A fixed point formula of Lefschetz type in Arakelov geometry I: statement and proof

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Abstract

We consider arithmetic varieties endowed with an action of the group scheme of n-th roots of unity and we define equivariant arithmetic K_0-theory for these varieties. We use the equivariant analytic torsion to define direct image maps in this context and we prove a Riemann-Roch theorem for the natural transformation of equivariant arithmetic K_0-theory induced by the restriction to the fixed point scheme; this theorem can be viewed as an analog, in the context of Arakelov geometry, of the regular case of the theorem proved by P. Baum, W. Fulton and G. Quart in [BaFQ]. We show that it implies an equivariant refinement of the arithmetic Riemann-Roch theorem, in a form conjectured by J.-M. Bismut (cf. [B2, Par. (l), p. 353] and also Ch. Soulé's question in [SABK, 1.5, p. 162]).

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1 Introduction

It is the aim of this article to prove a Lefschetz type fixed point theorem for some schemes endowed with the action of a diagonalisable group scheme, in the context of Arakelov geometry. This formula is similar to the formula [ASe, III, (4.6), p. 566] and to the formulae which are the main results of [BaFQ] and [T3]; it was originally worked out and conjectured by T. Chinburg, K. Köhler and K. Künnemann jointly. Its main analytic ingredient is the equivariant analytic torsion.

To make things more explicit, we shall briefly recall a special case of the main result of [BaFQ]. Let Y be a smooth projective variety defined over $\mathbf{C}$ and let g be an automorphism of finite order of Y. Let E be a vector bundle on Y. A g-linearisation on E is a morphism of vector bundles $g_E : g^*E \to E$ and the pair (E, g_E) is called an equivariant vector bundle. The cohomology groups $H^i(E)$ of E can naturally be equipped with the g-linearisations $H^i(g_E)$ (over a point). The equivariant vector bundles give rise to a K_0-theory group K_0^g similar to the usual K_0-theory group. This group carries a natural ring structure and furthermore the rule L that associates the linear combination $\sum_{i\geq 0}(-1)^i(H^i(E), H^i(g_E))$ to an equivariant vector bundle (E, g_E) induces a group morphism $L : K_0^g(Y) \to K_0^g(\mathrm{Pt})$ (Pt stands for the point). Suppose now that g is of finite order n. Let Y_g be the fixed point set of g; this set is a smooth projective subvariety of Y and g induces an g-linearisation on the normal bundle N_{Y/Y_g} of the immersion $Y_g \to Y$. Let $\rho : K_0^g(Y) \to K_0^g(Y_g)$ be the morphism arising from the rule that restricts equivariant bundles from Y to Y_g. There are natural isomorphisms $K_0^g(Y_g) \to K_0(Y_g) \otimes_{\mathbf{Z}} K_0^g(\mathrm{Pt})$ and $K_0^g(\mathrm{Pt}) \simeq \mathbf{Z}[\mathbf{C}]$ ($\mathbf{Z}[\mathbf{C}]$ is the $\mathbf{Z}$-module $\oplus_{z\in\mathbf{C}}\mathbf{Z}$, endowed with the ring structure arising from the multiplicative structure of $\mathbf{C}$; see [BaFQ, Par. 0.4]). Choose a $K_0^g(\mathrm{Pt})$-algebra $\mathcal{R}$ in which $1-\zeta$ is invertible for each non-trivial n-th root of unity ζ. The map $L : K_0^g(Y_g) \to K_0^g(\mathrm{Pt})$ naturally extends to a map $L : K_0(Y_g) \otimes_{\mathbf{Z}} \mathcal{R} \to K_0(\mathrm{Pt}) \otimes_{\mathbf{Z}} \mathcal{R} \simeq \mathcal{R}$. A special case of [BaFQ] then states that the equality

$$L(E) = L((\lambda_{-1}(N^\vee_{Y/Y_g}))^{-1}\rho(E)) \tag{1}$$

holds in $\mathcal{R}$ (note that we dropped all references to the underlying g-linearisations). Here $\lambda_{-1}(N^\vee_{Y/Y_g})$ is the alternating sum $\sum_{i\geq 0}(-1)^i\Lambda^i(N^\vee_{Y/Y_g})$, all whose terms are endowed with their natural linearisations. It is a part of the statement that $\lambda_{-1}(N^\vee_{Y/Y_g})$ has an inverse in $K_0(Y_g) \otimes_{\mathbf{Z}} \mathcal{R}$.

In order to carry out a similar reasoning in the field of arithmetic geometry, one has to give meaning to the formula (1) on a projective regular scheme $f : Y \to \mathrm{Spec}\ \mathbf{Z}$ over the integers (actually even slightly more general rings), when E is a hermitian vector bundle, i.e. a vector bundle on Y which is endowed with a (conjugation invariant) hermitian metric on the complex points $Y(\mathbf{C})$ of Y. In this context, we choose to suppose that Y is endowed with the action of the group scheme $\mu_n \to \mathrm{Spec}\ \mathbf{Z}$ of n-th roots of unity rather than with the

action of an automorphism of some order.
To justify this choice, let us define D to be the ring of integers of the cyclotomic field $\mathbf{Q}(\mu_n)$ and let C_n be the constant group scheme over $\mathbf{Z}$ which is associated to the cyclic group of order n; there is an isomorphism of group schemes $\mu_n \times_{\mathrm{Spec}\ \mathbf{Z}} \mathrm{Spec} D[\frac{1}{n}] \simeq C_n \times_{\mathrm{Spec}\ \mathbf{Z}} \mathrm{Spec} D[\frac{1}{n}]$ (recall that $D[\frac{1}{n}]$ is the ring D localised at the multiplicative subset generated by $1/n$). This is a consequence of the chinese remainder theorem. Thus, after a suitable base change, a μ_n-action is equivalent to the action of an automorphism of finite order, away from the fibers of the scheme that lie over the primes numbers dividing n. On such a fiber, the action of an automorphism of finite order can have a very irregular fixed scheme, whereas the fixed scheme of the action of a diagonalisable group scheme will be smooth (see the end of section 2). By choosing diagonalisable group schemes, we avoid having to deal with automorphisms of order not coprime with the characteristic of the ground field.
There is a closed subscheme of Y, the fixed point scheme $h : Z \to \mathrm{Spec}\ \mathbf{Z}$, which is maximal among the closed subschemes that inherit a trivial action from Y. One can prove that Z is also regular. We suppose then that the action of μ_n can be lifted to an action on E, which is compatible with the metric on $E_{\mathbf{C}}$. We call the vector bundle E together with its metric and its action a μ_n-equivariant hermitian vector bundle. One can define a K_0-theory $\widehat{K}_0^{\mu_n}(Y)$ for the equivariant hermitian vector bundles. Let now ω_Y be a μ_n-invariant Kähler metric on Y. There is a push-forward morphism $f_* : \widehat{K}_0^{\mu_n}(Y) \to \widehat{K}_0^{\mu_n}(\mathbf{Z})$, dependent on ω_Y and a restriction morphism $\rho : \widehat{K}_0^{\mu_n}(Y) \to \widehat{K}_0^{\mu_n}(Z)$. Fix a primitive n-th complex root of unity ζ_n. Let $R(\mu_n) \simeq \mathbf{Z}[T]/(1-T^n)$ be the Grothendieck group of μ_n-comodules. The primitive root ζ_n determines a ring homomorphism $R(\mu_n) \to \mathbf{C}$ and a holomorphic automorphism g of $Y(\mathbf{C})$. Our main result Th. 4.4 reads

$$f_*(\overline{E}) = h_*((\lambda_{-1}(\overline{N}^{\vee}_{Y/Z}))^{-1}\rho(\overline{E})) - \int_{Z(\mathbf{C})} \mathrm{Td}_g(TY_{\mathbf{C}}) R_g(N_{Y_{\mathbf{C}}/Z_{\mathbf{C}}}) \mathrm{ch}_g(E_{\mathbf{C}}), \quad (2)$$

where the equality holds in the ring $\widehat{K}_0^{\mu_n}(\mathrm{Spec}\ \mathbf{Z}) \otimes_{R(\mu_n)} \mathbf{C}$ (Th. 4.4 is in fact slightly more general in that not only complex coefficients are considered). The expression $\lambda_{-1}(\overline{N}^{\vee}_{Y/Z})$ stands for the alternating sum $\sum_{i\geq 0}(-1)^i \Lambda^i(\overline{N}^{\vee}_{Y/Z})$, where $\overline{N}_{Y/Z}$ is equipped with the metric it inherits from ω_Y; the expressions $\mathrm{ch}_g(E_{\mathbf{C}}), \mathrm{Td}_g(TY_{\mathbf{C}})$ and $R_g(TY_{\mathbf{C}})$ represent complex characteristic classes depending on g. It is a part of the statement that $\lambda_{-1}(\overline{N}^{\vee}_{Y/Z})$ is invertible in the ring $\widehat{K}_0^{\mu_n}(Z) \otimes_{R(\mu_n)} \mathbf{C}$.
It turns out that there is a natural map $\widehat{\deg}_{\mu_n} : \widehat{K}_0^{\mu_n}(\mathrm{Spec}\ \mathbf{Z}) \to \mathbf{C}$. To describe $\widehat{\deg}_{\mu_n}(f_*(\overline{E}))$, suppose for simplicity that f is a flat map and that the cohomology groups $R^i f_* E = 0$ for $i > 0$. The group $R^0 f_* E$ is then free; we endow it with the μ_n-action it inherits from E by functoriality and with the L_2-hermitian metric it inherits from E. The μ_n-action on $R^0 f_* E$ is then described

by a $\mathbf{Z}/(n)$-grading, whose terms are orthogonal. We write $(R^0f_*\overline{E})_k$ for the k-th term ($k \in \mathbf{Z}/(n)$), endowed with induced hermitian metric. In terms of this structure, we have

$$\widehat{\deg}_{\mu_n}(f_*(\overline{E})) = \sum_{k\in\mathbf{Z}/(n)} \zeta_n^k\, \widehat{\deg}((R^0f_*\overline{E})_k) - T_g(Y(\mathbf{C}), \overline{E}_{\mathbf{C}}).$$

Here $T_g(Y(\mathbf{C}), \overline{E}_{\mathbf{C}})$ is the equivariant analytic torsion of $E_{\mathbf{C}}$, a purely analytic term which depends on ω_Y and the metric on $E_{\mathbf{C}}$. It coincides with Ray-Singer's analytic torsion when the action is trivial. The symbol $\widehat{\deg}$ refers to the arithmetic degree of a hermitian $\mathbf{Z}$-module (it is a real number); see [Bo1, Par. 2.5] for the definition. We call the term $\sum_{k\in\mathbf{Z}/(n)} \zeta_n^k.\widehat{\deg}((R^0f_*\overline{E})_k)$ the arithmetic Lefschetz trace; as it happens in the geometric setting, the arithmetic Lefschetz trace coincides with the arithmetic Euler-Poincaré characteristic when the action is trivial (this is the quantity computed by the arithmetic Riemann-Roch theorem [GS8, 4.2.3]). Our main result Th. 4.4 thus computes the arithmetic Lefschetz trace of an equivariant hermitian vector bundle as a contribution of the fixed point scheme of the action of μ_n on Y and an anomaly term, the equivariant analytic torsion, which is purely analytic.

We now shortly discuss our method of proof of Th. 4.4. There are several different ways to prove a formula like (1); first it has been shown via index theory and topological K-theory ([ASe, III]), a second method uses the asymptotics of heat kernels for small times ([BeGeV, Chap. 6]) (these two only work over the complex numbers), a third one uses the Quillen localisation sequence for higher equivariant K-theory ([T3]) and a fourth one uses the deformation to the normal cone ([BaFQ]). The algebro-geometric part of our proof follows this last strategy whereas its differential geometric part relies heavily on the results of Bismut in [B3], who applies refined versions of the second method. On the group-scheme theoretic side, we prove in section 2 some results on the action of a diagonalisable group scheme on a projective space. On the analytic side, the main original ingredient entering the proof is the double complex formula Th. 3.14, which generalises a result of Bismut, Gillet and Soulé in [BGS5, Th 2.9, p. 279] to the equivariant case. The construction of the proof of Th. 4.4 is globally parallel to the construction of the proof given in [R1, Th. 3.7] of an Adams-Riemann-Roch theorem in Arakelov geometry. Some λ-ring-theoretic results of [R1] are also used. Although the algebro-geometric techniques of the present paper and [R1] are comparable, many points have been simplified here and replaced by arguments of homological algebra (e.g. Prop. 6.2).

We encourage the reader to begin with the section 4 containing the statement and refer to the sections 2 and 3 as necessary. In the last subsection of the paper, we translate Th. 4.4 into the language of the arithmetic Chow theory of Gillet and Soulé (see [GS2]). The result Th. 7.14 we obtain gives a positive answer to Bismut's question on the existence of an equivariant arithmetic Riemann-Roch theorem (see [B2, Par. (l), p. 353] and also Soulé's question in [SABK, 1.5, p.

162]).
The applications of the main result of this paper are or will be discussed elsewhere. They include a Bott-type residue formula for the height of arithmetic varieties endowed with the action of a diagonalisable torus [KR3], a new proof of the Jantzen sum formula for representations of Chevalley schemes [KK], a computation of the height of flag varieties [KK] and a computation of the Faltings height of certain abelian varieties (to appear).
The results of this paper are partially announced in [KR].
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2 Group scheme-theoretic preliminaries

Until the end of the paper, all schemes will be noetherian. We fix a base scheme S and we adopt the convention, in this section, that all schemes are S-schemes and all morphisms S-morphisms. We let **Schemes**/S denote the category of S-schemes and **Sets** the category of sets. Let now G be a flat group scheme over S. A G-action on an scheme Y is a morphism $m_Y : G \times_S Y \to Y$, satisfying some compatibility properties. We refer to [Mu, Def. 0.3] for the description of the latter. A scheme which is endowed with a G-action is said to be G-equivariant or a G-scheme. A morphism $r : Y \to X$ of G-schemes such that $m_X \circ (\mathrm{Id} \times r) = r \circ m_Y$ is said to be a G-map or to be G-equivariant. If r is a closed immersion (resp. open immersion) then Y is called a closed (resp. open) G-subscheme, or a G-equivariant closed (resp. open) subscheme of X. A G-action on a scheme Y is called **trivial** if the morphism m_Y describing the action is the natural projection on the second component. If $Y' \to Y$, $Y'' \to Y$ are equivariant morphisms of G-schemes, then the fiber product $Y' \times_Y Y''$ carries a G-action such that the natural projections are equivariant; this follows from the definition of a group scheme action and some diagram chasing. Let us now fix a scheme Y and a G-action m_Y on Y. Call $p_Y : G \times_S Y \to Y$ the natural projection. Let F be a coherent sheaf on Y. A G-action on F is a isomorphism of coherent sheaves $m_F : p_Y^* F \to m_Y^* F$ satisfying certain associativity properties. We refer to [Mu, Def. 1.6] (for a line bundle, but in fact valid without change for any coherent sheaf) or [Köck, 1., (1.1) Def.] for the description of the latter.

A coherent sheaf with a G-action is said to be a G-sheaf or a G-equivariant sheaf. If $Y = S$ and G (resp. S) is the spectrum of a ring B (resp. A), then F corresponds to a finitely generated module M over A. The structure induced on M by the G-action on F is called a B-comodule structure and M together with this structure is called a B-**comodule**.
To an S-morphism $y : T \to Y$, we can associate a map $G \times T \to Y \times T$, given in point set notation by the rule $g \times t \mapsto m_Y(g \times y(t)) \times t$. Let $Y(T)_{G(T)}$ be the set of S-morphisms y from T to Y such that the morphism $G \times T \to Y \times T$ induced by y is given by the composition $(y \times \mathrm{Id}) \circ p_T$, where $p_T : G \times T \to T$ is the natural projection.

Definition 2.1 *The functor of fixed points associated to Y is the functor* **Schemes**$/S \to$ **Sets** *described by the rule $T \mapsto Y(T)_{G(T)}$.*

The following proposition is proved in [SGA3, VIII, 6.5 d].

Proposition 2.2 *If G is diagonalisable over S and Y is separated over S, then the functor of fixed points of Y is representable by an S-scheme Y_G and the canonical immersion of functors $Y(\cdot)_{G(\cdot)} \to Y(\cdot)$ induces an equivariant closed immersion $i_G : Y_G \to Y$.*

We call the scheme Y_G the **fixed point scheme** of Y. By definition, if it exists, the scheme Y_G thus enjoys the following universal property: if $i : Y' \to Y$ is a closed G-subscheme of Y whose action is trivial, then there is a unique closed immersion $j : Y' \to Y_G$, such that $i_G \circ j = i$. It also follows from the preceding definition that if $i : Y' \to Y$ is a closed G-subscheme of Y, then Y' has a fixed point scheme and $i^* Y_G = Y'_G$.

Definition 2.3 *A G-scheme Y is called G-quasi-projective (resp. G-projective) if there is a G-immersion (resp. closed G-immersion) $i : Y \to \mathbf{P}^n_S$ into some projective space endowed with a G-action.*

Caution. This definition is more restrictive than the definition given in [Köck, Def. (3.2)].
Suppose now that we are given a G-action on the sheaf $E := \mathcal{O}_S^{\oplus\, n+1}$, the free sheaf of rank $n+1$ on S ($n \geq 0$). Identify $\mathbf{P}^n_S$ with $\mathrm{Proj}(\mathrm{Sym}(E^\vee))$. Using the functorial properties of the Proj symbol, we obtain a G-action on $\mathbf{P}^n_S$. A G-action on $\mathbf{P}^n_S$ thus arising will henceforth be called **global**. The following lemma is a special case of [Köck, Lemma (3.3) (a)].

Lemma 2.4 *Let Y be a G-projective scheme. If S is affine and G is a diagonalisable group scheme (over S), then the following statements are equivalent:*

(a) *the scheme Y admits a closed G-immersion into a projective space over S endowed with a global action;*

(b) *there is a very ample G-equivariant line bundle on Y.*

The next lemma shows that in a certain situation the conditions of the Lemma 2.4 are always fulfilled:

Lemma 2.5 *If S is affine, then on every G-projective scheme, there is a very ample G-equivariant line bundle.*

Proof: Let Y be a G-projective scheme. Choose an equivariant closed immersion i of Y into some G-equivariant projective space $p : \mathbf{P}^n_S \to S$ $(n \geq 0)$. Write P for $\mathbf{P}^n_S$. Let p_P be the natural projection $G \times P \to P$. The automorphism $G \times_S P \to G \times_S P$ arising from the G-action on P extends by functoriality to an automorphism of the sheaf of differentials $\omega_{G\times_S P/G} \simeq p_P^*\omega_{P/S}$. This automorphism defines a G-action on $\omega_{P/S}$ (see ([Köck, Ex. 1.2 (c)]). Consider now the dual of the determinant bundle of $\omega_{P/S}$; the restriction of this bundle to Y is equivariant and ample and thus some tensor power of it has the required properties. So we are done. **Q.E.D.**

Let us also notice the following facts. Let X, Y be G-schemes; let $a_X : G\times X \to G \times X$ and $a_Y : G \times Y \to G \times Y$ be the automorphisms arising from the respective G-actions. Suppose $r : X \to Y$ is a morphism of schemes. Then r is a G-morphism if and only if $a_Y \circ (\mathrm{Id} \times r) = (\mathrm{Id} \times r) \circ a_X$ (*****); moreover the automorphism a_Y is the identity if and only if the action on Y is trivial (******). This follows from the definition of a group scheme action, the universal properties of fiber products and some diagram chasing.

Lemma 2.6 *Let Y be a G-scheme and let $u_1 : U_1 \to Y, u_2 : U_2 \to Y, \ldots, u_l : U_l \to Y$ be G-equivariant open subschemes that cover Y. The following conditions are equivalent*

(a) *the G-action on Y is trivial;*

(b) *for each i $(1 \leq i \leq l)$, the G-action on U_i is trivial.*

Proof: Consider first the constant group scheme associated to an ordinary group M. To give an action of such a group scheme on a scheme X is equivalent to give a homomorphism of M into the group of scheme automorphisms of X; thus we see that the lemma holds for such a group scheme.
Returning to the general case, let us now consider the open immersions $\mathrm{Id}\times u_i : G\times U_i \to G\times Y$; the scheme $G\times U_i$ carries the action of $\mathbf{Z}$ via the automorphism a_{U_i} and the scheme $G \times Y$ carries the action of $\mathbf{Z}$ via the automorphism a_Y;

furthermore by the fact (*) mentioned above, the open immersions $\mathrm{Id} \times u_i : G \times U_i \to G \times Y$ satisfy the hypothesies of this same lemma, with $G \times Y$ in place of Y and with the constant group scheme associated to the group $\mathbf{Z}$ in place of G. The first paragraph of this proof then shows that the lemma holds in the latter situation and using the fact (**) we see that this is equivalent to the general case. **Q.E.D**.

Lemma 2.7 *Let Y be a G-scheme and let $u_1 : U_1 \to Y, u_2 : U_2 \to Y, \dots, u_l : U_l \to Y$ be G-equivariant open subschemes that cover Y. Suppose that $U_{i,G}$ exists for each i and that Y_G exists.*
If Y' is a closed equivariant subscheme of Y such that $u_i^* Y' = U_{i,G}$ for all $1 \leq i \leq l$, then $Y' = Y_G$.

Proof: Since $u_i^* Y' = U_{i,G}$, we can apply the Lemma 2.6 to conclude that there is a unique equivariant closed immersion $Y' \to Y_G$. On the other hand, by the Lemma 2.6 and the equivariance properties of fiber products, the restriction of this immersion to every U_i is an isomorphism. It is thus globally an isomorphism. **Q.E.D**.

Let us now suppose that S is the spectrum of a ring A. Let N be a finitely generated abelian group (written additively) and let $T_N := (\mathrm{Spec}\, \mathbf{Z}[N]) \times_{\mathbf{Z}} S$ be the associated diagonalisable group scheme over S (see [SGA3, VIII] for more details). A T_N-action on an A-module is equivalent to an A-module N-grading and a T_N-action on an A-algebra is equivalent to an A-algebra N-grading. We shall denote by $\deg_N(h) \in N$ the homogeneous degree of a homogeneous element h in an N-graded object. To simplify the discussion, we shall suppose that $N = \mathbf{Z}$ or that $N = \mathbf{Z}/(n)$ for some $n \in \mathbf{Z}$. Let $M = \oplus_{k \in N} M_k$ be an N-grading on an A-module M. In the functorial language, the corresponding T_N-action can be described as follows. Let C be an A-algebra. The set $T_N(C)$ then corresponds to the set of n-th roots of unity (if $N = \mathbf{Z}/(n)$) or to the set of units (if $N = \mathbf{Z}$); the action of $T_N(C)$ on $M \otimes_A C$ is given by the formula $u.(m_k)_{k \in N} = (u^k.m_k)_{k \in N}$.

Lemma 2.8 *Let $B := A[\mathbf{X}]$ be the polynomial ring with variables in the finite set $\mathbf{X}$. Let $w : \mathbf{X} \to N$ be a function. Endow B with the only A-algebra grading $B = \oplus_{k \in N} B_k$ such that $X \in B_k$ if $\deg_N(X) = w(X)$. Let I be the ideal of B generated by the set $\{X \in \mathbf{X} | \deg_N(X) \neq 0\}$. Then $(\mathrm{Spec} B)_{T_N} = \mathrm{Spec}\,(B/I)$.*

Proof: The ideal J of $(\mathrm{Spec} B)_{T_N}$ in B is by definition the largest homogeneous ideal with the property that if $b \in B_k$ and $k \neq 0$ then b lies in this ideal. By definition J contains the ideal generated by $\oplus_{k \in N, k \neq 0} B_k$; we have to prove that the reverse inclusion holds. So let $a.X_1 \dots X_l$ be a monomial in B_k, $k \neq 0$; by definition $\sum_{i=1}^{l} \deg_N(X_j) \neq 0$ and thus at least one of the $\deg_N(X_i)$ is not 0. Thus $a.X_1 \dots X_l \in$ lies in the ideal generated by $\{X \in \mathbf{X} | \deg_N(X) \neq 0\}$. As all the elements of B_k are sums of such monomials, the reverse inclusion is proved and we are done. **Q.E.D.**

So let M be a module over A, endowed with an N-grading $M = \oplus_{k\in N} M_k$, where the M_k are supposed free and finitely generated. Using the functorial properties of the **Proj** and Sym symbols we obtain a T_N-action on the scheme $\mathbf{P}(M) := \text{Proj}(\text{Sym}(M^\vee))$. By functoriality again, the inclusion $M_k \subseteq M$ ($k \in N$) induces an immersion $\mathbf{P}(M_k) \to \mathbf{P}(M)$.

Proposition 2.9 *The fixed point scheme of* $\mathbf{P}(M)$ *is the disjoint union of the closed subschemes* $\coprod_{k\in N} \mathbf{P}(M_k)$.

Proof: Let $m_0, \dots m_l$ be a basis of M consisting of homogeneous elements. Let B be the polynomial ring $B := A[X_1, \dots, X_l]$. For any affine S-scheme $S' = \text{Spec } C$, we have a canonical isomorphism between $(\text{Spec } B)(S')$ and $\oplus_{i=1}^{l} C$ and a canonical isomorphism between $(\mathbf{P}(M))(S')$ and the set of projective submodules of rank 1 of $\oplus_{i=0}^{l} C$. Fix $0 \leq l_0 \leq l$ and consider the map that sends $(x_1, \dots, x_l) \in \oplus_{i=1}^{l} C$ to the line generated by $(x_1, \dots, x_{l_0-1}, 1, x_{l_0}, \dots x_l)$. This map is functorial in C and defines the basic open immersion Spec $B \to$ P(M). Now let $u \in T_N(C)$ act on $\oplus_{i=1}^{l} C$ by the formula $(x_1, \dots, x_l) \mapsto (u^{\deg_N(m_1)-\deg_N(m_{l_0})}.x_1, \dots, u^{\deg_N(m_l)-\deg_N(m_{l_0})}.x_l$; by construction, this map is functorial in C and it defines a T_N-action on B, which commutes with the basic open immersion. By the discussion before the lemma, this T_N-action is equivalent to the unique N-grading on B, such that X_i has degree $\deg_N(m_i) - \deg_N(m_{l_0})$. Notice also that $\coprod_{k\in N} \mathbf{P}(M_k)(S')$ consists of projective submodules of rank 1 of $(x_1, \dots, x_l) \in \oplus_{i=0}^{l} C$ that lie in one of the subspaces $M_k \otimes_A C$ ($k \in N$). From this fact and the functorial description of the open immersion, one can see that that the restriction of $\coprod_{k\in N} \mathbf{P}(M_k)$ to the affine scheme Spec B, is the closed subscheme of Spec B representing the functor that associates $M_{\deg_N(m_{l_0})} \otimes_A C$ to C. One can check from the definition that this closed subscheme is defined by the ideal generated by the variables X_i such that $\deg_N(X_i) \neq \deg_N(m_{l_0})$. Using the Lemma 2.8, we see that the restriction of $\coprod_{k\in N} \mathbf{P}(M_k)$ to Spec B is the fixed point scheme of B. Now notice that if l_0 varies, the corresponding open immersions cover P(M). Thus we can apply Lemma 2.7 to conclude. **Q.E.D.**

Corollary 2.10 *Let* Y *be a scheme endowed with a trivial* T_N*-action and let* E *be a vector bundle on* Y *endowed with a* T_N*-action. Then the fixed scheme of* $\mathbf{P}(E)$ *is the closed subscheme* $\coprod_{k\in N} \mathbf{P}(E_k)$.

Proof: Let $\{U_i\}$ ($i \in I$) be an open affine covering of Y, such that each E_k is free on each U_i; this covering yields an open covering $\{\mathbf{P}(E)|_{U_i}\}$ of $\mathbf{P}(E)$. Consider now that by Prop. 2.9 $\coprod_{k\in N} \mathbf{P}(E_k)|_{U_i}$ corresponds to the fixed point scheme of $\mathbf{P}(E|_{U_i})$; we can thus apply Lemma 2.7 to conclude. **Q.E.D.**

Corollary 2.11 *Let Y be a T_N-projective scheme over A. Then there is a covering $\{u_i : U_i \to Y\}$ $(i \in I)$ of Y by open affine equivariant subschemes, such that $u_i^* Y_{T_N} = U_{i,T_N}$. Furthermore, let B be an A-algebra and let p_1 be the projection of $Y_B := Y \times_{\mathrm{Spec}\ A} \mathrm{Spec}\ B$ on the first factor; endow Y_B with the induced T_N-action. Then the closed subschemes $p_1^* Y_{T_N}$ and Y_{B,T_N} coincide.*

Proof: The first statement follows from the equivariance properties of fiber products. To prove the second statement, notice that if Y is a projective space over A, this follows from the explicit description given in the Prop. 2.9. The general case then follows, if one remembers that pull-back of ideal sheaves is an operation invariant under base change. **Q.E.D**.

The following proposition gives some informations about the regularity of the fixed scheme. Its proof can be found in [T3, Prop. 3.1, p. 455].

