \dots, m\}$, there is a natural closed immersion $\delta_j : \Delta^{m-1} \hookrightarrow \Delta^m$, whose image is the closed subscheme of Δ^m given by the equation $x_j = 0$. The morphism δ_j is called the j-th face map. A closed subscheme of Δ^m is called a face if it is the image by a morphism δ_j of either Δ^{m-1} or one of the faces of Δ^{m-1}. If X is a smooth scheme over k, a closed subscheme of $X \times \Delta^m$ is called a face if it has the form $X \times$ (face of Δ^m).

If X is a smooth scheme over k, let $z^i(X, m)$ be free abelian group generated by all the codimension i irreducible closed subvarieties C of $X \times \Delta^m$, such that C intersects all the faces of $X \times \Delta^m$ properly. Define a differential $\delta : z^i(X, m) \to z^i(X, m-1)$ by the formula

$$\delta(C) := \sum_{j=0}^{m} (-1)^j \delta_j^*.$$

With this differential, we obtain a chain complex $z^i(X, *)$. We define

$$\mathrm{CH}^i(X, m) := H_m(z^i(X, *)).$$

If $f : X \to Y$ a morphism between smooth schemes, we define a map $f^* : \mathrm{CH}^i(Y, m) \to \mathrm{CH}^i(X, m)$ in the following way.

If f is flat, then the pull-back map $f^* : \mathrm{CH}^i(Y,m) \to \mathrm{CH}^i(X,m)$ is the unique map of abelian groups $z^i(Y,m) \mapsto z^i(X,m)$, which sends a closed subvariety C of $X \times \Delta^m$ to the cycle defined by the closed subscheme $f^*(C)$ of $Y \times \Delta^m$ (see [Ful98, I.5] for the definition of a cycle defined by a closed subscheme). The following property (called homotopy invariance) is verified: the map f^* is an isomorphism if there is a covering of Y by open subsets $U \subseteq Y$ such that $f|_U$ is isomorphic to the natural projection of $\mathbb{A}^l \times_k U$ to U. Here $\mathbb{A}^l$ is the affine space of dimension l over k.

We now define pull-back maps in a more general situation. Let $\gamma_f : \Gamma_f \simeq X \hookrightarrow X \times Y$ be the graph of f. Define $z^i(Y,m)_f$ to be the free abelian group generated by all the irreducible closed subvarieties C of codimension i of $Y \times \Delta^m$, such that

- C meets all the faces of $Y \times \Delta^m$ properly;

- $X \times C \hookrightarrow X \times Y \times \Delta^m$ meets $\Gamma_f \times \Delta^m$ properly;

- $(\gamma_f \times \mathrm{Id}_{\Delta^m})^*(X \times C)$ meets all the faces of $X \times \Delta^m$ properly.

By a result of Levine (see [MVW06, Prop. 17.6]), $z^i(Y,m)_f$ defines a subcomplex of $z^i(Y,m)$, which is quasi-isomorphic to $z^i(Y,m)$, *provided Y is affine.* So if Y is affine, we define a map $z^i(Y,m)_f \mapsto z^i(X,m)$ by the formula

$$C \mapsto (\gamma_f \times \mathrm{Id}_{\Delta^m})^*(X \times C).$$

One can show that this map is a map of complexes and thus it induces a map

$$f^* : \mathrm{CH}^i(Y,m) \to \mathrm{CH}^i(X,m).$$

If Y is not affine, there exists a morphism $\phi_Y : Y' \to Y$, where Y' is an affine scheme and Y' can be endowed with the structure of a torsor under a vector bundle over Y. Let $X' := X \times_Y Y'$ and let $\phi_X : X' \to X$ be the first projection. Let $f_1 : X' \to Y'$ be the natural morphism. The morphisms ϕ_Y and ϕ_X are then flat and by homotopy invariance, the maps ϕ_Y^* and ϕ_X^* are invertible. We now define $f^* := (\phi_X^*)^{-1} \circ f_1^* \circ \phi_Y^*$. One can show that this definition of the pull-back map does not depend on the various choices that we made and that it coincides with the definition given above when f is flat.

If $f : X \to Y$ is a finite morphism of smooth schemes, there is a natural map of abelian groups

$$f_* : z^i(X,m) \to z^{i+\dim(Y)-\dim(X)}(Y,m),$$

called a push-forward map, such that $f_*(C) := [\kappa(C) : \kappa(f(C))](f \times \mathrm{Id}_{\Delta^m})_* C$ if C is closed subvariety of codimension i in $X \times \Delta^m$. This map induces a map of complexes $z^i(X,*) \to z^{i+\dim(Y)-\dim(X)}(Y,*)$

Let $r : X \hookrightarrow Y$ be a closed immersion of smooth schemes. Let c be the codimension of r. Let $U := Y \backslash X$ and let $u : U \to Y$ be the natural map. Consider the sequence of complexes

$$0 \to z^i(X,*) \xrightarrow{r_*} z^{i+c}(Y,*) \xrightarrow{u^*} z^{i+c}(U,*) \to 0$$

This sequence is exact everywhere but at the target of u^*. It is proven in [Blo94] that the cokernel of u^* is an exact complex. Hence the image of u^* is quasi-isomorphic to the complex $z^{i+c}(U,\bullet)$.