Proposition 2.12 *Let Y be a T_N-quasi-projective scheme over A. Suppose that Y is regular. Then Y_{T_N} is also regular and the normal bundle $N_{Y/Y_{T_N}}$ is a T_N-equivariant bundle with vanishing fixed subsheaf, i.e. $(N_{Y/Y_{T_N}})_0 = 0$.*

3 Differential-geometric preliminaries

3.1 Equivariant Determinants

Let g be an isometry of an hermitian vector space $\overline{E}$. Let Θ denote the set of eigenvalues ζ of g with associated eigenspaces $\overline{E}_\zeta$. The g-**equivariant determinant** of E is defined as

$$\det_g E := \bigoplus_{\zeta \in \Theta} \det E_\zeta.$$

The g-**equivariant metric** associated to the metric on E is the map

$$\begin{aligned} \log \|\cdot\|^2_{\det_g E} : \det_g E &\to \mathbf{C} \\ (s_\zeta)_\zeta &\mapsto \sum_{\zeta \in \Theta} \log \|s_\zeta\|^2_\zeta \cdot \zeta, \end{aligned}$$

where $\|\cdot\|^2_\zeta$ denotes the induced metric on $\det E_\zeta$. Let Γ be a finite group and let $\sigma : \Gamma \to \mathrm{End} E$ be a unitary representation. Denote the group of irreducible unitary representations (ρ, V_ρ) by $\hat{\Gamma}$. The Γ-**equivariant determinant** of E is defined as

$$\det_\Gamma E := \bigoplus_{\rho \in \hat{\Gamma}} \det(\mathrm{Hom}_\Gamma(V_\rho, E) \otimes V_\rho).$$

The associated Γ-**equivariant metric** [B3] is the map

$$\begin{array}{rcl}\log\|\cdot\|^2_{\det_\Gamma E} : \det_\Gamma E & \to & \mathbf{C}\Gamma\otimes\mathbf{C} \\ (s_\rho)_\rho & \mapsto & \sum_{\rho\in\hat\Gamma}\log\|s_\rho\|^2_\rho\frac{\chi_\rho}{\mathrm{rk}V_\rho},\end{array}$$

where $\|\cdot\|^2_\rho$ is the metric on $\det(\mathrm{Hom}_\Gamma(V_\rho,E)\otimes V_\rho)$ and χ_ρ denotes the character of ρ. Tensor products of equivariant determinant lines are defined as the sum of the products of lines corresponding to the same representations.

Now let $g\in\Gamma$ be an element of order N of a finite group and let $(E,\|\cdot\|^2)$ be a hermitian representation space of Γ. Let V_k, $1\le k\le N$, denote the one-dimensional unitary representations of the cyclic group generated by g, where g acts as ζ_N^k on V_k. As both versions of the equivariant metrics are used in the literature and in this article, we would like to emphasize that the difference between $\log\|\cdot\|^2_{\det_\Gamma E}(g)$ and $\log\|\cdot\|^2_{\det_g E}$ is entirely independent of E in the following sense: Let $\det_1$, $\det_2$ denote the canonical surjective maps from

$$\bigoplus_{\substack{\rho\in\Gamma^\vee\\ 1\le k\le N}}\det(\mathrm{Hom}_\Gamma(V_\rho,E)\otimes\mathrm{Hom}_g(V_k,V_\rho)\otimes V_k)$$

to

$$\det{}_\Gamma E=\bigoplus_{\rho\in\Gamma^\vee}\bigotimes_{k=1}^N\det(\mathrm{Hom}_\Gamma(V_\rho,E)\otimes\mathrm{Hom}_g(V_k,V_\rho)\otimes V_k)$$

and

$$\det{}_g E=\bigoplus_{k=1}^N\bigotimes_{\rho\in\Gamma^\vee}\det(\mathrm{Hom}_\Gamma(V_\rho,E)\otimes\mathrm{Hom}_g(V_k,V_\rho)\otimes V_k)$$

which map an N-tuple (resp. an $\#\Gamma$-tuple) to the tensor product of its components. Choose once and for all bases of the vector spaces $\mathrm{Hom}_g(V_k,V_\rho)\otimes V_k$.

Lemma 3.1 *Let $\alpha_\rho=(\alpha_{\rho,k})_k$ denote the multi index $(\dim\mathrm{Hom}_g(V_k,V_\rho))_k$. Then there is a canonical projection π, independent of the choice of the metric on E, and a map f which is independent of $\overline{E}$, such that the following diagram commutes*

$$\begin{array}{ccc}\bigoplus_{\substack{\rho\in\Gamma^\vee\\ 1\le k\le N}} & \det(\mathrm{Hom}_\Gamma(V_\rho,E)\otimes\mathrm{Hom}_g(V_k,V_\rho)\otimes V_k) & \\ \pi\downarrow & \searrow\log\|\det_1(\cdot)\|^2_{\det_\Gamma E}(g)-\log\|\det_2(\cdot)\|^2_{\det_g E} & \\ \prod_{\rho\in\Gamma^\vee}\mathbf{P}^{\alpha_\rho}\mathbf{C} & \xrightarrow{f}\mathbf{R} & \end{array}$$

where $\mathbf{P}^{\alpha_\rho}\mathbf{C}$ denotes the weighted projective space associated to α_ρ.

Proof: For $s \in \bigoplus_{\substack{\rho\in\Gamma^\vee \\ 1\leq k\leq N}} \det(\mathrm{Hom}_\Gamma(V_\rho, E)\otimes \mathrm{Hom}_g(V_k, V_\rho)\otimes V_k)$, $(t_\rho)_{\rho\in\Gamma^\vee} \in (\mathbf{R}^+)^{\#\Gamma}$ set $s' := (t_\rho^{\alpha_{\rho,k}} s_{\rho,k})_{\rho,k}$. Note that for any Γ-invariant metric $\|\cdot\|'^2$ on E there is such a tuple of scalars $(t_\rho)_{\rho\in\Gamma^\vee}$ such that for any s the induced metric on $\det_\Gamma E$ is given by

$$\log\|\det_1(s)\|'^2_{\det_\Gamma E} = \log\|\det_1(s')\|^2_{\det_\Gamma E} \ .$$

Now

$$\begin{aligned}\log\|\det_1(s')\|^2_{\det_\Gamma E}(g) &= \sum_{\rho\in\Gamma^\vee} \log(t_\rho^{2\sum_k \alpha_{\rho,k}} \|(\det_1(s))_\rho\|^2_\rho)\cdot \frac{\chi_\rho(g)}{\dim V_\rho} \\ &= \log\|\det_1(s)\|^2_{\det_\Gamma E}(g) + 2\sum_{\rho\in\Gamma^\vee} \log t_\rho \cdot \chi_\rho(g)\end{aligned}$$

and

$$\begin{aligned}\log\|\det_2(s')\|^2_{\det_g E} &= \sum_{k=1}^N \log(\prod_\rho t_\rho^{2\alpha_{\rho,k}}\cdot \|(\det_2(s))_k\|^2_k)\cdot \zeta_N^k \\ &= \log\|\det_2(s)\|^2_{\det_g E} + 2\sum_{\rho,k}\log t_\rho\cdot \zeta_N^k \alpha_{\rho,k} \\ &= \log\|\det_2(s)\|^2_{\det_g E} + 2\sum_{\rho\in\Gamma^\vee}\log t_\rho \cdot \chi_\rho(g) \ .\end{aligned}$$

Thus, $\log\|\det_1(\cdot)\|^2_{\det_\Gamma E}(g) - \log\|\det_2(\cdot)\|^2_{\det_g E}$ depends only on the projection of s to

$$\begin{aligned}&\prod_\rho\bigoplus_k \det(\mathrm{Hom}_\Gamma(V_\rho, E)\otimes\mathrm{Hom}_g(V_k, V_\rho)\otimes V_k)/s\sim t_\rho^{\alpha_\rho}s \\ \overset{\text{can.}}{\cong}\quad &\prod_\rho\bigoplus_k (\det\mathrm{Hom}_\Gamma(V_\rho, E))^{\dim\mathrm{Hom}_g(V_k,V_\rho)}/s\sim t_\rho^{\alpha_\rho}s \ .\end{aligned}$$

For an arbitrary complex line L, the space $\left(\bigoplus_k L^{\alpha_{\rho,k}}\right)/s\sim t_\rho^{\alpha_\rho}s$ is canonically isomorphic to $\prod_\rho \mathbf{P}^{\alpha_\rho}\mathbf{C}$. Hence

$$\prod_\rho\bigoplus_k \det(\mathrm{Hom}_\Gamma(V_\rho, E)\otimes\mathrm{Hom}_g(V_k, V_\rho)\otimes V_k)/s\sim t_\rho^{\alpha_\rho}s \overset{\text{can.}}{\cong} \prod_\rho \mathbf{P}^{\alpha_\rho}\mathbf{C} \ ,$$

thus the map $\log\|\det_1(\cdot)\|^2_{\det_\Gamma E}(g) - \log\|\det_2(\cdot)\|^2_{\det_g E}$ factors through $\prod_\rho \mathbf{P}^{\alpha_\rho}\mathbf{C}$ and it does not depend on the choice of the Γ-invariant metric on E. **Q.E.D.**

3.2 Equivariant Quillen-metrics

In this subsection we shall introduce the concept of equivariant Quillen metrics following Bismut [B3]. Let M be a compact n-dimensional hermitian manifold with associated Kähler form ω. Let $\overline{E}$ denote an hermitian holomorphic vector bundle on M and let

$$\bar{\partial} : \Gamma(\Lambda^q T^{*0,1} M \otimes E) \to \Gamma(\Lambda^{q+1} T^{*0,1} M \otimes E)$$

be the Dolbeault operator. Let N denote the number operator acting on $\Lambda^q T^{*0,1} M \otimes E$ by multiplication with q. Let Φ act on $\Lambda T^* M \otimes E$ by multiplication with $(2\pi i)^{-N/2}$. Supertraces shall be taken with respect to the grading given by N. As in [GS5], we equip $\mathfrak{A}^{0,q}(M,E) := \Gamma(\Lambda^q T^{*0,1} M \otimes E)$ with the hermitian L^2-metric

$$(\eta, \eta') := \int_M \langle \eta(x), \eta'(x) \rangle \frac{\omega^{\wedge n}}{(2\pi)^n n!}. \tag{3}$$

Here the metric on $\Lambda^q T^{*0,1} M \otimes E$ is the one induced by the metrics on TM and on E. Let $\bar{\partial}^*$ be the adjoint of $\bar{\partial}$ relative to this metric and let $\Box_q := (\bar{\partial} + \bar{\partial}^*)^2$ be the Kodaira-Laplacian acting on $\Gamma(\Lambda^q T^{*0,1} M \otimes E)$ with spectrum $\sigma(\Box_q)$. We denote by $\mathrm{Eig}_\lambda(\Box_q)$ the eigenspace of $\Box_q$ corresponding to an eigenvalue λ. Consider a holomorphic isometry g of M and assume given a holomorphic isometry $g^E : g^* E \to E$. The fixed point set of g shall be denoted by M_g. The element g induces an isometry g^* of the Dolbeault cohomology $H^{0,q}(M,E) := \ker \Box_q$ equipped with the restriction of the L^2-metric. Then the equivariant Quillen metric is defined via the zeta function

$$Z_g(s) := \sum_{q>0} (-1)^{q+1} q \sum_{\substack{\lambda \in \sigma(\Box_q) \\ \lambda \neq 0}} \lambda^{-s} \operatorname{Tr} g^*_{|\mathrm{Eig}_\lambda(\Box_q)}$$

for $\operatorname{Re} s \gg 0$. Classically, this zeta function has a meromorphic continuation to the complex plane which is holomorphic at zero ([Do]).

Definition 3.2 *Set $\lambda_g(M,E) := \left[\det_g H^{0,*}(M,E)\right]^{-1}$. The equivariant analytic torsion is defined as*

$$T_g(M, \overline{E}) := Z_g'(0)$$

([K1]). The equivariant Quillen metric on $\lambda_g(M,E)$ is defined as

$$\log \| \cdot \|^2_{Q,\lambda_g(M,E)} := \log \| \cdot \|^2_{L^2,\lambda_g(M,E)} - Z_g'(0). \tag{4}$$

We shall denote $(\lambda_g(M,E), \| \cdot \|^2_Q)$ by $\lambda_g(M, \overline{E})$. Similarly we define $\lambda_\Gamma(M, \overline{E})$ and $\lambda(M, \overline{E})$.

Lemma 3.3 *Let Γ denote a finite group acting on M by holomorphic and fixed point free isometries. Let E be a Γ-equivariant holomorphic hermitian vector bundle. For a unitary representation (V_ρ, ρ) of Γ with character χ_ρ let $E_\rho :=$*

$M\times_\rho V_\rho$ denote the associated flat hermitian vector bundle on M/Γ. Then there is a canonical isometry of equivariant determinants

$$\lambda_\Gamma(M,\overline{E}) \cong \bigoplus_{\rho\in\Gamma^\vee} \lambda(M/\Gamma, \overline{E}/\Gamma\otimes\overline{E}_\rho\otimes V_\rho) \ .$$

Proof: For a unitary representation ρ let P_ρ be the operator

$$P_\rho := \frac{1}{\#\Gamma}\sum_{g\in\Gamma} g^*\otimes\overline{\rho(g)}$$

which projects $\mathfrak{A}^{0,q}(M,E)\otimes V_\rho$ onto

$$\begin{aligned} \{s\otimes v\in\mathfrak{A}^{0,q}(M,E)\otimes V_\rho | g^*s\otimes v &= s\otimes\rho(g)v \text{ for all } g\in\Gamma\} \\ &= \mathfrak{A}^{0,q}(M/\Gamma, E/\Gamma\otimes E_\rho) \ . \end{aligned}$$

As g is a holomorphic isometry, this operator commutes with the Laplace operator. Hence it induces an isometry

$$P_\rho : H^{0,*}(M/\Gamma, E/\Gamma\otimes E_\rho) \to \operatorname{Hom}_\Gamma(H^{0,*}(M,E), V_\rho) \ ,$$

which induces an isometry of equivariant determinants

$$\det{}_\Gamma(H^{0,*}(M,E), |\cdot|^2_{L^2}) \cong \bigoplus_{\rho\in\Gamma^\vee} \det(H^{0,*}(M/\Gamma, E/\Gamma\otimes E_\rho)\otimes V_\rho, |\cdot|^2_{L^2}) \ .$$

Furthermore, when $P^\perp$ denotes the projection on the orthogonal complement of $\ker\Box$, for any q

$$\begin{aligned} \operatorname{Tr_s}\Box^{-s}P^\perp_{|\mathfrak{A}^{0,q}(M/\Gamma,E/\Gamma\otimes E_\rho)} &= \operatorname{Tr_s} P_\rho\Box^{-s}P^\perp_{|\mathfrak{A}^{0,q}(M,E)\otimes V_\rho} \\ &= \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\operatorname{Tr}\overline{\rho(g)}\operatorname{Tr_s} g^*\Box^{-s}P^\perp_{|\mathfrak{A}^{0,q}(M,E)\otimes V_\rho} \ . \end{aligned}$$

Thus the analytic torsion on M/Γ and the equivariant torsion are Fourier transforms of each other. More precisely,

$$T(M/\Gamma, \overline{E}/\Gamma\otimes\overline{E}_\rho) = \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\overline{\chi_\rho(g)}T_g(M,\overline{E})$$

and, equivalently,

$$T_g(M,\overline{E}) = \sum_{\rho\in\Gamma^\vee}\chi_\rho(g)T(M/\Gamma,\overline{E}/\Gamma\otimes\overline{E}_\rho) \qquad \forall g\in\Gamma \ .$$

Hence the above isometry of the determinants holds for the Quillen metrics, too. **Q.E.D**.

In particular, for $s \in \lambda_\Gamma(M,E)$ one finds the equations

$$\log \|s\|^2_{\lambda_\Gamma(M,\overline{E})}(g) = \sum_{\rho\in\Gamma^\vee} \chi_\rho(g) \log \|s\|^2_{\lambda(M/\Gamma,\overline{E}/\Gamma\otimes\overline{E}_\rho)} \qquad \forall g \in \Gamma$$

and

$$\log \|s\|^2_{\lambda(M/\Gamma,\overline{E}/\Gamma\otimes\overline{E}_\rho)} = \frac{1}{\#\Gamma} \sum_{g\in\Gamma} \overline{\chi_\rho(g)} \log \|s\|^2_{\lambda_\Gamma(M,\overline{E})}(g) \qquad \forall \rho \in \Gamma^\vee \ .$$

3.3 Equivariant secondary characteristic classes

Let $\mathfrak{A}^{p,q}(M) := \Gamma^\infty(M, \Lambda^p T^{1,0*}M \wedge \Lambda^q T^{0,1*}M)$ denote the space of (p,q)-forms and define

$$\widetilde{\mathfrak{A}}(M) := \bigoplus_{p=0}^{\dim M} \mathfrak{A}^{p,p}(M) / \left(\mathrm{im}\partial_{|\mathfrak{A}^{p-1,p}(M)} + \mathrm{im}\overline{\partial}_{|\mathfrak{A}^{p,p-1}(M)}\right) \ .$$

Let $\overline{E}$ be a hermitian holomorphic g-equivariant vector bundle on M. The hermitian vector bundle $\overline{E}$ splits on the fixed point set into a direct sum $\bigoplus_{\zeta\in S^1} \overline{E}_\zeta$, where the equivariant structure g^E of E acts on $\overline{E}_\zeta$ as ζ. We shall denote the g-invariant hermitian subbundle by $\overline{E}_g$ and its orthogonal complement by $\overline{E}_\perp$. Denote the rang of $\overline{E}_\zeta$ by r_ζ and the associated curvature form by $\Omega^{\overline{E}_\zeta} \in \mathfrak{A}^{1,1}(M_g)$. Consider a family $(\phi_\zeta)_{\zeta\in S^1}$ of ad $\mathbf{GL}(\mathbf{C})$-invariant formal power series

$$\phi_\zeta \in \mathbf{C}[[\mathfrak{gl}_{r_\zeta}(\mathbf{C})]] \qquad (\zeta \in S^1)$$

(i.e. $\phi_\zeta(hAh^{-1}) = \phi_\zeta(A)$ for any $h \in \mathbf{GL}_{r_\zeta}(\mathbf{C})$, $A \in \mathfrak{gl}_{r_\zeta}(\mathbf{C})$). For such a family $(\phi_\zeta)_{\zeta\in S^1}$ and every formal power series $f : \mathbf{C}[[\bigoplus_{\zeta\in S^1} \mathbf{C}]]$ we define

$$\phi_g(\overline{E}) := f\left((\phi_\zeta(-\frac{\Omega^{\overline{E}_\zeta}}{2\pi i}))_{\zeta\in S^1} \right)$$

as the Chern-Weil form associated to $(\phi_\zeta)_\zeta$ and f. Its class in $\widetilde{\mathfrak{A}}(M_g)$ is independent of the metric.

Theorem 3.4 *There is a unique way to attach to every short exact sequence $\mathcal{E} : 0 \to E' \to E \to E'' \to 0$ of holomorphic equivariant vector bundles equipped with arbitrary invariant metrics a class $\widetilde{\phi}_g(\overline{\mathcal{E}}) \in \widetilde{\mathfrak{A}}(M_g)$ such that*

1. *$\widetilde{\phi}_g(\overline{\mathcal{E}})$ provides the transgression*

$$\frac{\overline{\partial}\partial}{2\pi i} \widetilde{\phi}_g(\overline{\mathcal{E}}) = \phi_g(\overline{E}' \oplus \overline{E}'') - \phi_g(\overline{E}) \ ,$$

2. *for every holomorphic equivariant map $\xi : M' \to M$,*

$$\widetilde{\phi}_g(\xi^*\overline{\mathcal{E}}) = \xi^*\widetilde{\phi}_g(\overline{\mathcal{E}}) \ ,$$

3. *$\widetilde{\phi}_g(\overline{\mathcal{E}}) = 0$ if $\overline{\mathcal{E}}$ splits metrically.*

Proof: The exact sequence $\overline{\mathcal{E}}$ splits on X_g orthogonally into direct sequences

$$\mathcal{E}_\zeta : 0 \to E'_\zeta \to E_\zeta \to E''_\zeta \to 0$$

for all $\zeta \in S^1$. Using the non-equivariant Bott-Chern classes on X_g we define for $\zeta, \eta \in S^1$

$$\widetilde{(\phi_\zeta + \phi_\eta)}(\overline{\mathcal{E}}_\zeta, \overline{\mathcal{E}}_\eta) := \widetilde{\phi_\zeta}(\overline{\mathcal{E}}_\zeta) + \widetilde{\phi_\eta}(\overline{\mathcal{E}}_\eta)$$

and

$$\widetilde{(\phi_\zeta \phi_\eta)}(\overline{\mathcal{E}}_\zeta, \overline{\mathcal{E}}_\eta) := \widetilde{\phi_\zeta}(\overline{\mathcal{E}}_\zeta)\phi_\eta(\overline{E}_\eta) + \phi_\zeta(\overline{E}'_\zeta + \overline{E}''_\zeta)\widetilde{\phi_\eta}(\overline{\mathcal{E}}_\eta)$$

and similarly for arbitrary finite sums and products. Thus, we define secondary classes for a formal power series in the ϕ_ζ, evaluated at a formally infinite sum of sequences $(\overline{\mathcal{E}}_\zeta)_{\zeta \in S^1}$. We set $\widetilde{\phi}_g(\overline{\mathcal{E}}) := f(\widetilde{(\phi_\zeta)_{\zeta \in S^1}})((\overline{\mathcal{E}}_\zeta)_{\zeta \in S^1})$. Then the axiomatic characterization follows by the non-equivariant one [BGS1, Th. 1.29]. **Q.E.D.**

Remark. For longer exact sequences $\mathcal{E} : 0 \to E_0 \to E_1 \to \cdots \to E_m \to 0$ corresponding secondary classes $\widetilde{\phi}_g(\overline{\mathcal{E}})$ are constructed by splitting $\overline{\mathcal{E}}$ into direct sums of short exact sequences as in [BGS1, Section f]. The sign is chosen such that for an additive characteristic class ϕ_g

$$\frac{\overline{\partial}\partial}{2\pi i}\widetilde{\phi}_g(\overline{\mathcal{E}}) = \sum_{j=0}^{m}(-1)^j \phi_g(\overline{E}_j) \ .$$

The secondary class associated to the sequence $0 \to E \to E \to 0 \to 0$ is denoted by $\widetilde{\phi}_g(E, h^E, h^{E'})$, when the first E is equipped with a metric h^E and the second one with $h^{E'}$. Let Td and ch denote the formal power series given by Taylor expansions of $\det(\frac{A}{1-e^{-A}})$ and $\mathrm{Tr}\, e^A$ for matrices A. We define the Chern character form as

$$\begin{aligned} \mathrm{ch}_g(\overline{E}) &:= \sum_\zeta \zeta \mathrm{ch}(\overline{E}_\zeta) \\ &= \mathrm{Tr}\, g^E + \sum_\zeta \zeta c_1(\overline{E}_\zeta) + \sum_\zeta \zeta \left(\frac{1}{2}c_1^2(\overline{E}_\zeta) - c_2(\overline{E}_\zeta)\right) + \dots \ . \end{aligned}$$

Thus, $\widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}}) = \sum_\zeta \zeta \widetilde{\mathrm{ch}}(\overline{\mathcal{E}}_\zeta)$. As in [B3], we define the Todd form of an equivariant vector bundle as

$$\mathrm{Td}_g(\overline{E}) := \frac{c_{\mathrm{rk}\, E_g}(\overline{E}_g)}{\mathrm{ch}_g(\sum_{j=0}^{\mathrm{rk}\, E}(-1)^j \Lambda^j \overline{E}^*)} \ .$$

As in [Hi, Th. 10.1.1] one obtains

$$\mathrm{Td}_g(\overline{E}) \quad = \quad \mathrm{Td}(\overline{E}_g) \prod_{\zeta\neq 1} \det\Big(\frac{1}{1-\zeta^{-1}e^{\frac{\Omega^{\overline{E}_\zeta}}{2\pi i}}}\Big) \ .$$

Using the Taylor expansions in x at $x=0$

$$\frac{1}{1-\zeta^{-1}e^{-x}} = \frac{1}{1-\zeta^{-1}}\left(1-\frac{x}{\zeta-1}+\frac{x^2(\zeta+1)}{2(\zeta-1)^2}+O(x^3)\right)$$

for $\zeta\neq 1$ and $\frac{x}{1-e^{-x}} = 1+x/2+x^2/12+O(x^3)$, we find

$$\begin{aligned}\mathrm{Td}_g(\overline{E}) \quad = \quad & \frac{1}{\det(1-(g^{E_\perp})^{-1})}\Bigg[1-\sum_{\zeta\neq 1}\frac{c_1(\overline{E}_\zeta)}{\zeta-1}+\frac{1}{2}c_1(\overline{E}_g) \\ & -\sum_{\zeta\neq 1}\frac{\zeta c_2(\overline{E}_\zeta)}{(\zeta-1)^2}+\frac{1}{2}\sum_{\zeta\neq 1}\frac{c_1^2(\overline{E}_\zeta)}{\zeta-1}+\frac{1}{12}\left(c_1^2(\overline{E}_g)+c_2(\overline{E}_g)\right) \\ & +\left(\sum_{\zeta\neq 1}\frac{c_1(\overline{E}_\zeta)}{\zeta-1}-\frac{1}{2}c_1(\overline{E}_g)\right)\left(\sum_{\zeta\neq 1}\frac{c_1(\overline{E}_\zeta)}{\zeta-1}\right)+\dots\Bigg] \qquad (5)\end{aligned}$$

where $g^{E_\perp}$ denotes the non-trivial part of the action on $E_{|M_g}$. If $g^{\mathcal{E}}$ has the eigenvalues $\zeta_1,\dots,\zeta_m$, then

$$\widetilde{\mathrm{Td}}_g(\overline{\mathcal{E}}) = \sum_{i=1}^{m}\prod_{j=1}^{i-1}\mathrm{Td}_g(\overline{E}_{\zeta_j})\cdot\widetilde{\mathrm{Td}}_g(\overline{\mathcal{E}}_{\zeta_i})\cdot\prod_{j=i+1}^{m}\mathrm{Td}_g(\overline{E}'_{\zeta_j}\oplus\overline{E}''_{\zeta_j}) \ . \qquad (6)$$

Also, we define $((\mathrm{Td}_g)^{-1})'(\overline{E}) := \frac{\partial}{\partial b}_{|b=0}\left(\mathrm{Td}_g(b\,\mathrm{Id}-\frac{\Omega^{\overline{E}}}{2\pi i})^{-1}\right)$. For $\zeta\in S^1$ and $s>1$ consider the zeta function

$$L(\zeta,s) = \sum_{k=1}^{\infty}\frac{\zeta^k}{k^s}$$

and its meromorphic continuation to $s\in\mathbf{C}$. The function L is related to the classical Lerch zeta function Φ [WW, ch. XIII, p. 280] via $L(\zeta,s)=\zeta\Phi(\zeta,s,1)$. Define the formal power series in x

$$\widetilde{R}(\zeta,x) := \sum_{n=0}^{\infty}\left(\frac{\partial L}{\partial s}(\zeta,-n)+L(\zeta,-n)\sum_{j=1}^{n}\frac{1}{2j}\right)\frac{x^n}{n!}$$

Definition 3.5 *The Bismut equivariant R-class of an equivariant holomorphic hermitian vector bundle $\overline{E}$ with $\overline{E}_{|X_g}=\sum_\zeta\overline{E}_\zeta$ is defined as*

$$R_g(\overline{E}) := \sum_{\zeta\in S^1}\left(\mathrm{Tr}\,\widetilde{R}(\zeta,-\frac{\Omega^{\overline{E}_\zeta}}{2\pi i})-\mathrm{Tr}\,\widetilde{R}(1/\zeta,\frac{\Omega^{\overline{E}_\zeta}}{2\pi i})\right) \ .$$

Assume now that M is Kähler. Then there are two anomaly formulas satisfied by the Quillen metric:

Theorem 3.6 ([B3, Th. 2.5]) *Let h^{TM}, $h^{TM'}$ denote two equivariant Kähler metrics on M with associated Quillen metrics $\|\cdot\|_Q$, $\|\cdot\|'_Q$ on $\lambda_g(M,E)$. Then*

$$\widetilde{\mathrm{ch}}_g(\lambda_g(M,E),\|\cdot\|_Q^2,\|\cdot\|_Q'^2) = -\int_{M_g} \widetilde{\mathrm{Td}}_g(TM,h^{TM},h^{TM'})\mathrm{ch}_g(\overline{E}) \ .$$

The following formula is equivalent to [B3, Th. 0.1] when applied to the immersion of either the empty space or the full manifold itself:

Theorem 3.7 *Let $\mathcal{E}: 0 \to E' \to E \to E'' \to 0$ be a short exact sequence equipped with metrics as in Th. 3.4. The associated sequence of determinant lines*

$$\lambda_g(M,\mathcal{E}) : 0 \to \lambda_g(M,E) \to \lambda_g(M,E') \otimes \lambda_g(M,E'') \to 0 \to 0$$

equipped with Quillen metrics satisfies

$$\widetilde{\mathrm{ch}}_g(\lambda_g(M,\overline{\mathcal{E}})) = -\int_{M_g} \mathrm{Td}_g(\overline{TM})\widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}}) \ .$$

Remark. Let $\mathcal{E}: 0 \to E' \to E \to E'' \to 0$ denote a short exact sequence as above and let $\mathcal{H}: 0 \to H^0(M,E') \to H^0(M,E) \to H^0(M,E'') \to H^1(M,E') \to \cdots$ denote the corresponding long exact sequence in cohomology, equipped with the L^2-metric. Then

$$\begin{aligned} & \widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}}) + T_g(M,\overline{E}) - T_g(M,\overline{E}') - T_g(M,\overline{E}'') \\ = \ & \sum_\zeta \zeta\, \widetilde{c}_1(\overline{\mathcal{H}}_\zeta) + T_g(M,\overline{E}) - T_g(M,\overline{E}') - T_g(M,\overline{E}'') \qquad (7) \\ = \ & -\widetilde{\mathrm{ch}}_g(\lambda_g(M,\overline{\mathcal{E}})) \ . \end{aligned}$$

3.4 Equivariant Bott-Chern singular currents

In this subsection we repeat the definition of an equivariant Bott-Chern singular current given by Bismut in [B3, sect. VI] and we prove some properties of these currents. This construction generalizes the definition of the secondary Chern character $\widetilde{\mathrm{ch}}$ to certain coherent sheaves.

Let $i:(Y,h^{TY}) \hookrightarrow (X,h^{TX})$ be an equivariant isometric embedding of compact Hermitian g-manifolds with normal bundle $\overline{N}_{X/Y}$, the Hermitian metric on

$\overline{N}_{X/Y}$ being the quotient metric with respect to h^{TX} and h^{TY}. Let $\overline{\eta}$ be an equivariant holomorphic hermitian vector bundle on Y and let

$$(\xi, v) : 0 \to \xi_m \to \ldots \to \xi_0 \to 0$$

be a chain complex of equivariant holomorphic vector bundles on X which provides a resolution of the sheaf $i_* \mathcal{O}_Y(\eta)$ on X. Equip $\xi_\bullet$ with an hermitian metric. Let N_H denote the number operator acting on $\Lambda T^*X \otimes \xi_j$ by multiplication with j. Let F^k be the pullback of the cohomology vector bundle $H^k(\xi, v)$ over Y_g to N_{X_g/Y_g}. For $z \in N_{X_g/Y_g}$ let $\partial_z v$ denote the derivative of the chain map in a given local holomorphic trivialization of (ξ, v). As is shown in [B1, 1c], this map is independent of the choice of the trivialization and $(F^*, \partial_z v)$ forms a complex, which is isomorphic to the Koszul complex $(\Lambda^* N^\vee_{X/Y} \otimes \eta, \iota_z)$. Consider for an arbitrary equivariant metric on F^* the superconnection $B := \nabla^F + \partial_z v + (\partial_z v)^*$ on F^*. According to [B1, Prop. 3.1], the forms $\mathrm{Tr_s}\, g^* e^{-B^2}$ and $\mathrm{Tr_s}\, N_H g^* e^{-B^2}$ decay faster than $e^{-C|z|^2}$ for some $C > 0$ and $|z| \to \infty$, where the supertrace is taken with respect to the grading $N + N_H$. Let Φ denote the homomorphism of differential forms of even degree on N_{X_g/Y_g} mapping a form α of degree $2p$ to $(2\pi i)^{-p}\alpha$. We define

$$\theta_g(\overline{F}) := \int_{N_{X_g/Y_g}} \Phi \mathrm{Tr_s}\, g^* e^{-B^2} \text{ and } \theta'_g(\overline{F}) := \int_{N_{X_g/Y_g}} \Phi \mathrm{Tr_s}\, N_H g^* e^{-B^2} .$$

Bismut's **assumption (A)** is said to be satisfied if the isomorphism $F^* \cong \Lambda^* N^\vee_{X/Y} \otimes \eta$ is an isometry. Under this condition

$$\theta_g(\overline{F}) = \frac{\mathrm{ch}_g(\overline{\eta})}{\mathrm{Td}_g(\overline{N}_{X/Y})} \text{ and } \theta'_g(\overline{F}) = -(\mathrm{Td}_g^{-1})'(\overline{N}_{X/Y})\mathrm{ch}_g(\overline{\eta})$$

([B3, eq. (6.25),eq. (6.26)]). As is shown in [B1, Proposition 1.6], for any choice of smooth Hermitian metrics on $N_{X/Y}$ and η there exist metrics on $\xi_\bullet$ such that condition (A) is verified.

Let ∇^ξ be the hermitian holomorphic connection on $\overline{\xi}_\bullet$, let v^* be the adjoint of v and set $C_u := \nabla^\xi + \sqrt{u}(v + v^*)$ for $u \geq 0$. Now choose the metric on F^* to be the metric induced by the isomorphism $F^k \cong \ker(v + v^*)^2 \subseteq \xi_k$. Let δ_{Y_g} denote the current of integration on the orientable manifold Y_g. Then for $s \in \mathbf{C}$, $0 < \mathrm{Re}\, s < \frac{1}{2}$, the current-valued zeta function

$$Z_g(\overline{\xi})(s) := \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \left(\Phi \mathrm{Tr_s}\, g^* N_H e^{-C_u^2} - \theta'_g(\overline{F})\delta_{Y_g} \right) du$$

is well-defined on X_g and it has a meromorphic continuation to the complex plane which is holomorphic at $s = 0$ ([B3, eq. (6.22),sect. VI.d]).

Definition 3.8 *The equivariant singular current on* X_g *associated to* $\overline{\xi}$ *is defined as*

$$T_g(\overline{\xi}) := \frac{\partial}{\partial s}_{|s=0} Z_g(\overline{\xi})(s) \ .$$

Notice that the notation for the analytic torsion and the notation for the singular current are similar. We systematically include the manifold in the former notation to keep them different.

Theorem 3.9 *[B3, Th. 6.7] The current* $T_g(\overline{\xi})$ *is a sum of* (p,p)*-currents and it satisfies the transgression formula*

$$\frac{\bar{\partial}\partial}{2\pi i} T_g(\overline{\xi}) = \theta_g(\overline{F})\delta_{Y_g} - \mathrm{ch}_g(\overline{\xi}) \ .$$

Proof: This is shown in [B3, Th. 6.7]. The Kähler condition posed in [B3, III.d] is not necessary for this result similar to [BGS5]. It is formulated there under the assumption (A), but this assumption is not needed in the proof. **Q.E.D**.

The axiomatic characterization of Bott-Chern classes implies

Corollary 3.10 *If* $Y = \emptyset$ *then* $T_g(\overline{\xi}) = -\widetilde{\mathrm{ch}}_g(\overline{\xi})$.

Let $\widetilde{\mathrm{Td}}_g(\overline{TY}, \overline{TX}_{|Y})$ denote the Bott-Chern class which verifies the transgression formula

$$\frac{\overline{\partial}\partial}{2\pi i} \widetilde{\mathrm{Td}}_g(\overline{TY}, \overline{TX}_{|Y}) = \mathrm{Td}_g(\overline{TX}_{|Y}) - \mathrm{Td}_g(\overline{TY})\mathrm{Td}_g(\overline{N}_{X/Y})$$

associated to the short exact sequence

$$0 \to TY \to TX_{|Y} \to N_{X/Y} \to 0$$

with the induced metrics. This somehow unintuitive sign choice is due to a sign incompatibility between [B3] and [BGS1]. One of the most important tools in this article is the main result of [B3]:

Theorem 3.11 ([B3, Th. 0.1]) *Assume that the metrics on* X *and* Y *are Kähler and that the compatibility assumption (A) is verified. The sequence of equivariant determinant lines*

$$\lambda_g(\xi, \eta) : 0 \to \lambda_g(X, \xi) \to \lambda_g(Y, \eta) \to 0 \to 0$$

equipped with Quillen metrics satisfies

$$\begin{aligned} \widetilde{\mathrm{ch}}_g(\lambda_g(\overline{\xi}, \overline{\eta})) \quad &= \quad \int_{X_g} \mathrm{Td}_g(\overline{TX}) T_g(\overline{\xi}) - \int_{Y_g} \widetilde{\mathrm{Td}}_g(\overline{TY}, \overline{TX}_{|Y}) \frac{\mathrm{ch}_g(\overline{\eta})}{\mathrm{Td}_g(\overline{N}_{X/Y})} \\ &+ \int_{X_g} \mathrm{Td}_g(TX) R_g(TX) \mathrm{ch}_g(\xi) - \int_{Y_g} \mathrm{Td}_g(TY) R_g(TY) \mathrm{ch}_g(\eta) \ . \end{aligned}$$

To multiply the singular current with other currents we need to know its wave front set.

Theorem 3.12 *The wave front set* $\mathrm{WF}(T_g(\overline{\xi}))$ *of* $T_g(\overline{\xi})$ *is contained in* $N^*_{X_g/Y_g,\mathbf{R}}$.

Proof: As suggested in [B3, Remark 6.8], the proof proceeds as in [BGS4, Th. 2.5]. Set $\mathrm{ch}'_g(\overline{\xi}) := \sum_{j=0}^m (-1)^j j \mathrm{ch}_g(\overline{\xi}_j)$. In the explicit formula

$$\begin{aligned} T_g(\overline{\xi}) \quad &= \quad \int_0^1 \Phi \mathrm{Tr_s}\, g^* N_H \left(e^{-C_u^2} - e^{-C_0^2}\right) \frac{du}{u} \\ &+ \int_1^\infty \left(\Phi \mathrm{Tr_s}\, g^* N_H e^{-C_u^2} - \theta'_g(\overline{F})\delta_{Y_g}\right) \frac{du}{u} \\ &- \Gamma'(1) \left(\mathrm{ch}'_g(\overline{\xi}) - \theta'_g(\overline{F})\delta_{Y_g}\right) \ , \end{aligned}$$

the first summand is globally defined and smooth on X_g. As the last summand is smooth on the submanifold Y_g, its wave front set equals $N^*_{X_g/Y_g,\mathbf{R}}$ (see [Hö, Ex. 8.2.5]). Thus we are left with the middle term

$$\rho_\xi := \int_1^\infty \left(\Phi \mathrm{Tr_s}\, g^* N_h e^{-C_u^2} - \theta'_g(\overline{F})\delta_{Y_g}\right) \frac{du}{u} \ .$$

As $\Phi \mathrm{Tr_s}\, g N_h e^{-C_u^2}$ has exponential decay as u tends to infinity, this current is smooth on $X_g \backslash Y_g$. Consider U, Γ, ϕ, m and the associated seminorm $p_{U,\Gamma,\phi,m}(\rho_\xi)$ as in [B1, III.c]. With just the same proof as of [B1, Th. 3.2] one verifies that

$$p_{U,\Gamma,\phi,m}(\rho_\xi) \leq C \int_1^\infty \frac{du}{u^{3/2}} < \infty \ ,$$

thus $\mathrm{WF}(\rho_\xi) \subset N^*_{Y_g/X_g,\mathbf{R}}$ according to [Hö, Lemma 8.2.1]. **Q.E.D.**

Let $\tilde{X}$ be a compact connected complex manifold and consider an equivariant holomorphic map $f : \tilde{X} \to X$, which is transversal to Y in the sense of [BGS4, Def. 2.6]. As in [BGS4, Th. 2.7] $(f^*\xi, f^*v)$ provides an equivariant projective resolution of $f^*\eta$, and $T_g(f^*\overline{\xi}) = f^* T_g(\overline{\xi})$. The proof proceeds as in [BGS4, Th. 2.7] by approximating $T_g(\overline{\xi})$ with

$$\begin{aligned} T_g^a(\overline{\xi}) \quad &:= \quad \int_0^1 \Phi \mathrm{Tr_s}\, g^* N_H \left(e^{-C_u^2} - e^{-C_0^2}\right) \frac{du}{u} \\ &+ \int_1^a \left(\Phi \mathrm{Tr_s}\, g^* N_H e^{-C_u^2} - \theta'_g(\overline{F})\delta_{Y_g}\right) \frac{du}{u} \\ &- \Gamma'(1) \left(\mathrm{ch}'_g(\overline{\xi}) - \theta'_g(\overline{F})\delta_{Y_g}\right) \end{aligned}$$

for $1 \leq a < \infty$ and pulling this sum of smooth forms back.

For a smooth curve $\mathbf{R} \ni l \mapsto h_l^F = h_l^{F_g} \oplus h_l^{F_\perp}$ into the space of metrics on F define

$$\chi_g(F, h_0^F, h_1^F) := -\int_0^1 dl \int_{N_{X_g/Y_g}} \Phi \mathrm{Tr}_s\,(h_l^F)^{-1} \frac{dh_l^F}{dl} g^* e^{-B_{|F}^2} \ .$$

Lemma 3.13 *The class of $\chi_g(F, h_0^F, h_1^F)$ in $\widetilde{\mathfrak{A}}(Y_g)$ depends only on h_0 and h_1. It verifies the transgression formula*

$$\frac{\bar\partial\partial}{2\pi i} \chi_g(F, h_0^F, h_1^F) = \theta_g(F, h_1^F) - \theta_g(F, h_0^F) \ .$$

Proof: This follows by decomposing into the various subcomplexes F_ζ for $\zeta \in S^1$ and applying the first paragraph of the proof of [BGS5, Th. 2.4] to each summand. **Q.E.D.**

Let $\mathcal{D}'_{N_{X_g/Y_g}}$ denote the space of currents γ on X_g such that $WF(\gamma) \subseteq N^*_{X_g/Y_g,\mathbf{R}}$ and let P_Y^X denote the vector space generated by currents $\gamma \in \mathcal{D}'_{N_{X_g/Y_g}}$ of type (p,p) divided by its intersection with $\partial\mathcal{D}'_{N_{X_g/Y_g}} + \overline{\partial}\mathcal{D}'_{N_{X_g/Y_g}}$. We shall establish an equivariant analogue of [BGS5, 2.c]. Assume given an equivariant double complex of holomorphic vector bundles

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & 0 & & \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
0 & \to & \xi_m^0 & \xrightarrow{v} & \dots & \xrightarrow{v} & \xi_0^0 & \xrightarrow{r} & \eta^0 & \to & 0 \\
 & & \uparrow w & & & & \uparrow w & & \uparrow w & & \\
 & & \vdots & & & & \vdots & & \vdots & & \\
 & & \uparrow w & & & & \uparrow w & & \uparrow w & & \\
0 & \to & \xi_m^p & \xrightarrow{v} & \dots & \xrightarrow{v} & \xi_0^p & \xrightarrow{r} & \eta^p & \to & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
 & & 0 & & & & 0 & & 0 & &
\end{array}$$

on X (resp. Y), where r denotes the restriction map. Assume that the horizontal complexes (ξ^j, v) are resolutions of the η^j. The vertical complexes shall be acyclic. Let (ξ_i^j) and (η^j) be equipped with hermitian metrics $(h^{\xi_i^j})$ and (h^{η^j}) such that assumption (A) is satisfied by each line. Let $T_g(\overline{\xi}^j)$ denote the equivariant singular current associated to the resolution of η^j by each line.

Theorem 3.14 *In P_Y^X, the alternating sum of the $T_g(h^{\xi^j})$ is given by*

$$\sum_{j=0}^p (-1)^j T_g(\overline{\xi}^j) = i_* \frac{\widetilde{\mathrm{ch}}_g(\overline{\eta})}{\mathrm{Td}_g(\overline{N}_{X/Y})} - \sum_{i=0}^m (-1)^i \widetilde{\mathrm{ch}}_g(\overline{\xi}_i) \ .$$

Proof: As in [BGS1, Proof of Th. 1.29], one constructs an equivariant double complex $(\widetilde{\xi}, \widetilde{\eta})$ of hermitian holomorphic vector bundles on $X \times \mathbf{P}^1$ (resp. $Y \times \mathbf{P}^1$) such that its restriction to $X \times \{0\}$ (resp. $Y \times \{0\}$) equals the original complex

$$\widetilde{\xi}_{|X\times\{0\}} = \xi,\ \widetilde{\eta}_{|Y\times\{0\}} = \eta$$

and its restriction to $X \times \{\infty\}$ (resp. $Y \times \{\infty\}$) splits orthogonally and holomorphically in the vertical direction, i.e. there are vector bundles $(\overline{\eta'}^j)_{0\leq j\leq p}$ such that $(\widetilde{\eta}, \widetilde{w})_{|Y\times\{\infty\}}$ is isometric to the complex

$$0 \to {\eta'}^0 \to {\eta'}^0 \oplus {\eta'}^1 \to \ldots \to {\eta'}^{p-2} \oplus {\eta'}^{p-1} \to {\eta'}^{p-1} \to 0$$

and similarly for $(\widetilde{\xi}_i, \widetilde{w})_{|X\times\{\infty\}}$ for each $0 \leq i \leq m$. Let F^j_* denote the pullback of $H^*(\widetilde{\xi}^j, v)$ to $N_{X_g/Y_g} \times \mathbf{P}^1$. Let z denote the canonical coordinate on $\mathbf{P}^1$. As assumption (A) can only be guaranteed at $z = 0$, there are two natural metrics on F: One metric h^F_0 induced by the imbedding in $\overline{\widetilde{\xi}}$ and another metric h^F_1 induced via the isomorphism $F^j_* \cong \Lambda^* \overline{N^\vee_{X/Y}} \otimes \overline{\widetilde{\eta}}^j$.

As in [BGS5, p. 266], [Hö, Th. 8.2.10] shows that the wave front sets of the currents $T(\overline{\widetilde{\xi}^j})$ and $\log|z|^2$ do not intersect. Thus, their product is a well-defined current. By Th. 3.9 one finds

$$\begin{aligned}
& \frac{\bar\partial}{2\pi i}\left(\partial \log|z|^2 T_g(\overline{\widetilde{\xi}}^j)\right) + \frac{\partial}{2\pi i}\left(\log|z|^2 \bar\partial T_g(\overline{\widetilde{\xi}}^j)\right) \\
= {} & \left(\frac{\bar\partial\partial}{2\pi i}\log|z|^2\right) T_g(\overline{\widetilde{\xi}}^j) - \log|z|^2 \frac{\bar\partial\partial}{2\pi i} T_g(\overline{\widetilde{\xi}}^j) \qquad (8)\\
= {} & (\delta_{X\times\{0\}} - \delta_{X\times\{\infty\}}) T_g(\overline{\widetilde{\xi}}^j) + \log|z|^2 \cdot \left(\mathrm{ch}_g(\overline{\widetilde{\xi}}^j) - \theta_g(F^j, h^F_0)\delta_{Y_g\times\mathbf{P}^1}\right) .
\end{aligned}$$

The same way, by Lemma 3.13 one obtains

$$\begin{aligned}
& \frac{\bar\partial}{2\pi i}\left(\partial \log|z|^2 \chi_g(F^j, h^F_0, h^F_1)\right) + \frac{\partial}{2\pi i}\left(\log|z|^2 \bar\partial \chi_g(F^j, h^F_0, h^F_1)\right) \qquad (9)\\
= {} & (\delta_{X\times\{0\}} - \delta_{X\times\{\infty\}}) \chi_g(F^j, h^F_0, h^F_1) - \log|z|^2 \left(\theta_g(F^j, h^F_1) - \theta_g(F^j, h^F_0)\right) .
\end{aligned}$$

Integrating the sum of (8),(9) over $\mathbf{P}^1$ thus yields

$$\begin{aligned}
T_g(\overline{\widetilde{\xi}}^j) \quad + \quad & \chi_g(F^j_{|z=0}, h^F_0, h^F_1)\delta_{Y_g} = \int_{\mathbf{P}^1} \log|z|^2 \theta_g(F^j, h^F_1) \cdot \delta_{Y_g\times\mathbf{P}^1} \qquad (10)\\
& + T_g(\overline{\widetilde{\xi}}^j_{|z=\infty}) + \chi_g(F^j_{|z=\infty}, h^F_0, h^F_1)\delta_{Y_g} - \int_{\mathbf{P}^1} \log|z|^2 \mathrm{ch}_g(\overline{\widetilde{\xi}}^j) .
\end{aligned}$$

in $\widetilde{\mathfrak{A}}(Y_g)$. As $h^F_0 = h^F_1$ at $z = 0$ by assumption (A), we find $\chi_g(F^j_{|z=0}, h^F_0, h^F_1) = 0$. Furthermore, when taking the alternating sum over j we find by the splitting of $\widetilde{\xi}^j_{|X\times\{\infty\}}$ that

$$\begin{aligned}\sum_{j=0}^{p}(-1)^j T_g(\overline{\widetilde{\xi}}^j_{|z=\infty}) &= T_g(\overline{\xi'}^0)+\sum_{j=1}^{p-1}(-1)^j\left(T_g(\overline{\xi'}^{j-1})+T_g(\overline{\xi'}^j)\right)+(-1)^pT_g(\overline{\xi'}^{p-1})\\ &= 0\,.\end{aligned}$$

At $z=\infty$, (F^j,h_1^F) is isometric to $\Lambda\overline{N}_{X/Y}\otimes(\overline{\eta'}^j\oplus\overline{\eta'}^{j+1})$. Thus at $z=\infty$ both the complexes (F,h_1^F) and (F,h_0^F) split holomorphically and orthogonally in the vertical direction. Taking a linear interpolation $(h_l^F)_{0\le l\le 1}$ of h_0^F and h_1^F, we get

$$\sum_{j=0}^{p}(-1)^j\chi_g(\overline{F}^j_{|z=\infty},h_0^F,h_1^F)=0$$

as above. The alternating sum of (10) thus equals

$$\begin{aligned}\sum_{j=0}^{p}(-1)^jT_g(\xi^j,h^{\xi^j}) &\equiv \mathrm{Td}_g^{-1}(\overline{N}_{X/Y})\int_{\mathbf{P}^1}\log|z|^2\sum_{j=0}^{p}(-1)^j\mathrm{ch}_g(\widetilde{\eta}^j,h^{\widetilde{\eta}^j})\cdot\delta_{Y_g}\\ &\quad-\int_{\mathbf{P}^1}\log|z|^2\sum_{j=0}^{p}(-1)^j\mathrm{ch}_g(\widetilde{\xi}^j,h^{\widetilde{\xi}^j})\end{aligned}$$

which gives the desired result by the construction of Bott-Chern classes in [BGS1]. **Q.E.D.**

The equivariant and non-equivariant singular current are related by the following lemma:

Lemma 3.15 *Assume that there is an r-dimensional equivariant Hermitian holomorphic vector bundle $\overline{Q}$ over X with a g-invariant section σ which is transversal to X. If Y is the zero set of σ then there is a global Koszul resolution*

$$0\to\Lambda^rQ^\vee\stackrel{\iota_\sigma}{\to}\cdots\stackrel{\iota_\sigma}{\to}Q^\vee\stackrel{\iota_\sigma}{\to}\mathcal{O}_X\to i_*\mathcal{O}_Y\to 0$$

of the structure sheaf on Y. Assume furthermore that condition (A) holds, i.e. that $\overline{Q}_{|Y}$ is equivariant isometric to $\overline{N}_{X/Y}$. Let $T(\overline{\Lambda^\bullet Q_g^\vee}):=T_{\mathrm{id}}(\overline{\Lambda^\bullet Q_g^\vee})$ denote the non-equivariant singular current of the fixed part of the complex on X_g. Then the singular currents of $\overline{\Lambda^\bullet Q^\vee}$ and $\overline{\Lambda^\bullet Q_g^\vee}$ are related by the equation

$$T_g(\overline{\Lambda^\bullet Q^\vee})\mathrm{Td}_g(\overline{Q})=T(\overline{\Lambda^\bullet Q_g^\vee})\mathrm{Td}(\overline{Q_g})$$

in P_Y^X.

Note that $T(\overline{\Lambda^\bullet Q_g^\vee})$ is computed more explicitly in [BGS5, section 3]. In particular, it is shown in [BGS5, Th. 3.14], [BGS5, Th. 3.17] that $T(\overline{\Lambda^\bullet Q_g^\vee})\mathrm{Td}(\overline{Q_g})$

is represented by a current of type $(r-1,r-1)$ in P_Y^X. By abuse of language we call this element of P_Y^X the **Euler-Green current** of the section σ.

Proof of Lemma 3.15: The vector bundle $\overline{Q}$ splits on X_g into a direct sum of holomorphic Hermitian vector bundles $\overline{Q_g}\oplus\overline{Q_\perp}$. Let ∇, $\nabla^\perp$ denote the holomorphic Hermitian connections of $\overline{Q}$ and $\overline{Q_\perp}$. Let C_u and C_u^g be the holomorphic Hermitian superconnections associated to the complexes $(\overline{\Lambda^\bullet Q^\vee},\iota_\sigma)$ and $(\overline{\Lambda^\bullet Q_g^\vee},\iota_\sigma)$ and let N_H, $N_H^\perp$ and N_H^g denote the number operators acting on $\Lambda^\bullet Q^\vee$, $\Lambda^\bullet Q_\perp^\vee$ and $\Lambda^\bullet Q_g^\vee$, respectively. Then

$$\begin{aligned} C_u^2 &= \nabla^2+\sqrt{u}(\iota_{\nabla\sigma}+\nabla\sigma^*\wedge)+u\|\sigma\|^2\\ &= (\nabla^\perp)^2\otimes 1+1\otimes(C_u^g)^2 \end{aligned}$$

as $\Lambda^\bullet T^*X$-valued operators on $\Lambda^\bullet Q_\perp^\vee\hat{\otimes}\Lambda^\bullet Q_g^\vee$. Hence

$$\begin{aligned} \mathrm{Tr_s}\, g^*N_H e^{-C_u^2} &= \mathrm{Tr_s}\, g^* N_H^\perp e^{-(\nabla^\perp)^2}{}_{|\Lambda^\bullet Q_\perp^\vee}\cdot \mathrm{Tr_s}\, e^{-(C_u^g)^2}{}_{|\Lambda^\bullet Q_g^\vee}\\ &\quad +\mathrm{Tr_s}\, g^* e^{-(\nabla^\perp)^2}{}_{|\Lambda^\bullet Q_\perp^\vee}\cdot \mathrm{Tr_s}\, N_H^g e^{-(C_u^g)^2}{}_{|\Lambda^\bullet Q_g^\vee}\\ &= -\Phi^{-1}(\mathrm{Td}_g^{-1})'(\overline{Q_\perp})\mathrm{Tr_s}\, e^{-(C_u^g)^2}{}_{|\Lambda^\bullet Q_g^\vee}\\ &\quad +\Phi^{-1}\mathrm{Td}_g^{-1}(\overline{Q_\perp})\mathrm{Tr_s}\, N_H^g e^{-(C_u^g)^2}{}_{|\Lambda^\bullet Q_g^\vee}\ . \end{aligned}$$

The Leibniz rule shows

$$(\mathrm{Td}_g^{-1})'(\overline{N}_{X/Y})=\frac{(\mathrm{Td}_g^{-1})'(\overline{Q_\perp})}{\mathrm{Td}(\overline{N_{X_g/Y_g}})}+\frac{(\mathrm{Td}^{-1})'(\overline{N_{X_g/Y_g}})}{\mathrm{Td}_g(\overline{Q_\perp})}\ .$$

Thus the zeta function defining $T_g(\overline{\Lambda^\bullet Q^\vee})$ is given by

$$\begin{aligned} \zeta_g(s) &= \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\left\{\Phi\mathrm{Tr_s}\, g^*N_He^{-C_u^2}+(\mathrm{Td}_g^{-1})'(\overline{N}_{X/Y})\cdot\delta_{Y_g}\right\}du\\ &= -\frac{(\mathrm{Td}_g^{-1})'(\overline{Q_\perp})}{\Gamma(s)}\int_0^\infty u^{s-1}\left\{\Phi\mathrm{Tr_s}\, e^{-(C_u^g)^2}-\mathrm{Td}^{-1}(\overline{N_{X_g/Y_g}})\cdot\delta_{Y_g}\right\}du \qquad (11)\\ &\quad +\frac{\mathrm{Td}_g^{-1}(\overline{Q_\perp})}{\Gamma(s)}\int_0^\infty u^{s-1}\left\{\Phi\mathrm{Tr_s}\, N_H^g e^{-(C_u^g)^2}+(\mathrm{Td}^{-1})'(\overline{N_{X_g/Y_g}})\cdot\delta_{Y_g}\right\}du \end{aligned}$$

for $0<\mathrm{Re}\ s<1/2$. Using [B3, Th. 6.2] and [B3, Th. 6.7], one verifies that the first expression in equation (11) vanishes in P_Y^X. The second part equals the zeta function defining $T(\overline{\Lambda^\bullet Q_g^\vee})$ multiplied with $\mathrm{Td}(\overline{Q_g})/\mathrm{Td_g}(\overline{Q})$. **Q.E.D.**

4 The statement

Let D be a regular arithmetic ring. By this we mean a regular, excellent, Noetherian integral ring, together with a finite set $\mathcal{S}$ of injective ring homomorphisms of $D \to \mathbf{C}$, which is invariant under complex conjugation (see [GS2, Def. 3.1.1, p. 124]). We shall denote by μ_n the diagonalisable group scheme over D associated to $\mathbf{Z}/(n)$, the cyclic group of order n. We shall denote the set of complex n-th roots of unity by R_n and we choose once and for all a primitive n-th root of unity ζ_n. We shall call **equivariant arithmetic variety** a regular integral scheme, endowed with a μ_n-projective action over Spec D. Let $f : Y \to \text{Spec } D$ be an equivariant arithmetic variety of dimension d. We write $Y(\mathbf{C})$ for the set of complex points of the variety $\coprod_{e \in \mathcal{S}} Y \times_D \mathbf{C}$, which naturally carries the structure of a complex manifold. The groups R_n acts on $Y(\mathbf{C})$ by holomorphic automorphisms and we shall write g for the automorphism corresponding to ζ_n. By Prop. 2.12, the fixed point scheme Y_{μ_n} is regular and by Cor. 2.11 and the GAGA principle, there are natural isomorphisms of complex manifolds $Y_{\mu_n}(\mathbf{C}) \simeq (Y(\mathbf{C}))_g$ (recall that $(Y(\mathbf{C}))_g$ is the set of fixed points of Y under the action of R_n, cf. subsection 3.2). We write f^{μ_n} for the map $Y_{\mu_n} \to \text{Spec } D$ induced by f. Complex conjugation induces an antiholomorphic automorphism of $Y(\mathbf{C})$ and $Y_{\mu_n}(\mathbf{C})$, both of which we denote by F_∞. We write $\widetilde{\mathfrak{A}}(Y_{\mu_n})$ for $\widetilde{\mathfrak{A}}(Y(\mathbf{C})_g) := \bigoplus_{p \geq 0} (\mathfrak{A}^{p,p}(Y(\mathbf{C})_g)/(\text{Im}\,\partial + \text{Im}\,\overline{\partial}))$, where $\mathfrak{A}^{p,p}(\cdot)$ denotes the set of smooth complex differential forms ω of type (p,p), such that $F_\infty^* \omega = (-1)^p \omega$. (see the beginning of subsection 3.3; there the F_∞-invariance requirement is not stated because the manifolds are not assumed to have models over the real field).