We thus obtain a long exact sequence

$$\cdots \to \mathrm{CH}^{i+c}(X, m+1) \to \mathrm{CH}^{i}(X, m) \to \mathrm{CH}^{i+c}(X, m) \to \mathrm{CH}^{i+c}(U, m) \to \mathrm{CH}^{i}(X, m-1) \to \dots,$$

called the localisation sequence for r. We call 'residue maps' (as is customary) the connecting homomorphisms $\mathrm{CH}^{i+c}(U, m) \to \mathrm{CH}^{i}(X, m-1)$.

Finally, we recall the following difficult theorem of Suslin-Voevodsky (see [MVW06]): there is a natural isomorphism

$$\mathrm{CH}^{i}(X, m) \simeq H^{2i-m}_{\mathcal{M}}(X, i)$$

which is compatible with push-forward and pull-back maps.

Proposition 0.0.1. *Let*

$$\begin{tikzcd} X \arrow[r, hook, "i"] & Y \\ X_0 \arrow[u, "f"] \arrow[r, hook, "l"] & Y_0 \arrow[u, "g"'] \end{tikzcd}$$

be a cartesian diagram of smooth schemes over a field. Suppose that the horizontal morphisms are closed immersions. Let $c \in \mathbb{N}$ and suppose that the codimension of X in Y (resp. X_0 in Y_0) is c. Let $U := Y \backslash X$ and $U_0 := Y_0 \backslash X_0$. Then we have a commutative diagram of localisation sequences

$$\begin{tikzcd} \dots \arrow[r] & H^{\bullet-2c}_{\mathcal{M}}(X, *-c) \arrow[r] \arrow[d] & H^{\bullet}_{\mathcal{M}}(Y, *) \arrow[r] \arrow[d] & H^{\bullet}_{\mathcal{M}}(U, *) \arrow[r] \arrow[d] & H^{\bullet+1-2c}_{\mathcal{M}}(X, *-c) \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & H^{\bullet-2c}_{\mathcal{M}}(X_0, *-c) \arrow[r] & H^{\bullet}_{\mathcal{M}}(Y_0, *) \arrow[r] & H^{\bullet}_{\mathcal{M}}(U_0, *) \arrow[r] & H^{\bullet+1-2c}_{\mathcal{M}}(X_0, *-c) \arrow[r] & \dots \end{tikzcd}$$

where the vertical maps are the pull-back maps.

Proof. If the morphism g (and hence the morphism f) is flat, then the claim follows from [Qui73, Rem. 3.4, p. 120]. In view of this and of the construction of pull-back maps described above, we may assume (details left to the reader) that Y, and hence X are affine schemes. We now claim that the image of the map

$$i_* : z^i(X, *)_f \to z^{i+c}(Y, *)$$

lies inside $z^{i+c}(Y, *)_g \subseteq z^{i+c}(Y, *)$. To see this, consider the diagram of closed immersions

$$\begin{tikzcd} X_0 \times X \times \Delta^m \arrow[r, hook] & Y_0 \times Y \times \Delta^m \\ X_0 \times \Delta^m \arrow[u, "\gamma_f \times \mathrm{Id}_{\Delta^m}"] \arrow[r, hook] & Y_0 \times \Delta^m \arrow[u, "\gamma_g \times \mathrm{Id}_{\Delta^m}"'] \end{tikzcd}$$

The reader may verify that this diagram is cartesian. Let C is a closed subvariety of $X \times \Delta^m$, such that $X_0 \times C$ meets $(\gamma_f \times \mathrm{Id}_{\Delta^m})_*(X_0 \times \Delta^m)$ properly and such that $(\gamma_f \times \mathrm{Id}_{\Delta^m})^*(X_0 \times C)$ meets all the faces of $X_0 \times \Delta^m$ properly. A computation of codimensions shows that $Y_0 \times (i \times \mathrm{Id}_{\Delta^m})_*(C)$ meets $(\gamma_g \times \mathrm{Id}_{\Delta^m})_*(Y_0 \times \Delta^m)$ properly. Another computation of codimensions (details left to the reader) now shows that $(\gamma_g \times \mathrm{Id}_{\Delta^m})^*(Y_0 \times (i \times \mathrm{Id}_{\Delta^m})_*(C))$ also meets all the faces of $Y_0 \times \Delta^m$

properly. This implies that the image of the map i_* lies inside $z^{i+c}(Y,*)_g \subseteq z^{i+c}(Y,*)$. We thus obtain a diagram of morphisms of complexes

$$
\begin{tikzcd}
0 \arrow[r] & z^i(X,*)_f \arrow[r] \arrow[d] & z^{i+c}(Y,*)_g \arrow[r] \arrow[d] & z^{i+c}(U,*)_{g_{U_0}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & z^i(X_0,*) \arrow[r] & z^{i+c}(Y_0,*) \arrow[r] & z^{i+c}(U_0,*) \arrow[r] & 0
\end{tikzcd}
\tag{0.0.1}
$$