A hermitian equivariant sheaf (resp. vector bundle) on Y is a coherent sheaf (resp. a vector bundle) E on Y, assumed locally free on $Y(\mathbf{C})$, equipped with a μ_n-action which lifts the action of μ_n on Y and a hermitian metric h on $E_{\mathbf{C}}$, the bundle associated to E on the complex points, which is invariant under F_∞ and μ_n. We shall write (E, h) or $\overline{E}$ for an hermitian equivariant sheaf (resp. vector bundle). There is a natural $\mathbf{Z}/(n)$-grading $E|_{Y_{\mu_n}} \simeq \oplus_{k \in \mathbf{Z}/(n)} E_k$ on the restriction of E to Y_{μ_n}, whose terms are orthogonal, because of the invariance of the metric. We write $\overline{E}_k$ for the k-th term $(k \in \mathbf{Z}/(n))$, endowed with the induced metric. We also often write $\overline{E}_{\mu_n}$ for $\overline{E}_0$.

We write $\text{ch}_g(\overline{E})$ for the equivariant Chern character form (see after Th. 3.4) $\text{ch}_g((E_{\mathbf{C}}, h))$ associated to the restriction of $(E_{\mathbf{C}}, h)$ to $Y_{\mu_n}(\mathbf{C})$. Recall also that $\text{Td}_g(\overline{E})$ is the differential form $\text{Td}(\overline{E}_{\mu_n})\Big(\sum_{i \geq 0}(-1)^i \text{ch}_g(\Lambda^i(\overline{E}))\Big)^{-1}$. If $\mathcal{E} : 0 \to E' \to E \to E'' \to 0$ is an exact sequence of equivariant sheaves (resp. vector bundles), we shall write $\overline{\mathcal{E}}$ for the sequence $\mathcal{E}$ together with R_n- and F_∞-invariant hermitian metrics on $E'_{\mathbf{C}}$, $E_{\mathbf{C}}$ and $E''_{\mathbf{C}}$. To $\overline{\mathcal{E}}$ and ch_g is associated an equivariant Bott-Chern secondary class $\widetilde{\text{ch}}_g(\overline{\mathcal{E}}) \in \widetilde{\mathfrak{A}}(Y_{\mu_n})$, which satisfies the equation $\frac{\overline{\partial}\partial}{2\pi i}\widetilde{\text{ch}}_g(\overline{\mathcal{E}}) = \text{ch}_g(\overline{E}') + \text{ch}_g(\overline{E}'') - \text{ch}_g(\overline{E})$ (see Th. 3.4).

Definition 4.1 *The arithmetic equivariant Grothendieck group $\widehat{K}_0^{\mu_n}{}'(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$) of Y is the free abelian group generated by the elements of $\widetilde{\mathfrak{A}}(Y_{\mu_n})$ and by the equivariant isometry classes of hermitian equivariant sheaves (resp.*

vector bundles), together with the relations

(a) *for every exact sequence $\overline{\mathcal{E}}$ as above, $\widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}}) = \overline{E}' - \overline{E} + \overline{E}''$;*

(b) *if $\eta \in \widetilde{\mathfrak{A}}(Y_{\mu_n})$ is the sum in $\widetilde{\mathfrak{A}}(Y_{\mu_n})$ of two elements η' and η'', then $\eta = \eta' + \eta''$ in $\widehat{K}_0^{\mu_n}{}'(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$).*

Before we proceed, notice the following fact. Let M be a complex manifold and let ζ, κ be complex currents on M such that each of them is a sum of currents of type (p,p). If the wave front sets of ζ and κ are disjoint, then the cup products $(\frac{\overline{\partial}\partial}{2\pi i}\zeta) \wedge \kappa$ and $\zeta \wedge (\frac{\overline{\partial}\partial}{2\pi i}\kappa)$ are defined and we have an equality

$$(\frac{\overline{\partial}\partial}{2\pi i}\zeta) \wedge \kappa = \zeta \wedge (\frac{\overline{\partial}\partial}{2\pi i}\kappa) \tag{12}$$

in P_M^M (see after Lemma 3.13 for the definition of P_M^M). The proof follows from the equalities $\partial(\zeta \wedge \overline{\partial}\kappa) = \partial\zeta \wedge \overline{\partial}\kappa + \zeta \wedge \partial\overline{\partial}\kappa$ and $-\overline{\partial}(\partial\zeta \wedge \kappa) = \partial\zeta \wedge \overline{\partial}\kappa + \partial\overline{\partial}\zeta \wedge \kappa$. We shall now define a ring structure on $\widehat{K}_0^{\mu_n}{}'(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$). Let $\overline{V}$, $\overline{V}'$ be hermitian equivariant sheaves (resp. vector bundles) and let η, η' be elements of $\widetilde{\mathfrak{A}}(Y_{\mu_n})$. We define a product $\cdot$ on the generators of $\widehat{K}_0^{\mu_n}{}'(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$) by the rules $\overline{V} \cdot \overline{V}' := \overline{V} \otimes \overline{V}'$, $\overline{V} \cdot \eta = \eta \cdot \overline{V} := \mathrm{ch}_g(\overline{V}) \wedge \eta$ and $\eta \cdot \eta' := \frac{\overline{\partial}\partial}{2\pi i}\eta \wedge \eta'$ and we extend it by linearity. To see that it is well-defined, consider hermitian coherent sheaves $\overline{E}'$, $\overline{E}$ and $\overline{E}''$ (resp. vector bundles) and an exact sequence

$$\mathcal{E} : 0 \to E' \to E \to E'' \to 0.$$

We compute in $\widehat{K}_0^{\mu_n}{}'(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$):

$$\begin{aligned}(\overline{E} + \widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}})) \cdot \overline{V} &= \overline{E} \otimes \overline{V} + \widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}}) \wedge \mathrm{ch}_g(\overline{V}) \\ &= \overline{E} \otimes \overline{V} + \widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}} \otimes \overline{V}) = \overline{E}' \otimes \overline{V} + \overline{E}'' \otimes \overline{V}.\end{aligned}$$

and

$$(\overline{E} + \widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}})) \cdot \eta = \mathrm{ch}_g(\overline{E}) \wedge \eta + (\frac{\overline{\partial}\partial}{2\pi i}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}})) \wedge \eta = \mathrm{ch}_g(\overline{E}' \oplus \overline{E}'') \wedge \eta$$

From these computations, it follows that the product $\cdot$ is compatible with the defining relations of $\widehat{K}_0^{\mu_n}{}'(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$); furthermore it is associative and the trivial bundle endowed with the trivial metric is a unit for that product; these statements follows readily from the definitions. We thus obtain a ring structure on $\widehat{K}_0^{\mu_n}{}'(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$). Notice also that the definition of $\widehat{K}_0^{\mu_n}{}'(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$) implies that there is an exact sequence

$$\widetilde{\mathfrak{A}}(Y_{\mu_n}) \to \widehat{K}_0^{\mu_n}{}'(Y) \to K_0^{\mu_n}{}'(Y) \to 0 \tag{13}$$

(resp.

$$\widetilde{\mathfrak{A}}(Y_{\mu_n}) \to \widehat{K}_0^{\mu_n}(Y) \to K_0^{\mu_n}(Y) \to 0 \quad),$$

where $K_0^{\mu_n\prime}(Y)$ (resp. $K_0^{\mu_n}(Y)$) is the ordinary Grothendieck group of μ_n-equivariant coherent sheaves (resp. locally free sheaves) (see [Köck, Def. (2.1)]). Now let $\widetilde{A}(Y_{\mu_n})$ be the subgroup of $\widetilde{\mathfrak{A}}(Y_{\mu_n})$ consisting of elements that can be represented by real differential forms. If $\mu_n = \mu_1$ (the trivial group scheme) and one replaces $\widetilde{\mathfrak{A}}(Y)$ by $\widetilde{A}(Y)$ in the definition of $\widehat{K}_0^{\mu_n}(Y)$, one obtains the arithmetic Grothendieck group $\widehat{K}_0(Y)$ defined by Gillet and Soulé (see [GS3, II]). This ring can be equipped with a ring structure defined by the same rules as above and there is by construction a natural ring morphism $\widehat{K}_0(Y_{\mu_n}) \to \widehat{K}_0^{\mu_n}(Y_{\mu_n})$. Since every equivariant vector bundle is an equivariant sheaf, there is also natural morphism of rings $\widehat{K}_0^{\mu_n}(Y) \to \widehat{K}_0^{\mu_n\prime}(Y)$. Notice finally that there is a map from $\widehat{K}_0^{\mu_n'}(Y)$ to the space of complex closed differential forms, which is defined by the formula $\mathrm{ch}_g(\overline{E}+\kappa) := \mathrm{ch}_g(\overline{E}) + \frac{\bar{\partial}\partial}{2\pi i}\kappa$ ($\overline{E}$ an hermitian equivariant sheaf, $\kappa \in \widetilde{\mathfrak{A}}(Y_{\mu_n})$). One can see from the definition of the $\widehat{K}_0^{\mu_n'}$-groups that this map is well-defined and we shall denote it by $\mathrm{ch}_g(\cdot)$ as well.

Proposition 4.2 *The natural morphism $\widehat{K}_0^{\mu_n}(Y) \to \widehat{K}_0^{\mu_n\prime}(Y)$ is an isomorphism.*

Proof: We have to define a map which inverts the natural morphism. Let $\overline{E}$ be a hermitian equivariant sheaf. Let $\mathcal{O}(1)$ be a very ample equivariant line bundle on Y. By [H, Th. 8.8, p. 252], there is a surjective morphism of sheaves $f^*f_*(E \otimes \mathcal{O}(l)) \otimes \mathcal{O}(-l) \to E$ ($l \gg 0$), which is equivariant by construction. If we choose a surjective map of μ_n-comodules $M \to f_*(E \otimes \mathcal{O}(l))$, such that M is finitely generated and free, we obtain a surjective map $(f^*M) \otimes \mathcal{O}(-l) \to E \to 0$ of equivariant sheaves, where $(f^*M) \otimes \mathcal{O}(-l)$ is by construction locally free (recall that f is the structure map $Y \to \mathrm{Spec}\ D$). Repeating this process with the kernel of this surjection, we obtain an equivariant locally free resolution $\ldots \to V_i \to V_{i-1} \to \ldots \to V_0 \to E \to 0$ and by a dimension shifting argument $\ker(V_d \to V_{d-1})$ is locally free (see for ex. [FL, p. 101]). Thus we obtain a finite locally free equivariant resolution $\mathcal{V}$ of E. Endow each V_i with an invariant hermitian metric and write $\overline{\mathcal{V}}$ for $\mathcal{V}$ together with these metrics. We define the inverse map $I : \widehat{K}_0^{\mu_n\prime}(Y) \to \widehat{K}_0^{\mu_n}(Y)$ as the unique map of groups which sends differential forms on themselves and a hermitian equivariant sheaf $\overline{E}$ on the element $\sum_{i\geq 0}(-1)^{i+1}\overline{V}_i + \widetilde{\mathrm{ch}}_g(\overline{\mathcal{V}})$, where $\overline{\mathcal{V}}$ is any hermitian resolution of E as above. To prove that this map is well-defined and also a group map, consider the commutative diagram

$$
\begin{array}{ccccccccccc}
 & & & \mathcal{V}_m & & \mathcal{V}_{m-1} & \dots & \mathcal{V}_0 & & \mathcal{E} & \\
 & & & 0 & & 0 & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\mathcal{V}'' & 0 & \to & V''_m & \to & V''_{m-1} & \dots & V''_0 & \to & E'' & \to & 0 \\
 & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\mathcal{V} & 0 & \to & V_m & \to & V_{m-1} & \dots & V_0 & \to & E & \to & 0 \\
 & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\mathcal{V}' & 0 & \to & V'_m & \to & V'_{m-1} & \dots & V'_0 & \to & E' & \to & 0 \\
 & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
 & & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

Given an exact sequence of equivariant coherent sheaves $\mathcal{E}$, one can always construct a diagram as above, such that all the columns strictly to the left of $\mathcal{E}$ consist of locally free sheaves and such that its rows and columns are exact. If $E' = 0$, we say that $\mathcal{V}$ dominates $\mathcal{V}''$. Now endow all the sheaves in this diagram with invariant metrics and call $\overline{\mathcal{V}}'$, $\overline{\mathcal{V}}$ and $\overline{\mathcal{V}}''$ the rows together with the corresponding hermitian metrics. If we apply the double complex formula Th. 3.14, we see that

$$\widetilde{\mathrm{ch}}_g(\overline{\mathcal{V}}') + \widetilde{\mathrm{ch}}_g(\overline{\mathcal{V}}'') - \widetilde{\mathrm{ch}}_g(\overline{\mathcal{V}}) = \sum_{i \geq 0} \widetilde{\mathrm{ch}}_g(\overline{\mathcal{V}}^i)(-1)^{i+1} + \widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}}).$$

Applying this formula and the relations of equivariant arithmetic K_0-theory, we see that $\sum_{i \geq 0}(-1)^{i+1}\overline{V}_i + \widetilde{\mathrm{ch}}_g(\overline{\mathcal{V}}) = \sum_{i \geq 0}(-1)^{i+1}\overline{V}''_i + \widetilde{\mathrm{ch}}_g(\overline{\mathcal{V}}'')$, if $\mathcal{V}$ dominates $\mathcal{V}''$. Since for two resolutions there always exists a third one dominating both (see [L, p. 129]), we are done for well-definedness. To show that I is a morphism of groups, we consider again the above diagram and compute, using Th. 3.14, $I(\overline{E}) - I(\overline{E}') - I(\overline{E}'') + \widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}}) = \sum_{i \geq 0}(\widetilde{\mathrm{ch}}_g(\overline{\mathcal{V}}_i) + \overline{V}_i - \overline{V}'_i - \overline{V}''_i)(-1)^{i+1} = 0$. The map I is by construction an inverse of the natural morphism $\widehat{K}_0^{\mu_n}(Y) \to \widehat{K}_0^{\mu_n\prime}(Y)$ and so we are done. **Q.E.D.**

Fix a F_∞-invariant Kähler metric on $Y(\mathbf{C})$, with Kähler form ω_Y. We suppose that R_n acts by isometries with respect to this Kähler metric. Let $\overline{E} := (E, h)$ be an equivariant hermitian sheaf on Y; we write $T_g(Y, \overline{E})$ for the equivariant analytic torsion $T_g(Y(\mathbf{C}), (E_{\mathbf{C}}, h)) \in \mathbf{C}$ of $(E_{\mathbf{C}}, h)$ (see subsection 3.2). Let $f : Y \to \mathrm{Spec}\ D$ be the structure morphism. We let $R^i f_* \overline{E}$ be the i-th direct image sheaf, endowed with its natural equivariant structure and L_2-metric. We also write $\overline{H^i(Y, E)}$ for $R^i f_* \overline{E}$. Write $R^\cdot f_* \overline{E}$ for the linear combination $\sum_{i \geq 0}(-1)^i R^i f_* \overline{E}$. Let $\eta \in \widetilde{\mathfrak{A}}(Y_{\mu_n})$. Consider the rule which associates the element $R^\cdot f_* \overline{E} - T_g(Y, \overline{E})$ of $\widehat{K}_0^{\mu_n\prime}(D)$ to $\overline{E}$ and the element $\int_{Y(\mathbf{C})_g} \mathrm{Td}_g(\overline{TY})\eta \in \widehat{K}_0^{\mu_n\prime}(D)$ to η.

Proposition 4.3 *The above rule descends to a well defined group homomorphism* $f_*:\widehat{K}_0^{\mu_n}{}'(Y)\to\widehat{K}_0^{\mu_n}{}'(D)$.

Proof: Let $\overline{E}'$, $\overline{E}$ and $\overline{E}''$ be hermitian coherent sheaves on Y and suppose that there is an exact sequence

$$\mathcal{E}:0\to E'\to E\to E''\to 0.$$

Using the definition of f_* and the defining relations of $\widehat{K}_0^{\mu_n}(Y)$, we see that to prove our claim, it will be sufficient to prove that

$$\begin{aligned} &R^{\cdot}f_*\overline{E}-T_g(Y,\overline{E})+\int_{Y_{\mu_n}}\mathrm{Td}_g(\overline{TY})\widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}})-R^{\cdot}f_*\overline{E}' \\ &+T_g(Y,\overline{E}')-R^{\cdot}f_*\overline{E}''+T_g(Y,\overline{E}'')=0 \end{aligned} \tag{14}$$

in $\widehat{K}_0^{\mu_n}(D)$. According to Th. 3.7, the equation

$$\widetilde{\mathrm{ch}}_g(\lambda_g(Y(\mathbf{C}),\overline{\mathcal{E}}))=\int_{Y_{\mu_n}}\mathrm{Td}_g(\overline{TY})\widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}}) \tag{15}$$

holds in $\widetilde{\mathfrak{A}}(D)$. Denote by $R^{\cdot}f_*\mathcal{E}$ the long exact cohomology sequence of $\mathcal{E}$ with respect to f and let $R^{\cdot}f_*\overline{\mathcal{E}}$ be the sequence $R^{\cdot}f_*\mathcal{E}$ together with the L_2 hermitian metrics inherited from $\overline{\mathcal{E}}$ on each element. Using the defining relations of $\widehat{K}_0^{\mu_n}(Y)$, we see that

$$R^{\cdot}f_*\overline{E}+\widetilde{\mathrm{ch}}_g(R^{\cdot}f_*\overline{\mathcal{E}})-R^{\cdot}f_*\overline{E}'-R^{\cdot}f_*\overline{E}''=0 \tag{16}$$

in $\widehat{K}_0^{\mu_n}(D)$. Combining the remark (7), (15) and (16), we see that (14) holds. This ends the proof. **Q.E.D**.

Using the Prop. 4.3 and Prop. 4.2, we can define a map $\widehat{K}_0^{\mu_n}(Y)\to\widehat{K}_0^{\mu_n}(D)$, which we shall also call f_*. Finally, to formulate our fixed point theorem, we define the homomorphism $\rho:\widehat{K}_0^{\mu_n}(Y)\to\widehat{K}_0^{\mu_n}(Y_{\mu_n})$, which is obtained by restricting all the involved objects from Y to Y_{μ_n}. If $\overline{E}$ is a hermitian vector bundle on Y, we write $\lambda_{-1}(\overline{E}):=\sum_{k=0}^{\mathrm{rk}(E)}(-1)^k\Lambda^k(\overline{E})\in\widehat{K}_0^{\mu_n}(Y)$, where $\Lambda^k(\overline{E})$ is the k-th exterior power of $\overline{E}$, endowed with its natural hermitian and equivariant structure. Notice that if $\overline{E}$ is the orthogonal direct sum of two hermitian equivariant vector bundles $\overline{E}'$ and $\overline{E}''$, then $\lambda_{-1}(\overline{E})=\lambda_{-1}(\overline{E}').\lambda_{-1}(\overline{E}'')$; this follows from the very definition of the exterior power metric (see [BoGS, note to prop. 4.1.2]). A finer multiplicativity property will be proved later (see Lemma 7.1). Let $R(\mu_n)$ be the Grothendieck group of finitely generated projective μ_n-comodules. There are natural isomorphisms $R(\mu_n)\simeq K_0(D)[\mathbf{Z}/(n)]\simeq K_0(D)[T]/(1-T^n)$ (see [Se, Prop. 7, 3.4, p. 47]). Let $\overline{I}$ be the μ_n-comodule whose term of degree 1 is D endowed with the trivial metric and whose other terms are 0. We make

$\widehat{K}_0^{\mu_n}(D)$ an $R(\mu_n)$-algebra under the ring morphism which sends T to $\overline{I}$. In the next theorem (which is the main result), let $\mathcal{R}$ be any $R(\mu_n)$-algebra such that the elements $1-T^k$ ($k=1,\dots,n-1$) are invertible in $\mathcal{R}$. The algebra which is minimal with respect to this property is the ring $R(\mu_n)_{\{1-T^k\}_{k=1,\dots,n-1}}$, the localization of $R(\mu_n)$ at the multiplicative subset generated by the elements $\{1-T^k\}_{k=1,\dots,n-1}$. If $D=\mathbf{Z}$, we can make the complex numbers $\mathbf{C}$ an $R(\mu_n)$-algebra under the ring morphism which sends T to ζ_n; this gives a possible choice of $\mathcal{R}$ if $D=\mathbf{Z}$. Recall that $R_g(\cdot)$ is an equivariant additive characteristic class (see Def. 3.5); in the next theorem, we consider that its values lie in $\widetilde{\mathfrak{A}}(Y_{\mu_n})$. Recall furthermore that the quotient metric on normal bundles has been introduced in section 3.4.

Theorem 4.4 *Let $\overline{N}_{Y/Y_{\mu_n}}$ be the normal bundle of Y_{μ_n} in Y, endowed with its quotient equivariant structure and quotient metric structure (which is F_∞-invariant).*

(a) *The element $\Lambda := \lambda_{-1}(\overline{N}^\vee_{Y/Y_{\mu_n}})$ has an inverse in $\widehat{K}_0^{\mu_n}(Y_{\mu_n})\otimes_{R(\mu_n)}\mathcal{R}$;*

(b) *Let $\Lambda_R := \Lambda.(1-R_g(N_{Y/Y_{\mu_n}}))$; the diagram*

$$\begin{array}{ccc} \widehat{K}_0^{\mu_n}(Y) & \stackrel{\Lambda_R^{-1}.\rho}{\longrightarrow} & \widehat{K}_0^{\mu_n}(Y_{\mu_n})\otimes_{R(\mu_n)}\mathcal{R} \\ \downarrow\ f_* & & \downarrow\ f_*^{\mu_n} \\ \widehat{K}_0^{\mu_n}(D) & \stackrel{\mathrm{Id}\otimes 1}{\longrightarrow} & \widehat{K}_0^{\mu_n}(D)\otimes_{R(\mu_n)}\mathcal{R} \end{array}$$

commutes.

In the sequel, we shall also write $\lambda_{-1}^{-1}(\cdot)$ for $(\lambda_{-1}(\cdot))^{-1}$. Notice that if $n=2$, then we can choose $\mathcal{R}=\mathbf{Z}[\frac{1}{2}]$; thus the operation of tensoring with $\mathcal{R}$ does not necessarily imply a loss of information about the entire torsion subgroup of $\widehat{K}_0^{\mu_n}(D)$.
The part (a) of Th. 4.4 assures the existence of the inverse of $\lambda_{-1}(\overline{N}^\vee_{Y/Y_{\mu_n}})$, but does not describe an effective construction of this inverse. The proof of the next lemma provides an effective construction of the inverse of $\lambda_{-1}(\overline{E})$, when $\overline{E}$ is a hermitian equivariant vector bundle on Y_{μ_n}, such that $\overline{E}_{\mu_n}=0$. By Prop. 2.12, we know that $(\overline{N}^\vee_{Y/Y_{\mu_n}})_{\mu_n}=0$ and the proof of the next lemma thus provides an effective construction of the inverse of $\lambda_{-1}(\overline{N}^\vee_{Y/Y_{\mu_n}})$. In particular, it is an effective proof of part (a).
Now let Z be any arithmetic variety (without μ_n-action). In the proof of the next lemma, we shall make use of the following facts. There exist operations $\lambda^k:\widehat{K}_0(Z)\to\widehat{K}_0(Z)$ ($k\geq 0$) on the (non-equivariant) Grothendieck group $\widehat{K}_0(Z)$, that endow this group with a special λ-structure. We refer to [SGA6, Def. 2.1, p. 314] for the definition of this term and to [R1, Section 2] for details.

Let us just mention that if $\overline{E}$ is a hermitian vector bundle on Z, then $\lambda^k(\overline{E}) = \Lambda^k(\overline{E})$ in $\widehat{K}_0(Z)$; here Λ^k refers to the k-th exterior power of E, endowed with its natural hermitian structure. Let us define $\lambda_t : \widehat{K}_0(Z) \to \widehat{K}_0(Z)[[t]]$ by the rule $\lambda_t(x) := \sum_{k\geq 0} \lambda^k(x).t^k$. We then denote by $\gamma^k(x)$ the coefficient of t^k in the formal power series $\lambda_{\frac{t}{1-t}}(x)$. The operations γ^k are called the γ-operations (see [SGA6, 1, Exp. V] for more details). The ring $\widehat{K}_0(Z)$ also carries a natural augmentation morphism $\mathrm{rk} : \widehat{K}_0(Z) \to \mathbf{Z}$, which associates the rank of the underlying bundle to a hermitian vector bundle and the number 0 to an element of $\widetilde{\mathfrak{A}}(Z)$. The λ-structure together with this augmentation morphism give rise to a ring filtration $F^0\widehat{K}_0(Z) \supseteq F^1\widehat{K_0}(Z) \supseteq \ldots$ on $\widehat{K}_0(Z)$, called the γ-filtration; for $k = 0$, $F^0\widehat{K}_0(Z) = \widehat{K}_0(Z)$, for $k = 1$, $F^1\widehat{K}_0(Z)$ is the kernel of rk and for $k > 1$ $F^k\widehat{K}_0(Z)$ is the ideal generated by the elements $\gamma^{r_1}(x_1)\gamma^{r_2}(x_2)\ldots\gamma^{r_j}(x_j)$, where $x_1, \ldots, x_j \in F^1\widehat{K}_0(Z)$ and $r_1, \ldots, r_j$ are positive numbers such that $r_1 + \ldots + r_j \geq k$. It is proved in [R1, Section 4] that this filtration is locally nilpotent. A particular case of this result, which is the only one used in the proof of the coming lemma, is that if $x \in F^k\widehat{K}_0(Z)$ with $k > 0$, then there exists a natural number n, dependent on x, such that $x^n = 0$.

Lemma 4.5 *Let $\overline{E}$ be an equivariant hermitian vector bundle over Y_{μ_n}, such that $\overline{E}_{\mu_n} = 0$. Then the element $\lambda_{-1}(\overline{E})\otimes 1$ is invertible in $\widehat{K}_0^{\mu_n}(Y_{\mu_n})\otimes_{R(\mu_n)}\mathcal{R}$.*

Proof (of Lemma 4.5): By universality, we may assume that $\mathcal{R}$ is the localisation of the ring $R(\mu_n)$ at the multiplicative subset generated by the elements $T^i - 1$ ($1 \leq i < n$). Remember that if $\overline{E}$ is the orthogonal direct sum of two hermitian equivariant vector bundles $\overline{E}'$ and $\overline{E}''$, then $\lambda_{-1}(\overline{E}) = \lambda_{-1}(\overline{E}').\lambda_{-1}(\overline{E}'')$. Thus, since E is $\mathbf{Z}/(n)$-graded and the terms of the grading are pairwise orthogonal, we are reduced to prove that $\lambda_{-1}(\overline{E}_p)$ is invertible, where $p \in \mathbf{Z}/(n)$, $p \neq 0$ and $\overline{E}_p$ is an equivariant hermitian bundle on Y_{μ_n}, such that $E_k = 0$ if $k \neq p$. Now notice that

$$\lambda_{-1}(\overline{E}_p) \otimes 1 = \sum_{j=0}^{\mathrm{rk}(E_p)} (-1)^j \Lambda^j(\overline{E}_p') \otimes \zeta_n^{p.j} \tag{17}$$

where $\overline{E}_p'$ is the underlying hermitian bundle of $\overline{E}_p$, equipped with the trivial grading. This expression lies in the image of the natural ring morphism $\widehat{K}_0(Y_{\mu_n}) \otimes_{\mathbf{Z}} \mathcal{R} \to \widehat{K}_0^{\mu_n}(Y_{\mu_n}) \otimes_{R(\mu_n)} \mathcal{R}$. Now let $r = \mathrm{rk}(E_p)$ and suppose that $\overline{E}_p'$ is the sum $x_1 + x_2 + \ldots + x_r$ in $\widehat{K}_0(Y_{\mu_n})$ of line elements x_i (i.e. $\lambda^l(x_i) = 0$ if $l > 1$). By definition $\lambda^l(x_1 + x_2 + \ldots + x_r) = \sigma_l(x_1, \ldots, x_r)$, where σ_l is the l-th symmetric function ($l \geq 0$) and thus using (17), we see that

$$\lambda_{-1}(\overline{E}_p) \otimes 1 = \prod_i (1 - x_i \otimes \zeta_n^p).$$

We rewrite this last expression as

$$\prod_i (1 - 1 \otimes \zeta_n^p - (x_i - 1).(1 \otimes \zeta_n^p)).$$

This is a symmetric polynomial in the $x_i - 1$, with coefficients in $\mathcal{R}$ and thus by the theorem on symmetric functions and the definition of the γ-operations, there exists a polynomial P, with vanishing constant coefficient, such that

$$\lambda_{-1}(\overline{E}_p) \otimes 1 = (1 - 1 \otimes \zeta_n^p)^r - P(\gamma^1(\overline{E}'_p), \ldots, \gamma^r(\overline{E}'_p))$$

Now using the fact that $\widehat{K}_0(Y_{\mu_n})$ is a special λ-ring, [AT, p. 266] and the fact that $\mathcal{R}$ is a flat $R(\mu_n)$-module (see [Ma, p. 46]), we see that the last equality holds even without the hypothesis that $\overline{E}'_p$ is the sum of line elements. Now using the fact that the γ-filtration of $\widehat{K}_0(Y_{\mu_n})$ is locally nilpotent, we see that the sum

$$\sum_{l \geq 0} \frac{1}{(1 - 1 \otimes \zeta_n^p)^{(l-1).r}} (P(\gamma^1(\overline{E}'_p), \ldots, \gamma^r(\overline{E}'_p)))^l$$

is finite and provides an inverse of $\lambda_{-1}(\overline{E}_p) \otimes 1$. So we are done. **Q.E.D.**

For a refined multiplicativity property of $\lambda_{-1}(.) \otimes 1$, see Lemma 7.1.
The strategy of the proof of the part (b) of Th. 4.4 is as follows. In section 5, we state Bismut's immersion theorem Th. 3.11 in $\widehat{K}_0^{\mu_n}$-theoretic form; in section 6 we prove an analog of Th. 4.4 for closed immersions; in section 7 we use the anomaly formula Th. 3.6 of section 3 to prove that theorem Th. 4.4 is compatible with change of Kähler metrics and that the results of sections 5 and 6 combined implie that Th. 4.4 is compatible with immersions. In the same section, we show (using an argument of J.-B. Bost) that the compatibility with immersions implies that Th. 4.4 holds for projective spaces; from this we deduce Th. 4.4 in general.

5 $\widehat{K}_0^{\mu_n}$-theoretic form of Bismut's immersion theorem

Proposition 5.1 *Let $i : Y \to X$ be an equivariant closed immersion of equivariant arithmetic varieties and $f : Y \to$ Spec D, $p : X \to$ Spec D be the structure morphisms. Let*

$$\Xi : \ 0 \to \xi_m \to \xi_{m-1} \to \ldots \to \xi_0 \to i_*\eta \to 0$$

be an equivariant resolution by vector bundles on X of an equivariant vector bundle η on Y. Suppose that X is endowed with an F_∞- and R_n-invariant Kähler metric with Kähler form ω_X and that Y carries the induced Kähler *form $\omega_Y := i^*\omega_X$. Suppose that the normal bundle N of i carries the quotient metric and that η and the ξ_i are endowed with F_∞- and R_n-invariant hermitian metrics satisfying Bismut's condition (A) with respect to the metric of N. Then the equality*

$$\begin{aligned} f_*(\overline{\eta}) - \sum_{i=0}^{m}(-1)^i p_*(\overline{\xi}_i) &= \int_{Y_g} \mathrm{ch}_g(\eta) R_g(N)\mathrm{Td}_g(TX) + \int_{X_g} T_g(\overline{\xi}.)\mathrm{Td}_g(\overline{TX}) \\ &\quad - \int_{Y_g} \mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}_g}(\overline{TY}, \overline{TX})\mathrm{Td}_g^{-1}(\overline{N}) \end{aligned}$$

holds in $\widehat{K}_0^{\mu_n}(D)$.

Proof: Using the defining relations of arithmetic $K_0^{\mu_n}$-theory, we compute

$$\begin{aligned} f_*(\overline{\eta}) - \sum_{i=0}^{m}(-1)^i p_*(\overline{\xi}_i) &= R^{\cdot} f_*\overline{\eta} - T_g(Y, \overline{\eta}) - \sum_{i=0}^{m}(-1)^i (R^{\cdot} p_*\overline{\xi}_i - T_g(X, \overline{\xi}_i)) \\ &= \widetilde{\mathrm{ch}}(R^{\cdot} p_*\overline{\Xi}) - T(Y, \overline{\eta}) + \sum_{i=0}^{m}(-1)^i T_g(X, \overline{\xi}_i). \end{aligned}$$

Comparing the last expression with the formula Th. 3.11 yields the proof. **Q.E.D.**

6 A fixed point formula for closed immersions

In this section, we shall prove an analog of Th. 4.4 for closed immersions; the proof of this result involves the use Th. 3.11 (in the form of its $\widehat{K}_0^{\mu_n}$-theoretic form Prop. 5.1), but only in its non-equivariant form already proved in [BL].

6.1 The statement

Let Y, X be equivariant arithmetic varieties over D. Let $i : Y \to X$ be an equivariant closed immersion and let $f : Y \to \mathrm{Spec}\ D$, $p : X \to \mathrm{Spec}\ D$ be the structure morphisms. Endow $X(\mathbf{C})$ with an F_∞- and R_n-invariant Kähler metric and $Y(\mathbf{C})$ with the restricted metric. Let η be an equivariant vector bundle on Y and let

$$0 \to \xi_m \to \xi_{m-1} \to \ldots \to \xi_0 \to i_*\eta \to 0$$

be an equivariant locally free resolution of $i_*\eta$ on X (the beginning of the proof of Prop. 4.2 shows that such a resolution always exists). Let $\overline{N}_{X/Y}$ be the normal bundle of the immersion, endowed with its natural quotient metric. Let the ξ_i and η be endowed with F_∞- and R_n- invariant hermitian metrics satisfying Bismut's condition (A). We shall write $\overline{E}$ for the bundle $\oplus_{k\neq 0}(N_{X/Y})_k$, endowed with the induced metric.

Theorem 6.1 *The formula*

$$
\begin{aligned}
&f_*^{\mu_n}(\lambda_{-1}(\overline{E}^\vee)\rho(\overline{\eta})) - p_*^{\mu_n}(\Sigma_{j=0}^m(-1)^j\rho(\overline{\xi}_j)) \\
&-\int_{Y_g}\mathrm{Td}(TY_g)\mathrm{ch}_g(\rho(\eta)\otimes\lambda_{-1}(E^\vee))R(N_{X_g/Y_g}) \\
&-\int_{X_g}\mathrm{Td}(\overline{TX}_g)T_g(\overline{\xi}) + \int_{Y_g}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{X/Y})\widetilde{\mathrm{Td}}(\overline{TY}_g, \overline{TX}_g|_{Y_g}) = 0
\end{aligned}
$$

holds in $\widehat{K}_0^{\mu_n}(D)$.

We shall also later make use of the fact that this theorem also holds, if $i : Y \to X \simeq Y$ is the identity and the Kähler metric on $Y(\mathbf{C})$ differs from the Kähler metric on $X(\mathbf{C})$; this follows from the anomaly formula Th. 3.6.
The proof of Th. 6.1 in general forms the core of the paper and will occupy the next subsections. In the situation of Th. 6.1, we shall call $\delta(i, \overline{\xi}_., \overline{\eta})$ the left side of the equality in Th. 6.1. Notice that if X_{μ_n} has irreducible components that do not meet Y, then we might remove the contribution of these components in the expression for $\delta(i, \overline{\xi}_., \overline{\eta})$ without altering the value of $\delta(i, \overline{\xi}_., \overline{\eta})$; this follows from Cor. 3.10 and the definition of the equivariant arithmetic Grothendieck groups.

Proposition 6.2 *The element* $\delta(i, \overline{\xi}_., \overline{\eta})$ *is independent of the choice of* $\overline{\xi}_.$.

Proof: Consider a commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccccc}
 & & \Xi_m & & \Xi_{m-1} & \dots & \Xi_0 & & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \xi'_m & \to & \xi'_{m-1} & \dots & \xi'_0 & \to & i_*\eta & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \xi_m & \to & \xi_{m-1} & \dots & \xi_0 & \to & i_*\eta & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \xi''_m & \to & \xi''_{m-1} & \dots & \xi''_0 & \to & 0 & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

We endow the elements of the rows with invariant metrics satisfying the condition (A). From the definition of the equivariant arithmetic Grothendieck groups, we see that the equality $p_*^{\mu_n}(\rho(\xi'_i)) + p_*^{\mu_n}(\rho(\xi''_i)) = p_*^{\mu_n}(\rho(\xi_i)) + \int_{X_g}\mathrm{Td}(\overline{TX}_g)\widetilde{\mathrm{ch}}_g(\overline{\Xi}_i)$

is valid ($0 \leq i \leq m$). Using this equality and the double complex formula Th. 3.14, we obtain that $\delta(i, \overline{\xi}_., \overline{\eta}) = \delta(i, \overline{\xi}'_., \overline{\eta}) + \delta(i, \overline{\xi}''_., 0)$ and thus, using the definition of equivariant arithmetic K_0-theory again, that $\delta(i, \overline{\xi}'_., \overline{\eta}) = \delta(i, \overline{\xi}_., \overline{\eta})$. Now let us say that in the diagram above, the resolution $\overline{\xi}_.$ dominates the resolution $\overline{\xi}'_.$. We have proved that if $\overline{\xi}'_.$ dominates $\overline{\xi}_.$, then $\delta(i, \overline{\xi}_., \overline{\eta}) = \delta(i, \overline{\xi}'_., \overline{\eta})$. It is shown in [L, p. 129], that if $\xi_.$ and $\phi_.$ are two resolutions of $i_*\eta$, then there exists a resolution $\xi'_.$ that dominates both $\xi_.$ and $\phi_.$, so we are done. **Q.E.D.**

We shall thus henceforth write $\delta(i, \overline{\eta})$ for $\delta(i, \overline{\xi}_., \overline{\eta})$.

6.2 Algebro-geometric preliminaries

The two next subsections will describe the non-equivariant geometric setting of the proof. The third one will show how equivariance fits in this framework.

6.2.1 The deformation to the normal cone

Let Y, X be regular schemes and $Y \xrightarrow{i} X$ be a closed immersion over D. Let N denote the normal bundle of i. In the sequel, the notation $\mathbf{P}(E)$, where E is a vector bundle on any scheme, will refer to the space $\mathrm{Proj}(\mathrm{Sym}(E^\vee))$.

Definition 6.3 *The deformation to the normal cone $W(i)$ (or W) of the immersion i is the blow up of $X \times \mathbf{P}^1_D$ along $Y \times \{\infty\}$.*

We define p_X to be the projection $X \times \mathbf{P}^1 \to X$, p_Y the projection $Y \times \mathbf{P}^1 \to Y$ and π the blow-down map $W \to X \times \mathbf{P}^1$. Let also q_X be the projection $X \times \mathbf{P}^1 \to \mathbf{P}^1$ and q the map $q_X \circ \pi$. From the universality of the blow-up construction, we know that there is a canonical closed immersion $j : Y \times \mathbf{P}^1 \to W$ such that $\pi \circ j = i \times \mathrm{Id}$. We shall denote by i_X the natural immersion of X into W arising from the natural isomorphism $X \simeq \pi^*(X \times 0)$. The following is known about the structure of W; for the proof, see [F, Ch. 5] and [BaFM].

Proposition 6.4 *The closed subscheme $q^{-1}(\infty)$ has two irreducible components P and $\widetilde{X}$ that meet regularly. The component P is isomorphic to $\mathbf{P}(N \oplus 1)$ and the component $\widetilde{X}$ is isomorphic to the blow-up of X along Y. The component $\widetilde{X}$ does not meet $j(Y \times \mathbf{P}^1)$ and the scheme-theoretic intersection of $j(Y \times \mathbf{P}^1)$ and P is the image of the canonical section of $i_\infty : Y \to \mathbf{P}(N \oplus 1)$. Moreover, the map q is flat.*

The canonical section $i_\infty : Y \to \mathbf{P}(N \oplus 1)$ arises from the morphism of vector bundles $\mathcal{O}_Y \to N \oplus \mathcal{O}_Y$.
The embeddings of P and $\widetilde{X}$ in W will be denoted by i_P and $i_{\widetilde{X}}$. Let $k_Y : P \to Y$

be the projection and $\phi := p_X \circ \pi : W \to X$.
The interest of W comes from the possibility to control the rational equivalence class of the fibers $q^{-1}(p)$ ($p \in \mathbf{P}^1$). In the language of line bundles, this is expressed by the fact that $\mathcal{O}(X) \simeq \mathcal{O}(P + \widetilde{X}) \simeq \mathcal{O}(P) \otimes \mathcal{O}(\widetilde{X})$, which is an immediate consequence of the isomorphism $\mathcal{O}(\infty) \simeq \mathcal{O}(0)$ on $\mathbf{P}^1$.
This equality will enable us to reduce certain computations on X to computations on P, which is often much easier to handle. Indeed on P, the canonical quotient bundle Q has a canonical regular section s, which vanishes exactly on Y. Thus, the section s determines a global Koszul resolution

$$\mathfrak{K} : 0 \to \Lambda^{\dim Q}(Q^\vee) \to \ldots \to Q^\vee \to \mathcal{O}_P \to i_{\infty *}\mathcal{O}_Y \to 0$$

6.2.2 Deformation of resolutions

One of the difficulties of a Riemann-Roch formula for embeddings in $\widehat{K}_0$-theory comes from the impossibility to represent explicitly general coherent sheaves, in particular images of locally free sheaves by the embedding. One has to stick to certain explicit resolutions of these sheaves by locally free ones. The following proposition ensures that resolutions with pleasant geometric properties exist on W.

Lemma 6.5 *There exists a locally free resolution $\widetilde{\Xi}$ of $j_* p_Y^*(\eta)$ on W*

$$\widetilde{\Xi} : 0 \to \widetilde{\xi}_m \to \ldots \to \widetilde{\xi}_0 \to j_* p_Y^*(\eta) \to 0$$

such that the restriction $\widetilde{\Xi}|_{\widetilde{X}}$ is a split exact complex.

Proof: The next sublemma, which we shall need for the proof, describes a generalisation of a geometric construction of Bismut-Gillet-Soulé (see [BGS1, Par. f)]).

Sublemma 6.6 *Let $\sigma : \mathcal{O} \to L$ be a section of a line bundle on a scheme and let*

$$0 \to E_0 \xrightarrow{v} E_1 \to \ldots \xrightarrow{v} E_n \to 0$$

be an exact sequence of coherent sheaves on the same scheme. Let $F_j = \ker(E_j \to E_{j+1})$ and $\widetilde{F}_j = F_j \otimes L^{n-j+1}$; let also $\widetilde{E}_j = \mathrm{coker}(F_j \otimes L^{n-j} \to E_j \otimes L^{n-j} \oplus F_j \otimes L^{n-j+1})$ where the map is described by the rule $f_j \otimes l \mapsto v(f_j) \otimes l \oplus f_j \otimes l \otimes \sigma$ ($0 \leq j \leq n$). Then the map

$$\widetilde{F}_j \to \widetilde{E}_j$$

described by the rule $f_j \otimes l \mapsto 0 \oplus f_j \otimes l$ and the map

$$\widetilde{E}_j \to \widetilde{F}_{j+1}$$

described by the rule $e_j \otimes l \oplus f_j \otimes l' \to v(e_j) \otimes l$ are well-defined and yield exact sequences

$$0 \to \widetilde{F}_j \to \widetilde{E}_j \to \widetilde{F}_{j+1} \to 0.$$

Proof: Since the statement is local, we work over a ring and view all the sheaves as modules. The fact that the map $\widetilde{E}_j \to \widetilde{F}_{j+1}$ is well-defined follows from the fact that the image of $v(e_j)\otimes l \oplus f_j \otimes l \otimes \sigma$ is $v^2(e_j)\otimes l$, which is 0. The injectivity of the map $\widetilde{F}_j \to \widetilde{E}_j$ follows from the fact that if $0 \oplus f_j \otimes l = \sum_{r,s}[v(f_j^r) \otimes l^r \oplus f_j^r \otimes l^r \otimes \sigma]$, then $\sum_{r,s}[f_j^r \otimes l^r] = 0$ (because the map $F_j \otimes L^{n-j} \to E_j \otimes L^{n-j}$ is injective) and thus $f_j \otimes l = (\sum_{r,s}[f_j^r \otimes l^r]) \otimes \sigma = 0$. The surjectivity of the map $\widetilde{E}_j \to \widetilde{F}_{j+1}$ follows from the surjectivity of the map $E_j \to F_{j+1}$.
The sequence $0 \to \widetilde{F}_j \to \widetilde{E}_j \to \widetilde{F}_{j+1} \to 0$ is a complex and we still have to prove that $\mathrm{Im}(\widetilde{F}_j \to \widetilde{E}_j) = \ker(\widetilde{E}_j \to \widetilde{F}_{j+1})$. If for $e_j \otimes l \oplus f_j \otimes l'$, $v(e_j) \otimes l = 0$ then there exists $f_j'' \in F_j$ and $l'' \in L^{m-j}$ such that $e_j \otimes l = v(f_j'') \otimes l''$. Thus we can write $e_j \otimes l \oplus f_j \otimes l' = v(f_j'') \otimes l'' \oplus f_j'' \otimes l'' \otimes \sigma + 0 \oplus (f_j \otimes l' - f_j'' \otimes l'' \otimes \sigma)$, where the element before the + sign is by definition 0 in $\widetilde{E}_j$ and the element after the + sign lies in $\mathrm{Im}(\widetilde{F}_j \to \widetilde{E}_j)$. This concludes the proof. **Q.E.D.**

Notice that we can splice together the sequences $0 \to \widetilde{F}_j \to \widetilde{E}_j \to \widetilde{F}_{j+1} \to 0$ to obtain a sequence

$$0 \to \widetilde{E}_0 \to \widetilde{E}_1 \to \ldots \to \widetilde{E}_n \to 0$$

Let now $Z(\sigma)$ be the zero-scheme of σ. The restriction of the sequence $0 \to \widetilde{F}_j \to \widetilde{E}_j \to \widetilde{F}_{j+1} \to 0$ to the complement of $Z(\sigma)$ is isomorphic to the original sequence $0 \to F_j \to E_j \to F_{j+1} \to 0$. This can be seen as follows. On the complement of $Z(\sigma)$, $\widetilde{E}_j$ is isomorphic to $\mathrm{coker}(F_j \to E_j \oplus F_j)$, where the map is described by the rule $f_j \mapsto v(f_j) \oplus f_j$; we thus have an exact sequence $F_j \to E_j \oplus F_j \to E_j \to 0$, where the second map is described by the rule $e_j \oplus f_j \mapsto e_j - v(f_j)$.
Furthermore, by construction, if all the E_j are locally free in a neighborhood of $Z(\sigma)$, then the restriction of the $0 \to \widetilde{F}_j \to \widetilde{E}_j \to \widetilde{F}_{j+1} \to 0$ to $Z(\sigma)$ is isomorphic to the split sequence $0 \to \widetilde{F}_j \to \widetilde{F}_j \oplus \widetilde{F}_{j+1} \to \widetilde{F}_{j+1} \to 0$.
To obtain the resolution $\widetilde{\xi}_.$, we choose a section $\sigma_{\widetilde{X}}$ of $\mathcal{O}(\widetilde{X})$ vanishing on $\widetilde{X}$ and any locally free resolution of $0 \to \widetilde{\xi}'_m \to \widetilde{\xi}'_{m-1} \to \ldots \to \widetilde{\xi}'_0 \to j_* p_Y^*(\eta) \to 0$ on W. We then apply Lemma 6.5 to the sequence $\sigma_{\widetilde{X}}$ and to the sequence $0 \to \widetilde{\xi}'_m \to \widetilde{\xi}'_{m-1} \to \ldots \to \widetilde{\xi}'_0 \to j_* p_Y^*(\eta) \to 0$. **Q.E.D.**

We shall denote the complex $i_P^*(\widetilde{\Xi})$ by $\xi_.^{\infty}$.

6.2.3 Equivariance

We suppose now that the varieties Y and X are μ_n-equivariant and that the immersion i preserves the action. If we let μ_n act trivially on $\mathbf{P}^1_D$, we can extend the action of μ_n to $X \times \mathbf{P}^1_D$ and thus to the deformation to the normal cone (see [Köck, (1.6)]). We shall use the following fact.

Lemma 6.7 *The natural morphism $N_{X_{\mu_n}/Y_{\mu_n}} \to (N_{X/Y})_{\mu_n}$ is an isomorphism.*

Proof: Given a regular immersion $i' : Y' \to Y$, there is an exact sequence of locally free sheaves

$$0 \to N_{Y/Y'} \to N_{X/Y'} \to N_{X/Y} \to 0$$

induced by the various inclusions of ideal sheaves (see [FL, Prop. 3.4, p. 79]). Thus we have two exact sequences of locally free sheaves:

$$0 \to N_{Y/Y_{\mu_n}} \to N_{X/Y_{\mu_n}} \to N_{X/Y} \to 0$$

and

$$0 \to N_{X_{\mu_n}/Y_{\mu_n}} \to N_{X/Y_{\mu_n}} \to N_{X/X_{\mu_n}} \to 0$$

(we consider both sequences as restricted to Y_{μ_n}). Considering the 0-degree part of these sequences and using the last statement in Prop. 2.12, we get isomorphisms $N_{X_{\mu_n}/Y_{\mu_n}} \simeq (N_{X/Y_{\mu_n}})_{\mu_n}$ and $(N_{X/Y_{\mu_n}})_{\mu_n} \simeq (N_{X/Y})_{\mu_n}$. If we explicit the inclusions of ideals sheaves that are behind each of these isomorphisms, we see that the resulting isomorphism $N_{X_{\mu_n}/Y_{\mu_n}} \simeq (N_{X/Y})_{\mu_n}$ is induced by the inclusion (on Y_{μ_n}) of the ideal sheaf of the immersion $Y \to X$ in the ideal sheaf of the immersion $Y_{\mu_n} \to X_{\mu_n}$, i.e. it is the natural morphism. **Q.E.D.**

Proposition 6.8 *The immersions i_X, $i_{\widetilde{X}}$ and i_P are equivariant. The natural morphism of the deformation to the normal cone $W(i^{\mu_n})$ of the immersion $Y_{\mu_n} \to X_{\mu_n}$ into the fixed point scheme $W(i)_{\mu_n}$ of $W(i)$ is a closed embedding; this embedding induces the closed embeddings $\mathbf{P}(N_{\mu_n} \oplus 1) \to \mathbf{P}(N \oplus 1)_{\mu_n}$ and $\widetilde{X_{\mu_n}} \to \widetilde{X}_{\mu_n}$.*

Proof: The fact that the natural map $W(i^{\mu_n}) \to W(i)_{\mu_n}$ is a closed embedding follows from [H, Cor. 7.15, p. 165]. The other statements are direct consequences of the universality and base-change invariance of the blow-up construction. **Q.E.D.**

Proposition 6.9 *There exists an equivariant resolution $\widetilde{\Xi}$ of $j_* p_Y^* \eta$ such that the restriction $\widetilde{\Xi}|_{\widetilde{X}}$ is an equivariantly split exact complex.*

Proof: The construction of the sequence $\widetilde{\widetilde{\Xi}}$ is similar to the construction of the sequence $\widetilde{\Xi}$ appearing in Lemma 6.5. Each step of the construction given in the proof of Lemma 6.5 respects equivariance. **Q.E.D.**

6.3 Proof of the formula

In the next paragraphs, we shall often implicitly use the following fact. Let M' be a closed submanifold of a differentiable manifold M and E be a complex differentiable vector bundle on M; let h' be a hermitian metric on the restriction $E|_{M'}$. Then there always exists a hermitian metric h on E, that extends h'. This follows from a partition of unity argument.

6.3.1 A model for closed embeddings

Let $f : Y \to \text{Spec } D$ be an equivariant arithmetic variety and N_∞ an equivariant vector bundle on Y. In this subsection, we prove that Th. 6.1 holds for the closed immersion $i_\infty : Y \to \mathbf{P}(N_\infty \oplus 1)$ mentioned after Prop. 6.4. The deformation theorem of the next subsection will then show that a Lefschetz type formula for all regular immersions can be derived from that one. We suppose that $P = \mathbf{P}(N_\infty \oplus 1)$ is endowed with the equivariant structure arising from N_∞ and with an invariant Kähler metric and that Y carries the metric induced from P via i_∞. We let N_∞ carries the induced quotient metric. We let $\overline{E}_\infty$ be the bundle $\oplus_{k\neq 0} N_{\infty,k}$, endowed with the induced metric. We fix an invariant metric on Q (the universal quotient bundle on P) which yields the metric of N_∞, when restricted to Y. The resolution $\mathfrak{K}$ (see the end of subsubsection 5.2.1) carries the exterior product metrics of Q; as before, we let $\overline{\eta}$ be an equivariant hermitian bundle on Y. Notice first that by Cor. 2.10, the fixed point scheme of $\mathbf{P}(N_\infty \oplus 1)$ is the closed subscheme $\mathbf{P}(N_{\infty,\mu_n} \oplus 1) \coprod (\coprod_{k \in \mathbf{Z}/(n), k\neq 0} \mathbf{P}(N_{\infty,k}))$. Notice that by construction, the immersion $i_\infty^{\mu_n}$ factors through the closed subscheme $\mathbf{P}(N_{\infty,\mu_n} \oplus 1)$ and that the components $\mathbf{P}(N_{\infty,k})$, $k \neq 0$ do not meet Y (see the remark before Prop. 6.2). Thus the sequence $\mathfrak{K}_{\mu_n}$, obtained by taking the sequence in degree 0 associated to $\mathfrak{K}$, is a resolution of $\mathcal{O}_{Y_{\mu_n}}$ on $(\mathbf{P}(N_\infty \oplus 1))_{\mu_n}$.

Proposition 6.10 *The equality $\delta(i_\infty, \overline{\eta}) = 0$ holds.*

Proof: We shall need a formula comparing restrictions by i_∞ and direct-images by k_Y. This is the content of the next lemma.

Lemma 6.11 *The equality*

$$f_*^{\mu_n}(i_\infty^*(x)) \quad = \quad (f^{\mu_n} \circ k_Y^{\mu_n})_*(\lambda_{-1}(\overline{Q}_{\mu_n}^\vee)x) + \int_{Y_g} \mathrm{Td}(TY_g)\mathrm{ch}_g(i_\infty^*(x))R(N_{\infty,\mu_n})$$

$$+\int_{P_g}\mathrm{Td}(\overline{TP_g})T(\overline{\mathfrak{K}_{\mu_n}})\mathrm{ch}_g(x)$$
$$-\int_{Y_g}\mathrm{ch}_g(i_\infty^*(x))\mathrm{Td}^{-1}(\overline{N}_{\infty,\mu_n})\widetilde{\mathrm{Td}}(\overline{TY}_g,\overline{TP}_g|_{Y_g})$$

holds for any linear combination of hermitian bundles $x\in\widehat{K}_0^{\mu_n}(\mathbf{P}(N_\infty\oplus 1)_{\mu_n})$.

Proof (of Lemma 6.11): Let $x=\overline{V}$ and apply Prop. 5.1 to the sequence $\overline{\mathfrak{K}_{\mu_n}}\otimes\overline{V}$. Since both sides of the formula are additive, this yields the result. **Q.E.D.**

We now resume the proof of Prop. 6.10. We compute

$$\begin{aligned}
&(f^{\mu_n}\circ k_Y^{\mu_n})_*(\rho(k_Y^{\mu_n,*}(\overline{\eta}).\lambda_{-1}(\overline{Q}^\vee)))\\
=\;&(f^{\mu_n}\circ k_Y^{\mu_n})_*(\rho(k_Y^{\mu_n,*}(\overline{\eta}))\lambda_{-1}(\overline{Q}^\vee_{\mu_n})\lambda_{-1}(\oplus_{k\neq 0}\overline{Q}^\vee_k))\\
=\;&f_*^{\mu_n}(\rho(\overline{\eta})\lambda_{-1}(\overline{E_\infty}^\vee))-\int_{Y_g}\mathrm{Td}(TY_g)\mathrm{ch}_g(\overline{\eta}\cdot\lambda_{-1}(\overline{E_\infty}^\vee))R(N_\infty^{\mu_n})\\
&-\int_{P_g}\mathrm{Td}(\overline{TP_g})T(\overline{\mathfrak{K}_{\mu_n}})\mathrm{ch}_g(k_Y^*(\overline{\eta})\cdot\lambda_{-1}(\oplus_{k\neq 0}\overline{Q}^\vee_k))\\
&+\int_{Y_g}\mathrm{ch}_g(\overline{\eta}\cdot\lambda_{-1}(\overline{E_\infty}^\vee))\mathrm{Td}^{-1}(\overline{N}_{\infty,\mu_n})\widetilde{\mathrm{Td}}(\overline{TY}_g,\overline{TP}_g|_{Y_g}).
\end{aligned}$$

The proof is concluded, if we remember the definition of Td_g and Lemma 3.15. **Q.E.D.**

6.3.2 The deformation theorem

Let $i:Y\to X$ be an equivariant immersion of equivariant arithmetic varieties over Spec D. Let the terminology of subsection 5.2 hold.