One can see from the definitions that the right-hand square in this diagram is commutative. Furthermore, the left-hand diagram commutes because the equation

$$(\gamma_g \times \mathrm{Id}_{\Delta^m})^*(Y_0 \times (i \times \mathrm{Id}_{\Delta^m})_*(C)) = (l \times \mathrm{Id}_{\Delta^m})_*(\gamma_f \times \mathrm{Id}_{\Delta^m})^*(X_0 \times C).$$

(where C is as above), holds by [Ful75, 2.2(4)]. If we consider the diagram of long homology sequence associated with 0.0.1, we obtain the commutative diagram of localisation sequences advertised in the Proposition. $\square$

Proposition 0.0.2. *Let Y be a smooth scheme over k and let $i : X \hookrightarrow Y$ be a smooth closed subscheme. Let X_0 be a smooth closed subscheme of X. Let $c_X, c_Y \in \mathbb{N}$. Suppose that the codimension of X_0 in X (resp. in Y) is c_X (resp. c_Y). Then the diagram*

$$
\begin{tikzcd}
H^\bullet_{\mathcal{M}}(X\backslash X_0,*) \arrow[r] \arrow[d, "i_*"] & H^{\bullet+1-2c_X}(X_0,*-c_X) \arrow[d, "="] \\
H^{\bullet+2(c_Y-c_X)}_{\mathcal{M}}(Y\backslash X_0,*+c_Y-c_X) \arrow[r] & H^{\bullet+1-2c_X}(X_0,*-c_X)
\end{tikzcd}
$$

is commutative. Here the horizontal maps are the residue maps.

Proof. Consider the commutative diagram

$$
\begin{tikzcd}
Y\backslash X_0 \arrow[r, hook] & Y & X_0 \arrow[l] \\
X\backslash X_0 \arrow[r, hook] \arrow[u] & X \arrow[u] & X_0 \arrow[l] \arrow[u, "="']
\end{tikzcd}
$$

This diagram induces the diagram of complexes

$$
\begin{tikzcd}
0 \arrow[r] & z^i(X_0,*) \arrow[r] \arrow[d] & z^{i+c_X}(X,*) \arrow[r] \arrow[d] & z^{i+c_X}(X\backslash X_0,*) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & z^i(X_0,*) \arrow[r] & z^{i+c_Y}(Y,*) \arrow[r] & z^{i+c_Y}(Y\backslash X_0,*) \arrow[r] & 0
\end{tikzcd}
$$

where the vertical maps are push-forward maps. This diagram is commutative. The fact that the left-hand side square is commutative follows from the functoriality of the push-forward maps. The fact that the right-hand side commutes follows from the fact that the push-forward maps commute with base-change to an open subscheme. Thus we obtain a diagram of localisation sequences

$$\begin{tikzcd}
\dots \arrow[r] & H^{\bullet-2c_X}_{\mathcal{M}}(X_0, *-c_X) \arrow[r] \arrow[d] & H^{\bullet}_{\mathcal{M}}(X, *) \arrow[r] \arrow[d] & H^{\bullet}_{\mathcal{M}}(X\backslash X_0, *) \arrow[r] \arrow[d] & H^{\bullet+1-2c_X}_{\mathcal{M}}(X_0, *-c_X) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H^{\bullet-2c_X}_{\mathcal{M}}(X_0, *-c_X) \arrow[r] & H^{\bullet+2c}_{\mathcal{M}}(Y, *+c) \arrow[r] & H^{\bullet+2c}_{\mathcal{M}}(Y\backslash X_0, *+c) \arrow[r] & H^{\bullet+1-2c_X}_{\mathcal{M}}(X_0, *-c_X) \arrow[r] & \dots
\end{tikzcd}$$

where $c := c_Y - c_X$. This implies the result. □

References

[Blo86] Spencer Bloch. Algebraic cycles and higher K-theory. *Adv. in Math.*, 61(3):267–304, 1986.

[Blo94] S. Bloch. The moving lemma for higher Chow groups. *J. Algebraic Geom.*, 3(3):537–568, 1994.

[Ful75] William Fulton. Rational equivalence on singular varieties. *Inst. Hautes Études Sci. Publ. Math.*, (45):147–167, 1975.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel. *Lecture notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006.

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.