Definition 6.12 *A metric h on W is said to be normal to the deformation if*

(a) *It is invariant and Kähler;*

(b) *the restriction $h|_{j_*^{\mu_n}(Y_{\mu_n}\times\mathbf{P}^1)}$ is a product $h'\times h''$, where h' is a Kähler metric on Y_{μ_n} and h'' a Kähler metric on $\mathbf{P}^1$;*

(c) *the intersections of $i_{X*}X$ with $j_*(Y\times\mathbf{P}^1)$ and of $i_{P*}P$ with $j_*(Y\times\mathbf{P}^1)$ are orthogonal at the fixed points.*

Lemma 6.13 *There exists a metric on W, which is normal to the deformation.*

Proof: The existence of such a metric is proved in [R1, Lemma 6.14] if the action on W is trivial. Start with a metric h', whose existence is predicted by [R1, Lemma 6.14] and consider the metric $\frac{1}{n}\sum_{a\in R_n} a^*(h')$. This one has the required properties. **Q.E.D.**

We shall suppose that the $\widetilde{\xi}_i$ are endowed with metrics such that Bismut's assumption (A) is satisfied and such that the sequence $0 \to \widetilde{\xi}_m \to \widetilde{\xi}_{m-1} \to \dots \to \widetilde{\xi}_0 \to 0$ is orthogonally equivariantly split on $\widetilde{X_g}$. The proof of Th. 6.1 follows from the next theorem, which reduces the proof of Th. 6.1 to the case treated in the last subsubsection.

Theorem 6.14 (Deformation theorem) *Let W be endowed with a metric, which is normal to the deformation. Then the equality $\delta(i_\infty,\overline{\eta}) = \delta(i,\overline{\eta})$ holds.*

Proof: We work on the space $W(i^{\mu_n})$; the complex points $W(i^{\mu_n})(\mathbf{C})$ of this space form an open subset and thus a connected component of $W(i)(\mathbf{C})_g$ (the other components are the sets $P(N_{\infty,k}\oplus 1)(\mathbf{C})$, $k \neq 0$). This can be seen from Cor. 2.10 and the description of $W(i)(\mathbf{C})_g$ as a set of R_n-invariant points. We shall thus often implicitly restrict currents with any wave front set from $W(i)(\mathbf{C})_g$ to $W(i^{\mu_n})(\mathbf{C})$. We shall write $P^0_{\mu_n}$ for the scheme-theoretic intersection of P_{μ_n} with $W(i^{\mu_n})$. This intersection is the space $P(N_{\infty,\mu_n}\oplus 1)$ by Cor. 2.10. We choose once and for all sections of $\mathcal{O}(X_{\mu_n})$, $\mathcal{O}(P^0_{\mu_n})$, $\mathcal{O}(\widetilde{X_{\mu_n}})$ whose zero-schemes are X_{μ_n}, $P^0_{\mu_n}$, $\widetilde{X_{\mu_n}}$. If D is a Cartier divisor on W_{μ_n} and the bundle $\mathcal{O}(D)$ carries a hermitian metric, we shall often write $\mathrm{Td}(\overline{D})$ for $\mathrm{Td}(\overline{\mathcal{O}}(D))$ and $c_1(\overline{D})$ for $c_1(\overline{\mathcal{O}}(D))$. We shall also write $\rho(\overline{\xi}.)$ for $\sum_{i=0}^m(-1)^i\rho(\overline{\xi}_i)$. For the proof of the following lemma, see [R1, Lemma 6.16].

Lemma 6.15 *There are hermitian metrics on $\mathcal{O}(X_{\mu_n})$, $\mathcal{O}(P^0_{\mu_n})$, $\mathcal{O}(\widetilde{X_{\mu_n}})$ such that the isometry $\overline{\mathcal{O}}(X_{\mu_n}) \simeq \overline{\mathcal{O}}(P^0_{\mu_n})\otimes\overline{\mathcal{O}}(\widetilde{X_{\mu_n}})$ holds and such that the restriction of $\overline{\mathcal{O}}(X_{\mu_n})$ to X_{μ_n} yields the metric of $N_{W(i^{\mu_n})/X_{\mu_n}}$, the restriction of $\overline{\mathcal{O}}(\widetilde{X_{\mu_n}})$ to $\widetilde{X_{\mu_n}}$ yields the metric of $N_{W(i^{\mu_n})/\widetilde{X_{\mu_n}}}$ and the restriction of $\overline{\mathcal{O}}(P^0_{\mu_n})$ to $P^0_{\mu_n}$ induces the metric of $N_{W(i^{\mu_n})/P^0_{\mu_n}}$.*

We shall from now on suppose that $\mathcal{O}(X_{\mu_n})$, $\mathcal{O}(\widetilde{X_{\mu_n}})$ and $\mathcal{O}(P^0_{\mu_n})$ are endowed with hermitian metrics satisfying the hypotheses of Lemma 6.15. We shall compare direct images of restrictions to X_{μ_n} and $P^0_{\mu_n}$, by applying Prop. 5.1 to the resolutions

$$0 \to \mathcal{O}(-X_{\mu_n}) \to \mathcal{O}_{W(i^{\mu_n})} \to i_{X_{\mu_n}*}\mathcal{O}_{X_{\mu_n}} \to 0, \tag{18}$$

$$0 \to \mathcal{O}(-P^0_{\mu_n}) \to \mathcal{O}_{W(i^{\mu_n})} \to i_{P^0_{\mu_n}*}\mathcal{O}_{P^0_{\mu_n}} \to 0, \tag{19}$$

$$0 \to \mathcal{O}(-\widetilde{X_{\mu_n}}) \to \mathcal{O}_{W(i^{\mu_n})} \to i_{\widetilde{X_{\mu_n}}*}\mathcal{O}_{\widetilde{X_{\mu_n}}} \to 0, \tag{20}$$

and to the resolution which is the tensor product of (19) and (20):

$$0 \to \mathcal{O}(-X_{\mu_n}) \otimes \mathcal{O}(-P^0_{\mu_n}) \to \mathcal{O}(-X_{\mu_n}) \oplus \mathcal{O}(-P^0_{\mu_n}) \to \mathcal{O}_{W(i^{\mu_n})} \to i_{P^0_{\mu_n} \cap \widetilde{X_{\mu_n}},*} \mathcal{O}_{P^0_{\mu_n} \cap \widetilde{X_{\mu_n}}} \to 0 \tag{21}$$

All four resolutions are Koszul resolutions and we shall denote the Euler-Green currents of the three first ones by by $g_{X_{\mu_n}}$, $g_{P^0_{\mu_n}}$ and $g_{\widetilde{X_{\mu_n}}}$ respectively (see after Lemma 3.15). By [BGS5, Th. 2.7, p. 271], the Euler-Green current of the fourth one is then the current $c_1(\overline{\mathcal{O}}(P^0_{\mu_n}))g_{\widetilde{X_{\mu_n}}} + \delta_{\widetilde{X_{\mu_n}}} g_{P^0_{\mu_n}}$. Note now the equality

$$\rho(\overline{\overline{\xi}}.)((1-\overline{\mathcal{O}}(-X_{\mu_n}))-(1-\overline{\mathcal{O}}(-P^0_{\mu_n}))-(1-\overline{\mathcal{O}}(-\widetilde{X_{\mu_n}}))+(1-\overline{\mathcal{O}}(-P^0_{\mu_n}))(1-\overline{\mathcal{O}}(-\widetilde{X_{\mu_n}}))) = 0$$

in $\widehat{K}_0^{\mu_n}(W(i^{\mu_n}))$. We shall apply the push-forward map to both sides of the equality and show that the resulting equality is equivalent to the statement of the theorem. Using the non-equivariant version of Prop. 5.1, we compute that the equality implies that

$$\begin{aligned}
&p^{\mu_n}(\rho(\overline{\overline{\xi}}.)) - \int_{X_{\mu_n}} \mathrm{ch}_g(\rho(\xi.))R(N_{W(i^{\mu_n})/X_{\mu_n}})\mathrm{Td}(TX_{\mu_n}) \\
&- \int_{W(i^{\mu_n})} \mathrm{ch}_g(\overline{\overline{\xi}}.)\mathrm{Td}(\overline{TW(i^{\mu_n})})\mathrm{Td}^{-1}(\overline{X_{\mu_n}})g_{X_{\mu_n}} \\
&+ \int_{X_{\mu_n}} \mathrm{ch}_g(\rho(\overline{\xi}.))\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/X_{\mu_n}})\widetilde{\mathrm{Td}}(\overline{TX_{\mu_n}}, \overline{TW(i^{\mu_n})}|_{X_{\mu_n}}) \\
&-(f^{\mu_n} \circ k^{\mu_n})_*(\rho(\overline{\xi^\infty_.})) - \int_{P^0_{\mu_n}} \mathrm{ch}_g(\xi^\infty_.)R(N_{W(i^{\mu_n})/P^0_{\mu_n}})\mathrm{Td}(TP^0_{\mu_n}) \\
&- \int_{W(i^{\mu_n})} \mathrm{Td}(\overline{TW(i^{\mu_n})})\mathrm{ch}_g(\overline{\overline{\xi}}.)\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})g_{P^0_{\mu_n}} \\
&+ \int_{P^0_{\mu_n}} \mathrm{ch}_g(\overline{\xi^\infty_.})\widetilde{\mathrm{Td}}(\overline{TP^0_{\mu_n}}, \overline{TW(i^{\mu_n})}|_{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/P^0_{\mu_n}}) \\
&- \int_{W(i^{\mu_n})} \mathrm{Td}(\overline{TW(i^{\mu_n})})\mathrm{ch}_g(\overline{\overline{\xi}}.)\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})g_{\widetilde{X_{\mu_n}}} \\
&+ \int_{W(i^{\mu_n})} \mathrm{Td}(\overline{TW(i^{\mu_n})})\mathrm{ch}_g(\overline{\overline{\xi}}.)\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}}) \\
&\cdot[c_1(\overline{\mathcal{O}}(P^0_{\mu_n}))g_{\widetilde{X_{\mu_n}}} + \delta_{\widetilde{X_{\mu_n}}} g_{P^0_{\mu_n}}] = 0
\end{aligned}$$

(notice that we only used Prop. 5.1 in the non-equivariant setting here) where we used the remark after Lemma 3.15. We dropped all the terms where an integral is taken over $\widetilde{X_{\mu_n}}$, since $\mathrm{ch}_g(\overline{\overline{\xi}}.)$ vanishes on $\widetilde{X_{\mu_n}}$. For the same reason, we have

$$\int_{W(i^{\mu_n})} \mathrm{Td}(\overline{TW(i^{\mu_n})})\mathrm{ch}_g(\overline{\overline{\xi}}.)\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})\delta_{\widetilde{X_{\mu_n}}} g_{P^0_{\mu_n}} = 0.$$

For the next step, we shall need an Atiyah-Segal-Singer type formula for immersions. Let $j : M' \to M$ be an equivariant closed immersion of R_n-equivariant complex manifolds. Let $H(M_{R_n})$ be the complex de Rahm cohomology of the fixed point submanifold of M and let $K_0^{R_n}$ denote the K_0-theory of holomorphic R_n-equivariant vector bundles. In the next theorem, $j_*^{R_n} : H({M'}_{R_n}) \to H(M_{R_n})$ will stand for the push-forward in cohomology associated to j^{R_n} and j_* for the push-forward in $K_0^{R_n}$-theory.

Theorem 6.16 *Let N be the normal bundle of j. The equality*

$$j_*^{R_n}(\mathrm{Td}_g^{-1}(N)\mathrm{ch}_g(x)) = \mathrm{ch}_g(j_*(x))$$

holds in $H(M_{R_n})$, for all $x \in K_0^{R_n}(M')$.

For the proof, see [FL, p. 191 and p. 195]. Recall that we denoted by i the immersion $Y \to X$ and by i_∞ the immersion $Y \to \mathbf{P}(N_\infty \oplus 1)$ of the standard model. Using the projection formula in cohomology and Th. 6.16, we compute

$$\begin{aligned} & i_{X*}^{\mu_n}(\mathrm{ch}_g(\rho(\xi.))R(N_{W(i^{\mu_n})/X_{\mu_n}})) \\ & \qquad = i_{X*}^{\mu_n}(R(N_{W(i^{\mu_n})/X_{\mu_n}})i_*^{\mu_n}(\mathrm{Td}_g^{-1}(N_{X/Y})\mathrm{ch}_g(\eta))) \\ & \qquad = (i_X^{\mu_n} \circ i^{\mu_n})_*(R(N_{W(i^{\mu_n})/X_{\mu_n}})\mathrm{Td}_g^{-1}(N_{X/Y})\mathrm{ch}_g(\eta)). \end{aligned}$$

Now notice that the restriction of $N_{W(i^{\mu_n})/X_{\mu_n}}$ to Y_{μ_n} is trivial by construction and thus the last expression vanishes. An entirely analogous reasoning applies to the immersion $i_{P*}^{\mu_n}$ and we get

$$i_{P*}^{\mu_n}(\mathrm{ch}_g(\rho(\xi.^{\infty}))R(N_{W(i^{\mu_n})/P^0_{\mu_n}})) = 0.$$

Thus, we are left with the equality

$$\begin{aligned} & p_*^{\mu_n}(\rho(\overline{\xi}.)) - \int_{W(i^{\mu_n})} \mathrm{Td}(\overline{TW(i^{\mu_n})})\mathrm{ch}_g(\overline{\widetilde{\xi}}.)\mathrm{Td}^{-1}(\overline{X_{\mu_n}})g_{X_{\mu_n}} \\ & + \int_{X_{\mu_n}} \mathrm{ch}_g(\overline{\xi}.)\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/X_{\mu_n}})\widetilde{\mathrm{Td}}(\overline{TX_{\mu_n}}, \overline{TW(i^{\mu_n})}|_{X_{\mu_n}}) \\ & -\big(\ (f^{\mu_n} \circ k^{\mu_n})_*(\rho(\overline{\xi^{\infty}}.)) - \int_{W(i^{\mu_n})} \mathrm{Td}(\overline{TW(i^{\mu_n})})\mathrm{ch}_g(\overline{\widetilde{\xi}}.)\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})g_{P^0_{\mu_n}} \\ & + \int_{P^0_{\mu_n}} \mathrm{ch}_g(\overline{\xi^{\infty}}.)\widetilde{\mathrm{Td}}(\overline{TP^0_{\mu_n}}, \overline{TW(i^{\mu_n})}|_{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/P^0_{\mu_n}})\big) \\ & - \int_{W(i^{\mu_n})} \mathrm{Td}(\overline{TW(i^{\mu_n})})\mathrm{ch}_g(\overline{\widetilde{\xi}}.)\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})g_{\widetilde{X_{\mu_n}}} \\ & + \int_{W(i^{\mu_n})} \mathrm{Td}(\overline{TW(i^{\mu_n})})\mathrm{ch}_g(\rho(\overline{\widetilde{\xi}}.))\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X}})c_1(\overline{\mathcal{O}}(P^0_{\mu_n}))g_{\widetilde{X_{\mu_n}}} = 0. \end{aligned}$$

Gathering terms, we get

$$
\begin{aligned}
&p_*^{\mu_n}(\rho(\overline{\xi}.)) - (f^{\mu_n}\circ k^{\mu_n})_*(\rho(\overline{\xi^\infty}.)) = \\
&\quad \int_{W(i^{\mu_n})} \mathrm{Td}(\overline{TW(i^{\mu_n})})\mathrm{ch}_g(\rho(\overline{\overline{\xi}}.))\big(\ \mathrm{Td}^{-1}(\overline{X_{\mu_n}})g_{X_{\mu_n}} - \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})g_{P^0_{\mu_n}} \\
&\quad -\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})g_{\widetilde{X_{\mu_n}}} + \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{P^0_{\mu_n}})g_{\widetilde{X_{\mu_n}}}\big) \\
&\quad -\int_{X_{\mu_n}} \mathrm{ch}_g(\rho(\overline{\xi}.))\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/X_{\mu_n}})\widetilde{\mathrm{Td}}(\overline{TX_{\mu_n}},\overline{TW(i^{\mu_n})}|_{X_{\mu_n}}) \\
&\quad +\int_{P^0_{\mu_n}} \mathrm{ch}_g(\rho(\overline{\xi^\infty}.))\widetilde{\mathrm{Td}}(\overline{TP^0_{\mu_n}},\overline{TW(i^{\mu_n})}|_{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/P^0_{\mu_n}}). \qquad (22)
\end{aligned}
$$

Using the definition of the singular Bott-Chern current, we compute

$$
\begin{aligned}
&\mathrm{ch}_g(\rho(\overline{\overline{\xi}}.))\big(\mathrm{Td}^{-1}(\overline{X_{\mu_n}})g_{X_{\mu_n}} - \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})g_{P^0_{\mu_n}} - \mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})g_{\widetilde{X_{\mu_n}}} \\
&\qquad +\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{P^0_{\mu_n}})g_{\widetilde{X_{\mu_n}}}\big) \\
&= \ -(\frac{\overline{\partial}\partial}{2\pi i}T_g(\overline{\overline{\xi}}) - \mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbf{P}^1})\delta_{Y_{\mu_n}\times\mathbf{P}^1}). \\
&\qquad \big(\mathrm{Td}^{-1}(\overline{X_{\mu_n}})g_{X_{\mu_n}} - \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})g_{P^0_{\mu_n}} - \mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})g_{\widetilde{X_{\mu_n}}} \\
&\qquad +\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{P^0_{\mu_n}})g_{\widetilde{X_{\mu_n}}}\big) \\
&= \ -\big(\mathrm{Td}^{-1}(\overline{X_{\mu_n}})\frac{\overline{\partial}\partial}{2\pi i}T_g(\overline{\overline{\xi}})g_{X_{\mu_n}} - \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\frac{\overline{\partial}\partial}{2\pi i}T_g(\overline{\overline{\xi}})g_{P^0_{\mu_n}} \\
&\qquad -\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})\frac{\overline{\partial}\partial}{2\pi i}T_g(\overline{\overline{\xi}})g_{\widetilde{X_{\mu_n}}} + \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})\frac{\overline{\partial}\partial}{2\pi i}T_g(\overline{\overline{\xi}})c_1(\overline{P^0_{\mu_n}})g_{\widetilde{X_{\mu_n}}}\big) \\
&\qquad +\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbf{P}^1})\delta_{Y_{\mu_n}\times\mathbf{P}^1}.\big(\mathrm{Td}^{-1}(\overline{X_{\mu_n}})g_{X_{\mu_n}} \\
&\qquad -\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})g_{P^0_{\mu_n}} - \mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})g_{\widetilde{X_{\mu_n}}} + \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\widetilde{X_{\mu_n}})c_1(\overline{P^0_{\mu_n}})g_{\widetilde{X_{\mu_n}}}\big).
\end{aligned}
$$

The next lemma will evaluate the first part of the last expression.

Lemma 6.17 *The equality*

$$
\begin{aligned}
&\mathrm{Td}^{-1}(\overline{X_{\mu_n}})\frac{\overline{\partial}\partial}{2\pi i}T_g(\overline{\overline{\xi}})g_{X_{\mu_n}} - \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\frac{\overline{\partial}\partial}{2\pi i}T_g(\overline{\overline{\xi}})g_{P^0_{\mu_n}} \\
&\qquad -\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})\frac{\overline{\partial}\partial}{2\pi i}T_g(\overline{\overline{\xi}})g_{\widetilde{X_{\mu_n}}} + \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})\frac{\overline{\partial}\partial}{2\pi i}T_g(\overline{\overline{\xi}})c_1(\overline{P^0_{\mu_n}})g_{\widetilde{X_{\mu_n}}} \\
&= \ T_g(\overline{\overline{\xi}})\big(\mathrm{Td}^{-1}(\overline{X_{\mu_n}})\delta_{X_{\mu_n}} - \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\delta_{P^0_{\mu_n}} - \mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})\delta_{\widetilde{X_{\mu_n}}} \\
&\qquad +\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{P^0_{\mu_n}})\delta_{\widetilde{X_{\mu_n}}}\big)
\end{aligned}
$$

holds.

Proof (of Lemma 6.17): For the proof, we shall need the following identity. Let $\overline{E}$ be a (non-equivariant) hermitian bundle of rank r. The equality of forms

$$\mathrm{Td}(\overline{E})\mathrm{ch}(\lambda_{-1}(\overline{E}^{\vee})) = c_r(\overline{E})$$

holds. This is proved in [R1, Lemma 6.19]. Using (12), we compute that the left hand of the equality gives

$$\begin{aligned}
&T_g(\overline{\overline{\xi}})\big(\mathrm{Td}^{-1}(\overline{X_{\mu_n}})(\delta_{X_{\mu_n}} - c_1(\overline{X_{\mu_n}})) - \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})(\delta_{P^0_{\mu_n}}\\
&-c_1(\overline{P^0_{\mu_n}})) - \mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})(\delta_{\widetilde{X_{\mu_n}}} - c_1(\overline{\widetilde{X_{\mu_n}}}))\\
&+\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{P^0_{\mu_n}})(\delta_{\widetilde{X_{\mu_n}}} - c_1(\overline{\widetilde{X_{\mu_n}}}))\big)
\end{aligned}$$

Using the above identity, we compute that

$$\begin{aligned}
\mathrm{Td}^{-1}(\overline{X_{\mu_n}})c_1(\overline{X_{\mu_n}}) - \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})c_1(\overline{P^0_{\mu_n}}) - \mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{\widetilde{X_{\mu_n}}})\\
+\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{P^0_{\mu_n}}) = 0 \quad (23)
\end{aligned}$$

which completes the proof. **Q.E.D.**

Lemma 6.18 *The equality*

$$\begin{aligned}
&\int_{W(i^{\mu_n})} \mathrm{Td}(\overline{TW(i^{\mu_n})})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbf{P}^1})\delta_{Y_{\mu_n}\times\mathbf{P}^1}\\
&\cdot\big(\mathrm{Td}^{-1}(\overline{X_{\mu_n}})g_{X_{\mu_n}} - \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})g_{P^0_{\mu_n}} - \mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})g_{\widetilde{X_{\mu_n}}}\\
&+\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{P^0_{\mu_n}})g_{\widetilde{X_{\mu_n}}})\\
= &\int_{Y_{\mu_n}} \mathrm{ch}_g(\lambda_{-1}(\overline{E}^{\vee}))\mathrm{Td}^{-1}(\overline{N}_{\mu_n})\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}(\overline{TY}_g, \overline{TX}_g|_{Y_g})\\
&-\int_{Y_{\mu_n}} \mathrm{ch}_g(\lambda_{-1}(\overline{E_\infty}^{\vee}))\mathrm{Td}^{-1}(\overline{N_{\infty,\mu_n}})\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}(\overline{TY}_g, \overline{TP}_g|_{Y_g})\\
&+f_*^{\mu_n}(\lambda_{-1}(\overline{E}^{\vee})\rho(\overline{\eta})) - f_*^{\mu_n}(\lambda_{-1}(\overline{E_\infty}^{\vee})\rho(\overline{\eta}))
\end{aligned}$$

holds.

Proof (of Lemma 6.18): Using the definition of $\widetilde{\mathrm{Td}}$ (see after Cor. 3.10) and (12), we can rewrite the left side of the equality as

$$\begin{aligned}
&\int_{W(i^{\mu_n})} \big(\frac{\overline{\partial}\partial}{2\pi i}\widetilde{\mathrm{Td}}(\overline{T(Y_{\mu_n}\times\mathbf{P}^1)}, \overline{TW(i^{\mu_n})}|_{Y_{\mu_n}\times\mathbf{P}^1})\\
&+\mathrm{Td}_g(\overline{N}_{W/Y\times\mathbf{P}^1})\mathrm{ch}_g(\lambda_{-1}(\overline{N}^{\vee}_{W/Y\times\mathbf{P}^1}))\mathrm{ch}_g^{-1}(\lambda_{-1}(\overline{N}^{\vee}_{W(i^{\mu_n})/Y_{\mu_n}\times\mathbf{P}^1}))
\end{aligned}$$

$$
\begin{aligned}
&\cdot \mathrm{Td}(\overline{T(Y_{\mu_n}\times \mathbf{P}^1)})\big).\big(\mathrm{Td}^{-1}(\overline{X_{\mu_n}})g_{X_{\mu_n}} - \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})g_{P^0_{\mu_n}} \\
&-\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})g_{\widetilde{X_{\mu_n}}} + \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{P^0_{\mu_n}})g_{\widetilde{X_{\mu_n}}}\big) \\
&\cdot \delta_{Y_{\mu_n}\times \mathbf{P}^1}\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times \mathbf{P}^1}) \\
= &\int_{W(i^{\mu_n})}\widetilde{\mathrm{Td}}(\overline{T(Y_{\mu_n}\times \mathbf{P}^1)},\overline{TW(i^{\mu_n})}|_{Y_{\mu_n}\times \mathbf{P}^1})\delta_{Y_{\mu_n}\times \mathbf{P}^1}\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times \mathbf{P}^1}). \\
&\cdot\big(\mathrm{Td}^{-1}(\overline{X_{\mu_n}})(\delta_{X_{\mu_n}} - c_1(\overline{X_{\mu_n}})) - \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})(\delta_{P^0_{\mu_n}} - c_1(\overline{P^0_{\mu_n}})) \\
&-\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})(\delta_{\widetilde{X_{\mu_n}}} - c_1(\overline{\widetilde{X_{\mu_n}}})) + \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{P^0_{\mu_n}})(\delta_{\widetilde{X_{\mu_n}}} - c_1(\overline{\widetilde{X_{\mu_n}}}))\big) \\
&+\int_{W(i^{\mu_n})}\mathrm{ch}_g(\lambda_{-1}(\overline{N}^\vee_{W/Y\times \mathbf{P}^1}))\mathrm{ch}_g^{-1}(\lambda_{-1}(\overline{N}^\vee_{W(i^{\mu_n})/Y_{\mu_n}\times \mathbf{P}^1})) \\
&\cdot \mathrm{Td}_g(\overline{N}_{W/Y\times \mathbf{P}^1})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times \mathbf{P}^1})\mathrm{Td}(\overline{T(Y_{\mu_n}\times \mathbf{P}^1)}) \\
&\cdot \mathrm{ch}_g(p_Y^*\overline{\eta})\delta_{Y_{\mu_n}\times \mathbf{P}^1}.\big(\mathrm{Td}^{-1}(\overline{X_{\mu_n}})g_{X_{\mu_n}} - \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})g_{P^0_{\mu_n}} - \mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})g_{\widetilde{X_{\mu_n}}} \\
&+\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{P^0_{\mu_n}})g_{\widetilde{X_{\mu_n}}}\big).
\end{aligned}
$$

By Def. 6.12, we have $\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times \mathbf{P}^1})|_{Y_{\infty,\mu_n}} = \mathrm{Td}_g^{-1}(\overline{N_\infty})$, $\mathrm{Td}(\overline{P^0_{\mu_n}})|_{Y_{\infty,\mu_n}} = 1$ and $\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times \mathbf{P}^1})|_{Y_{0,\mu_n}} = \mathrm{Td}_g^{-1}(\overline{N_0})$, $\mathrm{Td}(\overline{X_{\mu_n}})|_{Y_{0,\mu_n}} = 1$. Furthermore, recall that $\delta_{Y_{\mu_n}\times \mathbf{P}^1}\wedge\delta_{\widetilde{X_{\mu_n}}} = 0$, $\delta_{Y_{\mu_n}\times \mathbf{P}^1}\wedge\delta_{P^0_{\mu_n}} = \delta_{Y_{\infty,\mu_n}}$, $\delta_{Y_{\mu_n}\times \mathbf{P}^1}\wedge\delta_{X_{\mu_n}} = \delta_{Y_{0,\mu_n}}$. With these equalities in hand and (23), we can evaluate the expression after the last equality as

$$
\begin{aligned}
&\int_{Y_{\mu_n}}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N_0})\widetilde{\mathrm{Td}}(\overline{T(Y_{\mu_n}\times \mathbf{P}^1)},\overline{TW(i^{\mu_n})}|_{Y_{\mu_n}\times \mathbf{P}^1}) \\
&-\int_{Y_{\mu_n}}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N_\infty})\widetilde{\mathrm{Td}}(\overline{T(Y_{\mu_n}\times \mathbf{P}^1)},\overline{TW(i^{\mu_n})}|_{Y_{\mu_n}\times \mathbf{P}^1}) \\
&+\int_{Y_{\mu_n}\times \mathbf{P}^1}\mathrm{ch}_g(\lambda_{-1}(\overline{N}^\vee_{W/Y\times \mathbf{P}^1}))\mathrm{ch}_g^{-1}(\lambda_{-1}(\overline{N}^\vee_{W(i^{\mu_n})/Y_{\mu_n}\times \mathbf{P}^1})) \\
&\cdot \mathrm{Td}(\overline{T(Y_{\mu_n}\times \mathbf{P}^1)}))\mathrm{ch}_g(p_Y^*(\overline{\eta}))\big(\mathrm{Td}^{-1}(\overline{X_{\mu_n}})g_{X_{\mu_n}} - \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})g_{P^0_{\mu_n}} \\
&-\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})g_{\widetilde{X_{\mu_n}}} + \mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{P^0_{\mu_n}})g_{\widetilde{X_{\mu_n}}}\big).
\end{aligned}
$$

Consider now that there is an exact commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & TY_g & \to & TP^0_g & \to & N_{\mu_n,\mathbf{C}} & \to & 0 \\ & & \downarrow Id & & \downarrow & & \downarrow & & \\ 0 & \to & T(Y_g\times\mathbf{P}^1_{\mathbf{C}}) & \to & TW_g & \to & N_{W_g/Y_g\times\mathbf{P}^1_{\mathbf{C}}} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & T\mathbf{P}^1_{\mathbf{C}} & \overset{Id}{\simeq} & T\mathbf{P}^1_{\mathbf{C}} & & 0 & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

where the various maps are induced by the corresponding immersions of complex manifolds. To see this, notice first that the intersection of P_g with $(Y_g\times\mathbf{P}^1_{\mathbf{C}})$ is transversal; this follows from the fact that the map $q_{\mathbf{C}}$ is a submersion and thus the map $q_{W,\mathbf{C}}$ is a submersion on a neighborhood of $(Y_g\times\mathbf{P}^1_{\mathbf{C}})$. This implies the natural map $N_{\mu_n,\mathbf{C}}\to N_{W_g/Y_g\times\mathbf{P}^1_{\mathbf{C}}}$ is an isomorphism and proves our claim. Furthermore, notice that the first and second non-vanishing column of the diagram are split complexes. For the first one, this follows from the existence of the immersion $Y_g\to Y_g\times\mathbf{P}^1$ at ∞ and the second one is automatically split if the first one is. From the orthogonality statement in Def. 6.12 and the double complex formula Th. 3.14 applied to the invariant subdiagram (obtained by restricting all the bundles to the corresponding fixed point sets and taking their invariant subbundles) of the above diagram, we deduce that $\widetilde{\mathrm{Td}}(\overline{T(Y_{\mu_n}\times\mathbf{P}^1)},\overline{TW(i^{\mu_n})}|_{Y_{\mu_n}\times\mathbf{P}^1})|_{Y_{\infty,\mu_n}}=\widetilde{\mathrm{Td}}(\overline{TY_{\mu_n}},\overline{TP^0_{\mu_n}}|_{Y_{\mu_n}})$. A completly similar argument shows that $\widetilde{\mathrm{Td}}(\overline{T(Y_{\mu_n}\times\mathbf{P}^1)},\overline{TW(i^{\mu_n})}|_{Y_{\mu_n}\times\mathbf{P}^1})|_{Y_{0,\mu_n}}=\widetilde{\mathrm{Td}}(\overline{TY_{\mu_n}},\overline{TX_{\mu_n}}|_{Y_{\mu_n}})$. Furthermore, we can compute

$$\begin{aligned}&\int_{Y_{\mu_n}\times\mathbf{P}^1}\mathrm{ch}_g(\lambda_{-1}(\overline{N}^{\vee}_{W/Y\times\mathbf{P}^1}))\mathrm{ch}_g^{-1}(\lambda_{-1}(\overline{N}^{\vee}_{W(i^{\mu_n})/Y_{\mu_n}\times\mathbf{P}^1}))\mathrm{Td}(\overline{T(Y_{\mu_n}\times\mathbf{P}^1)})\\ &\cdot\mathrm{ch}_g(p_Y^*\overline{\eta})\big(\mathrm{Td}^{-1}(\overline{X_{\mu_n}})g_{X_{\mu_n}}-\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})g_{P^0_{\mu_n}}-\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})g_{\widetilde{X_{\mu_n}}}\\ &+\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\mathrm{Td}^{-1}(\overline{\widetilde{X_{\mu_n}}})c_1(\overline{P^0_{\mu_n}})g_{\widetilde{X_{\mu_n}}}\big)\\ =\ &f^{\mu_n}_*(\lambda_{-1}(\overline{E_0}^{\vee})\rho(\overline{\eta}))-f^{\mu_n}_*(\lambda_{-1}(\overline{E_\infty}^{\vee})\rho(\overline{\eta})).\end{aligned}\tag{24}$$

To see this, notice that that there are natural isomorphisms $j_0^{\mu_n *}\mathcal{O}(-X_{\mu_n})\simeq\mathcal{O}(-Y_{\mu_n,0})$ and $j_\infty^{\mu_n *}\mathcal{O}(-P^0_{\mu_n})\simeq\mathcal{O}(-Y_{\mu_n,\infty})$. Thus we have resolutions

$$0\to j^{\mu_n *}\mathcal{O}(-X_{\mu_n})\to\mathcal{O}_{Y_{\mu_n}\times\mathbf{P}^1}\to i_{Y_{\mu_n},0*}\mathcal{O}_{Y_{\mu_n}}\to 0$$

and

$$0\to j^{\mu_n *}\mathcal{O}(-P^0_{\mu_n})\to\mathcal{O}_{Y\times\mathbf{P}^1}\to i_{Y_{\mu_n},\infty*}\mathcal{O}_{Y_{\mu_n}}\to 0$$

where $i_{Y_{\mu_n},0}$ is the embedding $Y_{\mu_n}\to Y_{\mu_n}\times\mathbf{P}^1_D$ at 0 and $i_{Y_{\mu_n},\infty}$ is the embedding $Y_{\mu_n}\to Y_{\mu_n}\times\mathbf{P}^1_D$ at ∞. The normal sequences of $i_{Y_{\mu_n},0}$ and $i_{Y_{\mu_n},\infty}$ are clearly split orthogonal, the normal bundles of $i_{Y_{\mu_n},0}$ and $i_{Y_{\mu_n},\infty}$ are trivial and the bundle $j^{\mu_n *}\mathcal{O}(-\widetilde{X_{\mu_n}})$ is trivial. Thus, if we apply Th. 3.11 to the equality

$$\begin{aligned}\rho(p_Y^*(\overline{\eta})\lambda_{-1}(\overline{N}^{\vee}_{W/Y\times\mathbf{P}^1})\lambda_{-1}^{-1}(\overline{N}^{\vee}_{W(i^{\mu_n})/Y_{\mu_n}\times\mathbf{P}^1}))\big((1-j^{\mu_n *}\overline{\mathcal{O}}(-X_{\mu_n}))\\ -(1-j^{\mu_n *}\overline{\mathcal{O}}(-P^0_{\mu_n}))-(1-j^{\mu_n *}\overline{\mathcal{O}}(-\widetilde{X_{\mu_n}}))\\ +(1-j^{\mu_n *}\overline{\mathcal{O}}(-P^0_{\mu_n}))(1-j^{\mu_n *}\overline{\mathcal{O}}(-\widetilde{X_{\mu_n}}))\big)=0\end{aligned}$$

as at the beginning of the proof of the deformation theorem, we obtain (24). **Q.E.D.**

The next lemma is concerned with the two last lines of (22).

Lemma 6.19 *The equalities*

$$\begin{aligned}&\int_{X_{\mu_n}}\mathrm{ch}_g(\rho(\overline{\xi}.))\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/X_{\mu_n}})\widetilde{\mathrm{Td}}(\overline{TX_{\mu_n}},\overline{TW(i^{\mu_n})}|_{X_{\mu_n}})\\ =\ &\int_{X_{\mu_n}}T_g(\overline{\xi})\mathrm{Td}(\overline{TX_{\mu_n}})-\int_{X_{\mu_n}}T_g(\overline{\xi})\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/X_{\mu_n}})\mathrm{Td}(\overline{TW(i^{\mu_n})})\end{aligned}$$

and

$$\begin{aligned}&\int_{P^0_{\mu_n}}\mathrm{ch}_g(\rho(\overline{\xi^{\infty}}.))\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/P^0_{\mu_n}})\widetilde{\mathrm{Td}}(\overline{TP^0_{\mu_n}},\overline{TW(i^{\mu_n})}|_{P^0_{\mu_n}})\\ =\ &\int_{P^0_{\mu_n}}T_g(\overline{\xi^{\infty}})\mathrm{Td}(\overline{TP^0_{\mu_n}})-\int_{P^0_{\mu_n}}T_g(\overline{\xi^{\infty}})\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/P^0_{\mu_n}})\mathrm{Td}(\overline{TW(i^{\mu_n})})\end{aligned}$$

hold.

Proof (of Lemma 6.19): We shall only prove the second one, the proof of the first one being similar. Using the definition of the singular Bott-Chern current, we compute

$$\begin{aligned}&\int_{P^0_{\mu_n}}\mathrm{ch}_g(\rho(\overline{\xi^{\infty}}.))\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/P^0_{\mu_n}})\widetilde{\mathrm{Td}}(\overline{TP^0_{\mu_n}},\overline{TW(i^{\mu_n})}|_{P^0_{\mu_n}})\\ =\ &-\int_{P^0_{\mu_n}}(\frac{\overline{\partial}\partial}{2\pi i}T_g(\overline{\xi^{\infty}})-\mathrm{Td}_g^{-1}(\overline{N_{\infty}})\mathrm{ch}_g(\overline{\eta})\delta_{Y_{\mu_n}})\\ &\cdot\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/P^0_{\mu_n}})\widetilde{\mathrm{Td}}(\overline{TP^0_{\mu_n}},\overline{TW(i^{\mu_n})}|_{P^0_{\mu_n}})\\ =\ &-\int_{P^0_{\mu_n}}(\frac{\overline{\partial}\partial}{2\pi i}T_g(\overline{\xi_{\infty}}))\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/P^0_{\mu_n}})\widetilde{\mathrm{Td}}(\overline{TP^0_{\mu_n}},\overline{TW(i^{\mu_n})}|_{P^0_{\mu_n}})\\ &+\int_{P^0_{\mu_n}}\mathrm{Td}_g^{-1}(\overline{N_{\infty}})\mathrm{ch}_g(\overline{\eta})\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/P^0_{\mu_n}})\widetilde{\mathrm{Td}}(\overline{TP^0_{\mu_n}},\overline{TW(i^{\mu_n})}|_{P^0_{\mu_n}}).\end{aligned}$$

The integral after the last + sign vanishes, since the normal sequence of $P^0_{\mu_n}$ in $W(i^{\mu_n})$ is split orthogonal on $Y_{\mu_n}\times\infty$. Applying (12), we obtain that the last expression is equal to

$$\int_{P^0_{\mu_n}} T_g(\overline{\xi^\infty})\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/P^0_{\mu_n}})$$
$$\cdot\big(\mathrm{Td}(\overline{N}_{W(i^{\mu_n})/P^0_{\mu_n}})\mathrm{Td}(\overline{TP^0_{\mu_n}})-\mathrm{Td}(\overline{TW(i^{\mu_n})})\big)$$

which is the result. **Q.E.D.**

If we combine (22) with the three last lemmata in their order of appearance, we get

$$p^{\mu_n}_*(\rho(\overline{\xi}.))-(f^{\mu_n}\circ k^{\mu_n})_*(\rho(\overline{\xi^\infty}.))$$
$$= -\int_{W(i^{\mu_n})}\mathrm{Td}(\overline{TW(i^{\mu_n})})T_g(\overline{\overline{\xi}})\big(\mathrm{Td}^{-1}(\overline{X_{\mu_n}})\delta_{X_{\mu_n}}$$
$$+\mathrm{Td}^{-1}(\overline{P^0_{\mu_n}})\delta_{P^0_{\mu_n}}\big)-\int_{Y_{\mu_n}}\mathrm{ch}_g(\lambda_{-1}(\overline{E}^\vee))\mathrm{Td}^{-1}(\overline{N}_{\mu_n})\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}(\overline{TY}_g,\overline{TX}_g|_{Y_g})$$
$$-\int_{Y_{\mu_n}}\mathrm{ch}_g(\lambda_{-1}(\overline{E_\infty}^\vee))\mathrm{Td}^{-1}(\overline{N_{\infty,\mu_n}})\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}(\overline{TY}_g,\overline{TP}_g|_{Y_g})$$
$$+f^{\mu_n}_*(\lambda_{-1}(\overline{E}^\vee)\rho(\overline{\eta}))-f^{\mu_n}_*(\lambda_{-1}(\overline{E_\infty}^\vee)\rho(\overline{\eta}))-\int_{X_{\mu_n}}T_g(\overline{\xi})\mathrm{Td}(\overline{TX_{\mu_n}})$$
$$+\int_{X_{\mu_n}}T_g(\overline{\xi})\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/X_{\mu_n}})\mathrm{Td}(\overline{TW(i^{\mu_n})})+\int_{P^0_{\mu_n}}T_g(\overline{\xi^\infty})\mathrm{Td}(\overline{TP^0_{\mu_n}})$$
$$-\int_{P^0_{\mu_n}}T_g(\overline{\xi^\infty})\mathrm{Td}^{-1}(\overline{N}_{W(i^{\mu_n})/P^0_{\mu_n}})\mathrm{Td}(\overline{TW(i^{\mu_n})})$$

Notice that we dropped the integrals involving $\delta_{\widetilde{X_{\mu_n}}}$, because $T_g(\overline{\overline{\xi}})$ vanishes on $\widetilde{X_{\mu_n}}$. This is due to Th. 3.4, Cor. 3.10 and to the fact that the restriction to $\widetilde{X_{\mu_n}}$ of the complex of hermitian bundles $\overline{\overline{\xi}}.$ is by construction a split orthogonal complex. From this equality and the fact that the integral involving the R-genus contributes the same quantity in both $\delta(i_\infty,\overline{\eta})$ and $\delta(i,\overline{\eta})$ (because the normal bundle of i is by construction equivariantly isomorphic to the normal bundle of i_∞), the deformation theorem follows.

7 Proof of the main theorem

In this section, we shall prove Th. 4.4. To do this, we first prove the compatibility of the error term of Th. 4.4 with a change of Kähler metrics; here the anomaly formula Th. 3.6 is used. We then prove the compatibility of the error term with immersions (Th. 7.4); here Prop. 5.1 and Th. 6.1 both play an essential role. Thirdly, we prove that Th. 4.4 holds for projective spaces; to do this Th. 7.4 is applied to a special immersion. Finally we combine the result for projective spaces and Th. 7.4 to conclude. The notation is the same as in section 3. Let $y \in \widehat{K}_0^{\mu_n}(Y)$. We define the error term of Th. 4.4 as follows:

$$\delta(f, y, \omega_Y) := f_*(y) - f_*^{\mu_n}(\lambda_{-1}^{-1}(\overline{N}_{Y/Y_{\mu_n}}^\vee)(1 - R_g(N_{Y/Y_{\mu_n}}))\rho(y)).$$

(recall that ω_Y is an invariant Kähler form on $Y(\mathbf{C})$)
Notice that the definition of the torsion immediately implies that $\delta(f, y.f^*(y'), \omega_Y) = y'.\delta(f, y, \omega_Y)$ for any $y' \in \widehat{K}_0^{\mu_n}(D)$.

7.1 Compatibility of the error term with change of Kähler metrics

The following lemma states a refined multiplicativity property of $\lambda_{-1}^{-1}(\cdot)$.

Lemma 7.1 *Let*

$$\overline{\mathcal{E}} : 0 \to E' \to E \to E'' \to 0$$

be a short exact sequence of equivariant hermitian bundles, such that E'_{μ_n}, E_{μ_n} and E''_{μ_n} vanish. Then the equality

$$\lambda_{-1}^{-1}(\overline{E}'^\vee \oplus \overline{E}''^\vee) - \lambda_{-1}^{-1}(\overline{E}^\vee) = \widetilde{\mathrm{Td}}_g(\overline{\mathcal{E}})$$

holds in $\widehat{K}_0^{\mu_n}(Y) \otimes_{R(\mu_n)} \mathcal{R}$.

Proof (of Lemma 7.1): By Th. 3.4, $\frac{\overline{\partial}\partial}{2\pi i}\widetilde{\mathrm{Td}}_g(\overline{\mathcal{E}}) = \mathrm{ch}_g(\lambda_{-1}^{-1}((\overline{E}' \oplus \overline{E}'')^\vee)) - \mathrm{ch}_g(\lambda_{-1}^{-1}(\overline{E}^\vee))$. Now consider the exterior product bundle $E'(1) := E' \otimes \mathcal{O}(1)$ on $Y_{\mu_n} \times \mathbf{P}_D^1$. Let σ be the canonical section of $\mathcal{O}(1)$, which vanishes only at ∞. It defines an equivariant map of vector bundles $E' \to E'(1)$. Define the bundle $\widetilde{E}$ as $(E \oplus E'(1))/E'$. Let j_0 (resp. j_∞) be the immersion of Y at 0 (resp. ∞) in $Y \times \mathbf{P}_D^1$. We have an exact sequence on $Y \times \mathbf{P}_D^1$

$$\widetilde{\mathcal{E}} : 0 \to E'(1) \to \widetilde{E} \to E'' \to 0$$

(see [BGS1, Par. f)]; this is a special case the construction appearing in Lemma 6.6) and equivariant isomorphisms $j_0^*\widetilde{E} \simeq E$, $j_\infty^*\widetilde{E} \simeq E' \oplus E''$. Endow $\widetilde{E}$ with an equivariant metric making these isomorphisms isometric. Endow $\mathcal{O}(1)$ with the Fubini-Study metric and $E'(1)$ with the product metric. Denote by p the projection $Y_{\mu_n} \times \mathbf{P}_D^1 \to Y_{\mu_n}$. As in [GS2, Theorem, 4.4.6, p. 161], we can now compute

$$
\begin{aligned}
&\lambda_{-1}^{-1}((\overline{E}'\oplus\overline{E}'')^\vee)-\lambda_{-1}^{-1}(\overline{E}^\vee)\\
&\qquad = j_\infty^*\lambda_{-1}^{-1}(\overline{\widetilde{E}}^\vee)-j_0^*\lambda_{-1}^{-1}(\overline{\widetilde{E}}^\vee)\\
&\qquad = -\int_{\mathbf{P}^1}\mathrm{ch}_g(\lambda_{-1}^{-1}(\overline{\widetilde{E}}^\vee))\log|z|^2\\
&\qquad = \int_{\mathbf{P}^1}\big(\mathrm{ch}_g(\lambda_{-1}^{-1}((\overline{E}'(1)\oplus\overline{E}'')^\vee))-\mathrm{ch}_g(\lambda_{-1}^{-1}(\overline{\widetilde{E}}^\vee))\big)\log|z|^2
\end{aligned}
$$

The last equality is justified by the fact that

$$\int_{\mathbf{P}^1}\mathrm{ch}_g(\lambda_{-1}^{-1}((\overline{E}'(1)\oplus\overline{E}'')^\vee))\log|z|^2=0.$$

Indeed $\log|1/z|^2=-\log|z|^2$ and the term $\mathrm{ch}_g(\lambda_{-1}^{-1}((\overline{E}'(1)\oplus\overline{E}'')^\vee))$ is by construction invariant under the change of variable $z\to 1/z$. Therefore the integral changes sign under that change of variable. Resuming our computations, we get

$$
\begin{aligned}
&\int_{\mathbf{P}^1}\big(\mathrm{ch}_g(\lambda_{-1}^{-1}((\overline{E}'(1)\oplus\overline{E}'')^\vee))-\mathrm{ch}_g(\lambda_{-1}^{-1}(\overline{\widetilde{E}}^\vee))\big)\log|z|^2\\
&= \int_{\mathbf{P}^1}\frac{\overline{\partial}_z\partial_z}{2\pi i}(\widetilde{\mathrm{Td}}_g(\overline{\widetilde{\mathcal{E}}}))\log|z|^2=\int_{\mathbf{P}^1}\widetilde{\mathrm{Td}}_g(\overline{\widetilde{\mathcal{E}}})\frac{\overline{\partial}_z\partial_z}{2\pi i}(\log|z|^2)\\
&= j_0^*\widetilde{\mathrm{Td}}_g(\overline{\widetilde{\mathcal{E}}})-j_\infty^*\widetilde{\mathrm{Td}}_g(\overline{\widetilde{\mathcal{E}}})=\widetilde{\mathrm{Td}}_g(\overline{\mathcal{E}})
\end{aligned}
$$

which ends the proof. **Q.E.D**.

Proposition 7.2 *If ω_Y, ω'_Y are two invariant Kähler forms on Y and $y_0\in\widehat{K}_0^{\mu_n}(Y)$ then $\delta(f,\omega_Y,y_0)=\delta(f,\omega'_Y,y_0)$.*

Proof (of Prop. 7.2) In order to emphasize the dependence on the Kähler form, we shall in this proof write $f_*^{\omega_Y}$ for the pushforward map $\widehat{K}_0^{\mu_n}(Y)\to\widehat{K}_0^{\mu_n}(D)$ associated to f and to a Kähler form ω_Y. Let us write $\overline{\mathcal{MC}}$ for the sequence

$$0\to Tf_{\mathbf{C}}\xrightarrow{Id}Tf_{\mathbf{C}}\to 0\to 0$$

where the second term carries the metric induced by ω_Y and the third one the metric induced by ω'_Y.

Lemma 7.3 *For any $y\in\widehat{K}_0^{\mu_n}(Y)$, the formula $f_*^{\omega'_Y}(y)-f_*^{\omega_Y}(y)=\int_{Y_{\mu_n}}\mathrm{ch}_g(y)\widetilde{\mathrm{Td}}_g(\overline{\mathcal{MC}})$ holds.*

Proof (of Lemma 7.3): since the Grothendieck group of vector bundles $K_0^{\mu_n}(Y)$ is generated by f-acyclic vector bundles and both sides of the equality to be proved are additive, we can assume that $y=\overline{E}$, where $\overline{E}$ is a f-acyclic hermitian equivariant vector bundle or that $y=\kappa\in\widetilde{\mathfrak{A}}(Y_{\mu_n})$. We write $T^{\omega_Y}f_{\mathbf{C}}$ for the bundle $Tf_{\mathbf{C}}$ endowed with the hermitian metric induced by ω_Y. For $y=\kappa$, we compute

$$
\begin{aligned}
f_*^{\omega'_Y}(\kappa)-f_*^{\omega_Y}(\kappa) &= \int_{Y_{\mu_n}}(\mathrm{Td}_g(T^{\omega'_Y}f_{\mathbf{C}})-\mathrm{Td}_g(T^{\omega_Y}f_{\mathbf{C}}))\kappa \\
&= \int_{Y_{\mu_n}}\frac{\overline{\partial}\partial}{2\pi i}\widetilde{\mathrm{Td}}_g(\overline{\mathcal{MC}})\kappa=\int_{Y_{\mu_n}}\widetilde{\mathrm{Td}}_g(\overline{\mathcal{MC}})\frac{\overline{\partial}\partial}{2\pi i}\kappa \\
&= \int_{Y_{\mu_n}}\widetilde{\mathrm{Td}}_g(\overline{\mathcal{MC}})\mathrm{ch}_g(\kappa).
\end{aligned}
$$

(remember that the range of ch_g has been extended before Prop. 4.2)
For $y=\overline{E}=(E,h^E)$, we compute using Th. 3.6

$$
\begin{aligned}
f_*^{\omega'_Y}(\overline{E})-f_*^{\omega_Y}(\overline{E}) &= (f_*E,f_*^{\omega'_Y}h^E)-T_g(Y,\omega'_Y,(E,h^E))-(f_*E,f_*^{\omega_Y}h^E)+T_g(Y,\omega_Y,(E,h^E)) \\
&= -T_g(Y,\omega'_Y,(E,h^E))+T_g(Y,\omega_Y,(E,h^E))+\widetilde{\mathrm{ch}}_g(f_*^{\omega_Y}h^E,f_*^{\omega'_Y}h^E) \\
&= \int_{Y_{\mu_n}}\widetilde{\mathrm{Td}}_g(\overline{\mathcal{MC}})\mathrm{ch}_g(\overline{E}).
\end{aligned}
$$

Here the expression $\widetilde{\mathrm{ch}}_g(f_*^{\omega_Y}h^E,f_*^{\omega'_Y}h^E)$ refers to the $\widetilde{\mathrm{ch}}_g$ secondary class of the sequence

$$0\to f_*E\to f_*E\to 0\to 0$$

where the second term is endowed with the L_2-metric induced by h^E and ω_Y and the third term with L_2-metric induced by h^E and ω'_Y. Combining our computations, we get the result. **Q.E.D.**

We resume the proof of Prop. 7.2. We write N for the bundle $N_{Y/Y_{\mu_n}}$ and N^{ω_Y} for the bundle $N_{Y/Y_{\mu_n}}$, endowed with hermitian metric induced by ω_Y. We compute

$$
\begin{aligned}
&\delta(f,\omega'_Y,y_0)-\delta(f,\omega_Y,y_0)= \\
&\quad=\int_{Y_{\mu_n}}\mathrm{ch}_g(y_0)\widetilde{\mathrm{Td}}_g(\overline{\mathcal{MC}})- \\
&\qquad(\ f_*^{\omega'_Y}(\lambda_{-1}^{-1}(N^{\omega'_Y,\vee})(1-R_g(N))\rho(y_0))-f_*^{\omega_Y}(\lambda_{-1}^{-1}(N^{\omega_Y,\vee})(1-R_g(N))\rho(y_0))\ )
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&f_*^{\omega'_Y}(\lambda_{-1}^{-1}(N^{\omega'_Y,\vee})(1-R_g(N))\rho(y_0))-f_*^{\omega_Y}(\lambda_{-1}^{-1}(N^{\omega_Y,\vee})(1-R_g(N))\rho(y_0)) \\
&\quad= f_*^{\omega'_Y}(\lambda_{-1}^{-1}(N^{\omega'_Y,\vee})\rho(y_0))-f_*^{\omega_Y}(\lambda_{-1}^{-1}(N^{\omega_Y,\vee})\rho(y_0)) \\
&\quad= (\ f_*^{\omega'_Y}(\lambda_{-1}^{-1}(N^{\omega'_Y,\vee})\rho(y_0))-f_*^{\omega_Y}(\lambda_{-1}^{-1}(N^{\omega'_Y,\vee})\rho(y_0))\ ) \\
&\qquad-(\ f_*^{\omega_Y}(\lambda_{-1}^{-1}(N^{\omega_Y,\vee})\rho(y_0))-f_*^{\omega_Y}(\lambda_{-1}^{-1}(N^{\omega'_Y,\vee})\rho(y_0))\ )
\end{aligned}
$$

Using Th. 3.6 and Lemma 7.1, we can see that the expression after the last equality equals

$$\int_{Y_{\mu_n}} \mathrm{ch}_g(\lambda_{-1}^{-1}(N^{\omega'_Y,\vee}))\mathrm{ch}_g(y_0)\widetilde{\mathrm{Td}}(\overline{\mathcal{MC}}_{\mu_n}) - \int_{Y_{\mu_n}} \mathrm{Td}(TY_{\mu_n})\mathrm{ch}_g(y_0)\widetilde{\mathrm{Td}}_g(\oplus_{k\in\mathbf{Z}/(n),\ k\neq 0}\overline{\mathcal{MC}}_k)$$

Using the equality (6), we see that the last expression equals $\int_{Y_{\mu_n}} \mathrm{ch}_g(y_0)\widetilde{\mathrm{Td}}_g(\overline{\mathcal{MC}})$ and we can thus conclude. **Q.E.D.**

In view of the last proposition, we shall from now on write $\delta(f,y)$ for $\delta(f,\omega_Y,y)$.

7.2 Compatibility of the error term with immersions

We know use the notation of section 6.1.

Theorem 7.4

$$\sum_{i\geq 0}(-1)^i\delta(p,\overline{\xi}_i) - \delta(f,\overline{\eta}) = 0$$

Proof: We compute

$$\begin{aligned}
&\sum_{i\geq 0}(-1)^i\delta(p,\overline{\xi}_i) - \delta(f,\overline{\eta})\\
&= \sum_{i\geq 0}(-1)^i\big(p_*(\overline{\xi}_i) - p_*^{\mu_n}(\lambda_{-1}^{-1}(\overline{N}^{\vee}_{X/X_{\mu_n}})(1-R_g(N_{X/X_{\mu_n}}))\rho(\overline{\xi}_i))\big) - \delta(f,\omega_Y,\overline{\eta})\\
&= -\int_{Y_g}\mathrm{ch}_g(\eta)R_g(N_{X/Y})\mathrm{Td}_g(TY) - \int_{X_g} T_g(\overline{\xi})\mathrm{Td}_g(\overline{TX})\\
&\quad + \int_{Y_g}\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{TY},\overline{TX}|_Y)\mathrm{Td}_g^{-1}(\overline{N}_{X/Y}) + f_*(\overline{\eta}) - f_*^{\mu_n}(\lambda_{-1}(\overline{E}^{\vee})\rho(\overline{\eta}).\lambda_{-1}^{-1}(\overline{N}^{\vee}_{X/X_{\mu_n}}))\\
&\quad + \int_{Y_g}\mathrm{Td}(TY_g)\mathrm{ch}_g(\eta.\lambda_{-1}(E^{\vee}))R(N_{Y_g/X_g})\mathrm{ch}_g(\lambda_{-1}^{-1}(N^{\vee}_{X/X_{\mu_n}}))\\
&\quad + \int_{X_g}\mathrm{Td}(\overline{TX_g})T_g(\overline{\xi})\mathrm{ch}_g(\lambda_{-1}^{-1}(\overline{N}^{\vee}_{X/X_{\mu_n}}))\\
&\quad - \int_{Y_g}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{X/Y})\widetilde{\mathrm{Td}}(\overline{TY_g},\overline{TX_g}|_{Y_g})\mathrm{ch}_g(\lambda_{-1}^{-1}(\overline{N}^{\vee}_{X/X_{\mu_n}}))\\
&\quad + \sum_{i\geq 0}(-1)^i\int_{X_g}\mathrm{ch}_g(\xi_i)\mathrm{ch}_g^{-1}(\lambda_{-1}(N^{\vee}_{X/X_{\mu_n}}))R_g(N_{X/X_{\mu_n}})\mathrm{Td}(TX_{\mu_n})\\
&\quad - f_*(\overline{\eta}) + f_*^{\mu_n}(\lambda_{-1}^{-1}(\overline{N}^{\vee}_{Y/Y_{\mu_n}})\rho(\overline{\eta})) - \int_{Y_g}\mathrm{Td}(TY_g)R_g(N_{Y/Y_{\mu_n}})\\
&\quad \cdot\mathrm{ch}_g(\eta)\mathrm{ch}_g^{-1}(\lambda_{-1}(\overline{N}^{\vee}_{Y/Y_{\mu_n}})).
\end{aligned}$$

Here we used Prop. 5.1 for the first one and a half lines after the last equality and then Th. 6.1. To compare $f_*^{\mu_n}(\lambda_{-1}(\overline{E}^\vee)\rho(\overline{\eta}).\lambda_{-1}^{-1}(\overline{N}^\vee_{X/X_{\mu_n}}))$ and $f_*^{\mu_n}(\lambda_{-1}^{-1}(\overline{N}^\vee_{Y/Y_{\mu_n}})\rho(\overline{\eta}))$, we shall use Lemma 7.1. Consider the sequence

$$H : 0 \to N_{Y/Y_{\mu_n}} \to N_{X/X_{\mu_n}} \to E \to 0$$

In view of Lemma 7.1, the equality

$$\begin{aligned} & f_*^{\mu_n}(\lambda_{-1}(\overline{E}^\vee)\lambda_{-1}^{-1}(\overline{N}^\vee_{X/X_{\mu_n}})\rho(\overline{\eta})) \\ &\qquad = f_*^{\mu_n}(\lambda_{-1}(\overline{E}^\vee)(\lambda_{-1}^{-1}(\overline{E}^\vee)\lambda_{-1}^{-1}(\overline{N}^\vee_{Y/Y_{\mu_n}}) - \widetilde{\mathrm{Td}}_g(\overline{H}))\rho(\overline{\eta})) \\ &\qquad = f_*^{\mu_n}(\lambda_{-1}^{-1}(\overline{N}^\vee_{Y/Y_{\mu_n}})\rho(\overline{\eta})) \\ &\qquad\quad - \int_{Y_g} \mathrm{Td}(TY_g)\mathrm{ch}_g(\lambda_{-1}(\overline{E}^\vee))\widetilde{\mathrm{Td}}_g(\overline{H})\mathrm{ch}_g(\overline{\eta}) \end{aligned}$$

holds. Thus we get

$$\begin{aligned} & \sum_{i\geq 0}(-1)^i\delta(p,\overline{\xi}_i) - \delta(f,\overline{\eta}) \\ &= \int_{Y_g}\big(- \mathrm{ch}_g(\eta)R_g(N_{X/Y})\mathrm{Td}_g(TY) + \mathrm{Td}(TY_g)\mathrm{ch}_g(\eta)\mathrm{ch}_g(\lambda_{-1}(E^\vee))R(N_{Y_g/X_g}) \\ &\quad \cdot\mathrm{ch}_g^{-1}(\lambda_{-1}(N^\vee_{X/X_{\mu_n}})) - \mathrm{ch}_g(\eta)\mathrm{Td}(TY_g)\mathrm{ch}_g^{-1}(\lambda_{-1}(N^\vee_{Y/Y_g}))R_g(N_{Y/Y_g})\big) \\ &\quad + \sum_{i\geq 0}(-1)^i\int_{X_g}\mathrm{ch}_g(\xi_i)\mathrm{ch}_g^{-1}(\lambda_{-1}(N^\vee_{X/X_{\mu_n}}))R_g(N_{X/X_{\mu_n}})\mathrm{Td}(TX_{\mu_n}) \qquad (25) \\ &\quad + \int_{Y_g}\big(\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{TY},\overline{TX}|_Y)\mathrm{Td}_g^{-1}(\overline{N}_{X/Y}) - \mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{X/Y})\widetilde{\mathrm{Td}}(\overline{TY_g},\overline{TX_g}|_{Y_g}) \\ &\quad \cdot\mathrm{ch}_g^{-1}(\lambda_{-1}(\overline{N}^\vee_{X/X_{\mu_n}})) + \mathrm{Td}(TY_g)\mathrm{ch}_g(\lambda_{-1}(\overline{E}^\vee))\widetilde{\mathrm{Td}}_g(\overline{H})\mathrm{ch}_g(\overline{\eta})\big) \qquad (26) \\ &\quad + \int_{X_g}\big(- T_g(\overline{\xi})\mathrm{Td}_g(\overline{TX}) + \mathrm{Td}(\overline{TX_g})T_g(\overline{\xi})\mathrm{ch}_g^{-1}(\lambda_{-1}(\overline{N}^\vee_{X/X_{\mu_n}}))\big) \qquad (27) \end{aligned}$$

It readily follows from the definitions that (27) vanishes. We shall show that (25) vanish. Using the cohomological equivariant Riemann-Roch formula, the sequence H and the additivity of the R_g genus, we compute that (25) equals the integral over Y_g of

$$\begin{aligned} & \mathrm{ch}_g(\eta)[-R_g(N_{X/Y})\mathrm{Td}_g(TY) + \mathrm{Td}(TY_g)\mathrm{Td}_g^{-1}(\oplus_{k\neq 0}N_{X/Y,k})R(N_{X_g/Y_g})\mathrm{Td}_g(N_{X/X_g}) \\ &\qquad + \mathrm{Td}_g^{-1}(N_{X/Y})\mathrm{Td}_g(N_{X/X_g})R_g(N_{X/X_g})\mathrm{Td}(TX_g) - \mathrm{Td}(TY_g)\mathrm{Td}_g(N_{Y/Y_g})R_g(N_{Y/Y_g})] \\ &= \mathrm{ch}_g(\eta)\mathrm{Td}_g(TY)[-R_g(N_{X/Y}) + R(N_{X_g/Y_g}) + R_g(N_{X/X_g}) - R_g(N_{Y/Y_g})] = 0. \end{aligned}$$

We now turn to the vanishing of (26), which will conclude the proof of Th. 7.4. This follows from the formula (6), applied to the sequence

$$0 \to TY_{\mathbf{C}} \to TX_{\mathbf{C}} \to N_{X_{\mathbf{C}}/Y_{\mathbf{C}}} \to 0.$$

Another way to check that (26) vanishes is to use the axiomatic characterisation mentioned in Th. 3.4. **Q.E.D**.

7.3 Proof of the theorem for projective spaces

In this subsection, we prove that Th. 4.4 holds for projective spaces. To do this, we adapt to the equivariant situation a diagonal immersion argument described by J.-B. Bost in [Bo2].

Proposition 7.5 *Endow* $f : \mathbf{P}^r_{\mathbf{Z}} \to \mathrm{Spec}\ \mathbf{Z}$ *with a global* μ_n*-action. Let* $\mathcal{O}_{\mathbf{P}^r_{\mathbf{Z}}}$ *be the trivial bundle on* $\mathbf{P}^r_{\mathbf{Z}}$*, endowed with the trivial equivariant structure. Then* $\delta(f, \mathcal{O}_{\mathbf{P}^r_{\mathbf{Z}}}) = 0$.

Proof: We shall first prove the following sublemma.

Sublemma 7.6 *For every equivariant bundle* E *on* $\mathbf{P}^r_{\mathbf{Z}}$*, the element* $\delta(f, \overline{E})$ *lies in* $\widetilde{\mathfrak{A}}(\mathbf{Z})$.

Proof (of Sublemma 7.6): Consider the diagram

$$\begin{array}{ccc} K_0^{\mu_n}(\mathbf{P}^r_{\mathbf{Z}}) & \xrightarrow{(\lambda_{-1}(N^{\vee}_{\mathbf{P}^r_{\mathbf{Z}}/\mathbf{P}^r_{\mathbf{Z},\mu_n}}))^{-1}\rho(.)} & K_0^{\mu_n}(\mathbf{P}^r_{\mathbf{Z},\mu_n}) \otimes_{R(\mu_n)} \mathcal{R} \\ \downarrow\ f_* & & \downarrow\ f_*^{\mu_n} \\ K_0^{\mu_n}(\mathbf{Z}) & \longrightarrow & K_0^{\mu_n}(\mathbf{Z}) \otimes_{R(\mu_n)} \mathcal{R} \end{array}$$

where $K_0^{\mu_n}(\cdot)$ is the ordinary K_0-group of μ_n-equivariant vector bundles and f_*, $f_*^{\mu_n}$ refer to the corresponding functors. In view of the sequence (13), the commutativity of this diagram is equivalent to the statement of the sublemma. Now consider that this diagram is base-change invariant; furthermore since the base-change maps $K_0^{\mu_n}(\mathbf{P}^r_{\mathbf{Z}}) \to K_0^{\mu_n}(\mathbf{P}^r_{\mathbf{C}})$ and $K_0^{\mu_n}(\mathbf{Z}) \to K_0^{\mu_n}(\mathbf{C})$ are isomorphisms, we are reduced to prove that the diagram is commutative, when the symbol $\mathbf{Z}$ is replaced by the symbol $\mathbf{C}$. In that case, this is a slight variant of the main result of [BaFQ], so we are done. **Q.E.D**.

Let now $\pi_i : Y_i \to \mathrm{Spec}\ \mathbf{Z}$ $(i = 1, 2)$ be two schemes that are μ_n-projective and smooth over Spec $\mathbf{Z}$ and let ω_i be (conjugation invariant) μ_n-invariant Kähler forms on $Y_i(\mathbf{C})$. Let $\overline{N}_i$ be the normal bundle of the immersion of Y_{i,μ_n} into Y_i, endowed with its natural metric. Let $\overline{E}_i$ be equivariant hermitian bundles on the Y_i. By construction, the scheme $\pi : Y = Y_1 \times_{\mathbf{Z}} Y_2 \to \mathrm{Spec}\ \mathbf{Z}$ is naturally smooth and μ_n-projective over Spec $\mathbf{Z}$. Let p_i be the natural projection $Y \to Y_i$ and let $\overline{E}$ be the bundle $p_1^* E_1 \otimes p_2^* E_2$, endowed with its natural equivariant and hermitian structure. We claim that

$$\delta(\pi, E) = L(E_{1,\mathbf{C}}, \pi_{1,\mathbf{C}})\delta(\pi_2, E_2) + L(E_{2,\mathbf{C}}, \pi_{2,\mathbf{C}})\delta(\pi_1, E_1)$$

where $L(E_{i,\mathbf{C}},\pi_{i,\mathbf{C}})$ is the holomorphic Lefschetz number of $E_{i,\mathbf{C}}$, which is viewed as an element of $\widetilde{\mathfrak{A}}(\mathbf{Z})\simeq\mathbf{C}$. We now compute in the group $\widehat{K}_0^{\mu_n}(\mathbf{Z})\otimes_{R(\mu_n)}\mathcal{R}$:

$$\begin{aligned}\delta(\pi,E) &= R^{\cdot}p_{1*}\overline{E}_1\otimes R^{\cdot}p_{2*}\overline{E}_2-T_g(Y,\overline{E})-\pi_*^{\mu_n}(\rho(\pi_1^*\overline{E}_1)\otimes\rho(\pi_2^*\overline{E}_2)\\ &\quad\cdot\lambda_{-1}^{-1}(p_1^*\overline{N}_1^{\vee}\oplus p_2^*\overline{N}_2^{\vee})(1-R_g(p_1^*N_1\oplus p_2^*N_2)))\\ &= R^{\cdot}\pi_{1*}\overline{E}_1\otimes R^{\cdot}\pi_{2*}\overline{E}_2-T_g(Y,\overline{E})-\pi_*^{\mu_n}(\rho(p_1^*\overline{E}_1)\lambda_{-1}^{-1}(p_1^*\overline{N}_1)\\ &\quad\cdot(1-R_g(p_1^*N_1))\rho(p_2^*\overline{E}_2)\lambda_{-1}^{-1}(p_2^*\overline{N}_2^{\vee})(1-R_g(p_2^*N_2)))\\ &= R^{\cdot}\pi_{1*}\overline{E}_1\otimes R^{\cdot}\pi_{2*}\overline{E}_2-T_g(Y,\overline{E})-\pi_{1*}^{\mu_n}(\rho(\overline{E}_1)\lambda_{-1}^{-1}(\overline{N}_1)(1-R_g(N_1))\\ &\quad\cdot\pi_{2*}^{\mu_n}(\rho(\overline{E}_2)\lambda_{-1}^{-1}(\overline{N}_2^{\vee})(1-R_g(N_2)))\end{aligned}$$

Using [K2, Lemma 2, p. 95], we compute that the last expression is equal to

$$\begin{aligned}& R^{\cdot}\pi_{1*}\overline{E}_1\otimes R^{\cdot}\pi_{2*}\overline{E}_2-L(E_{1,\mathbf{C}},\pi_{1,\mathbf{C}})T_g(Y_2,\overline{E}_2)-L(E_{2,\mathbf{C}},\pi_{2,\mathbf{C}})T_g(Y_1,\overline{E}_1)\\ &\quad-\pi_{1*}^{\mu_n}(\rho(\overline{E}_1)\lambda_{-1}^{-1}(\overline{N}_1)(1-R_g(N_1)).\pi_{2*}^{\mu_n}(\rho(\overline{E}_2)\lambda_{-1}^{-1}(\overline{N}_2^{\vee})(1-R_g(N_2)))\\ &= \pi_{1*}(\overline{E}_1)\pi_{2*}(\overline{E}_2)-\pi_{1*}^{\mu_n}(\rho(\overline{E}_1)\lambda_{-1}^{-1}(\overline{N}_1)(1-R_g(N_1))\\ &\quad\cdot\pi_{2*}^{\mu_n}(\rho(\overline{E}_2)\lambda_{-1}^{-1}(\overline{N}_2^{\vee})(1-R_g(N_2)))\\ &= [\pi_{1*}(\overline{E}_1)-\pi_{1*}^{\mu_n}(\rho(\overline{E}_1)\lambda_{-1}^{-1}(\overline{N}_1^{\vee})(1-R_g(N_1)))].\pi_{2*}(\overline{E}_2)\\ &\quad+[\pi_{2*}(\overline{E}_1)-\pi_{2*}^{\mu_n}(\rho(\overline{E}_2)\lambda_{-1}^{-1}(\overline{N}_2^{\vee})(1-R_g(N_2)))].\pi_{1*}(\overline{E}_2)\end{aligned}$$

where we used the fact that $\delta(\pi_1,E_1).\delta(\pi_2,E_2)=0$. This last fact follows from the sublemma and the fact that the square of any element of $\widetilde{\mathfrak{A}}(\mathbf{Z})$ vanishes, by the definition of the ring structure (see after Def. 4.1). By definition $\mathrm{ch}_g(\pi_{i*}(\overline{E}_i))=L(E_{i,\mathbf{C}},\pi_{i,\mathbf{C}})$ and thus we have proved the claim. Let now $Y_i=\mathbf{P}^r_{\mathbf{Z}}$ and let $\Delta:\mathbf{P}^r_{\mathbf{Z}}\to\mathbf{P}^r_{\mathbf{Z}}\times_{\mathbf{Z}}\mathbf{P}^r_{\mathbf{Z}}$ be the diagonal embedding. Let Q be the universal quotient bundle on $\mathbf{P}^r_{\mathbf{Z}}$. There is a canonical equivariant section of $\mathcal{E}=p_1^*\mathcal{O}(1)\otimes p_2^*Q$, whose zero-scheme is the diagonal (see [F, Ex. 8.4.2, (c), p. 146]). This section arises from the composition of equivariant morphisms $p_1^*\mathcal{O}(-1)\to p_1^*E\simeq p_2^*E\to p_2^*Q$, where Q is the quotient bundle on $\mathbf{P}^r_{\mathbf{Z}}$. It yields an exact Koszul complex

$$0\to\Lambda^n\mathcal{E}^{\vee}\to\ldots\to\mathcal{E}^{\vee}\to\mathcal{O}_{\mathbf{P}^r\times\mathbf{P}^r}\to\Delta_*\mathcal{O}_{\mathbf{P}^r}\to 0.$$

By Th. 7.4, we have the equality

$$\delta(f,\mathcal{O}_{\mathbf{P}^r})=\sum_{i=0}^{n}(-1)^i\delta(\pi,\Lambda^i(\mathcal{E}^{\vee}))\qquad(28)$$

where $\mathcal{O}(1)$ is equipped with the Fubini-Study metric, Q with the quotient metric and $\mathcal{O}_{\mathbf{P}^r}$ with the trivial metric. Furthermore

$$\delta(\pi,\Lambda^i(\mathcal{E}^{\vee}))\quad=\quad\delta(\pi,p_1^*\mathcal{O}(-i)\otimes p_2^*\Lambda^i(Q^{\vee}))$$

$$= \quad L(\mathcal{O}(-i)_{\mathbf{C}}, f_{\mathbf{C}})\delta(f, \Lambda^i(Q^\vee)) + L(\Lambda^i(Q^\vee_{\mathbf{C}}), f_{\mathbf{C}})\delta(f, \mathcal{O}(-i)) \tag{29}$$

Lemma 7.7 *The equalities* $L(\mathcal{O}_{\mathbf{C}}, f_{\mathbf{C}}) = 1$, $L(\mathcal{O}(-i)_{\mathbf{C}}, f_{\mathbf{C}}) = 0$, $L(\Lambda^i(Q^\vee_{\mathbf{C}}), f_{\mathbf{C}}) = 0$ *hold* ($1 \leq i \leq n$).

Proof (of Lemma 7.7): Only the third statement requires proof. Let V be the complex vector space of global sections of the bundle $\mathcal{O}(1)_{\mathbf{C}}$ on $\mathbf{P}^r_{\mathbf{C}}$, endowed with the equivariant structure arising from the global action. There is a fundamental equivariant sequence

$$0 \to \mathcal{O}(-1)_{\mathbf{C}} \to f^*_{\mathbf{C}} V^\vee \to Q_{\mathbf{C}} \to 0$$

on $\mathbf{P}^r_{\mathbf{C}}$. Thus in the ring of formal power series $K_0^{R_n}(\mathrm{Pt})[[t]]$, we compute

$$\begin{aligned} L(\Lambda_t(Q^\vee_{\mathbf{C}})) &= L\Big(\frac{\Lambda_t(V)}{1+\mathcal{O}(1)_{\mathbf{C}} t}\Big) = L(\Lambda_t(V)[\sum_{i\geq 0}(-1)^i \mathcal{O}(i)_{\mathbf{C}} t^i]) \\ &= \Lambda_t(V)\sum_{i\geq 0}(-1)^i L(\mathcal{O}(i)_{\mathbf{C}})t^i = \Lambda_t(V)\sum_{i\geq 0}(-1)^i S^i(V)t^i \end{aligned}$$

(where $\Lambda_t(\cdot)$ is the power series $\sum_{i\geq 0}\Lambda^i(\cdot).t^i$). The next sublemma shows that in $K_0^{R_n}(\mathrm{Pt})$, the equality $\Lambda_t(V).\sum_{i\geq 0}(-1)^i S^i(V) = 1$ holds. From this the statement follows.

Sublemma 7.8 *For all* $i \geq 1$ *there is an equivariant exact sequence*

$$0 \to \Lambda^{r+1}V \otimes S^{i-r-1}V \stackrel{d_{r+1}}{\to} \dots \stackrel{d_2}{\to} \Lambda^1 V \otimes S^{i-1}V \stackrel{d_1}{\to} S^i V \to 0 \tag{30}$$

of R_n*-modules.*

Proof (of Sublemma 7.8): In [L, Cor. 10.14, p. 602] a sequence is described, which corresponds to the sequence (30) stripped of its equivariant structure. In this sequence, the morphism d_l ($1 \leq l \leq r+1$) is described by the formula $d_l((x_1\wedge\ldots\wedge x_l)\otimes y) = \sum_{j=1}^l(-1)^{j-1}(x_1\wedge\ldots\wedge\widehat{x_j}\wedge\ldots\wedge x_l)\otimes(x_j\otimes y)$. One can check from this definition that d_l is equivariant and so we are done. **Q.E.D.**

If we apply (29) and the Lemma 7.7 to explicit (28), we obtain $\delta(f, \mathcal{O}_{\mathbf{P}^r}) = 2\delta(f, \mathcal{O}_{\mathbf{P}^r})$ and thus $\delta(f, \mathcal{O}_{\mathbf{P}^r}) = 0$, which concludes the proof. **Q.E.D.**

Before proceeding, we recall that for any arithmetic ring D, $K_0^{\mu_n}(\mathbf{P}^r_D)$ is generated over $K_0^{\mu_n}(D)$ by the bundles $\mathcal{O}(l)$ ($0 \leq l \leq r-1$) (see [T2, Th. 3.1, p. 549]; this holds in fact in a more general situation). In particular, this implies the following: if L is an equivariant line bundle on $\mathbf{P}^r_{\mathbf{Z}}$ whose underlying line bundle is isomorphic to $\mathcal{O}(l)$ for some $l \in \mathbf{Z}$, then there is an isomorphism of equivariant bundles $L \simeq \mathcal{O}(l)\otimes f^*M$, for some one-dimensional projective μ_n-comodule M. Thus if $\delta(f, \mathcal{O}(l))$ vanishes for some μ_n-action on $\mathcal{O}(l)$, it vanishes for any μ_n-action on $\mathcal{O}(l)$.

Proposition 7.9 *The element $\delta(f,E)$ vanishes for every equivariant vector bundle on $\mathbf{P}^r_{\mathbf{Z}}$ ($r \geq 0$).*

Proof: We prove the statement by induction on r. If $r = 0$ the statement is true, because the map f is the identity. We now carry out the inductive step and suppose that $r > 0$ and that the statement holds for $r-1$. Choose a section s of $\mathcal{O}(1)$ on $\mathbf{P}^r_{\mathbf{Z}}$ such that s is homogeneous for the $\mathbf{Z}/(n)$-grading of $H^0(\mathbf{P}^r_{\mathbf{Z}}, \mathcal{O}(1))$. This section induces an equivariant resolution

$$0 \to \mathcal{O}_{\mathbf{P}^r}(-1) \to \mathcal{O}_{\mathbf{P}^r} \to i_*\mathcal{O}_{\mathbf{P}^{r-1}} \to 0.$$

Tensoring this sequence with $\mathcal{O}(l)$, we obtain

$$0 \to \mathcal{O}_{\mathbf{P}^r}(l-1) \to \mathcal{O}_{\mathbf{P}^r}(l) \to i_*\mathcal{O}_{\mathbf{P}^{r-1}}(l) \to 0.$$

Applying induction on l together with Prop. 7.5, Th. 7.4 and the remarks before the proposition, we see that $\delta(f, \mathcal{O}(l)) = 0$, for any $l \in \mathbf{Z}$ and for any equivariant structure on $\mathcal{O}(l)$. Using the structure of $K_0^{\mu_n}(\mathbf{P}^r_{\mathbf{Z}})$ given before the proposition, we can finish the inductive step and the proof. **Q.E.D.**

Corollary 7.10 *If D is any arithmetic ring, $f : \mathbf{P}^r_D \to \operatorname{Spec} D$ is the structural map and E any equivariant vector bundle on $\mathbf{P}^r_D$, then $\delta(f,E) = 0$.*

Proof: Follows from base-change invariance and the structure of $K_0^{\mu_n}(\mathbf{P}^r_D)$. **Q.E.D.**

To complete the proof of Th. 4.4, we have to prove that $\delta(f,\eta) = 0$, where $f : Y \to \operatorname{Spec} D$ is an equivariant arithmetic variety over D and η an equivariant vector bundle on Y. Choose an equivariant embedding $i : Y \to X$, where X is some projective space over D equipped with a global action and combine Th. 7.4 and Cor. 7.10 to conclude.

7.4 Complement: arithmetic characteristic classes

For simplicity's sake, until the end of the paper we shall suppose that $D = \mathbf{Z}$. In this subsection, we combine Th. 4.4 with the arithmetic Riemann-Roch formula of Bismut-Gillet-Soulé and use it to express the arithmetic Lefschetz trace as a function of arithmetic characteristic classes of some hermitian bundles living on the fixed point scheme. Let V be a finitely generated $\mathbf{Z}$-module; the complex conjugation F_∞ acts on $V_{\mathbf{C}} := V \otimes_{\mathbf{Z}} \mathbf{C}$ via the formula $v \otimes z \mapsto v \otimes \overline{z}$ ($v \in V, z \in \mathbf{C}$). Identify $V_{\mathbf{R}} := V \otimes_{\mathbf{Z}} \mathbf{R}$ with the real vector space, which corresponds to the subset of $V_{\mathbf{C}}$ fixed under F_∞. Endow $V_{\mathbf{C}}$ with a hermitian metric h_V invariant under F_∞, so that its restriction to $V_{\mathbf{R}}$ yields a real metric. Now choose a basis $v_1, \ldots, v_r$ of the free part of V. The **covolume** $\operatorname{covol}(\overline{V})$ of

$\overline{V} := (V, h_V)$ is the norm of the element $(v_1 \otimes 1) \wedge \ldots \wedge (v_r \otimes 1)$ in $\mathrm{Det}(V_{\mathbf{C}})$, computed with the exterior product metric. It is not difficult to see that this definition does not depend on the choice of the basis of V. To understand the geometric meaning of the covolume, let us choose an orthonormal basis of $V_{\mathbf{R}}$ and use it to identify $V_{\mathbf{R}}$ with $\mathbf{R}^r$. Under this identification, the covolume of $\overline{V}$ is the volume (for the Lebesgue measure) of the cube spanned by the vectors $v_1 \otimes 1, \ldots, v_r \otimes 1$.
In the next lemma, we view the complex numbers $\mathbf{C}$ as an $R(\mu_n)$-module, with the $R(\mu_n)$-module structure described before the statement of Th. 4.4.

Lemma 7.11 *The mapping rule that associates the element $(V, -\sum_{k \in \mathbf{Z}/(n)} \zeta_n^k \cdot \log(\mathrm{covol}(\overline{V}_k)))$ to a hermitian μ_n-comodule $\overline{V}$ and the element $(0, \eta)$ to the element $\eta \in \widetilde{\mathfrak{A}}(\mathbf{Z})$ induces an isomorphism of $R(\mu_n)$-modules $\widehat{K}_0^{\mu_n}(\mathbf{Z}) \simeq R(\mu_n) \oplus \mathbf{C}$.*

Proof: We first have to check that the mapping rule described in the lemma is compatible with the relations of arithmetic equivariant K_0-theory. It decomposes into a degree 0 part, whose target is $R(\mu_n)$ and into degree 1 part, whose target is $\mathbf{C}$. The degree 0 part is compatible with the relations because it is the rule that forgets the hermitian structure. To see that its degree 1 part is compatible with the relations, let

$$\mathcal{V} : 0 \to V' \to V \to V'' \to 0$$

be an exact sequence of μ_n-comodules, where V', V and V'' are endowed with (conjugation invariant) μ_n-invariant hermitian metrics. This sequence carries a natural $\mathbf{Z}/(n)$-grading by subsequences $\mathcal{V}_k$ $(k \in \mathbf{Z}/(n))$, where the terms of the grading are orthogonal to each other. By the equation preceding (6), we have $\widetilde{\mathrm{ch}}_g(\overline{\mathcal{V}}) = \sum_{k \in \mathbf{Z}/(n)} \zeta_n^k \widetilde{\mathrm{ch}}(\overline{\mathcal{V}}_k)$; on the other hand, in [GS3, Prop. 2.5] it is proved that

$$\frac{1}{2}\widetilde{\mathrm{ch}}(\overline{\mathcal{V}}_k) := \log(\mathrm{covol}(\overline{V}'_k)) + \log(\mathrm{covol}(\overline{V}''_k)) - \log(\mathrm{covol}(\overline{V})).$$

Using the two last equations, we obtain that $\widehat{\deg}_{\mu_n}(\overline{V}) + \frac{1}{2}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{V}}) = \widehat{\deg}_{\mu_n}(\overline{V}') + \widehat{\deg}_{\mu_n}(\overline{V}'')$. This proves that the defining relations of the group $\widehat{K}_0^{\mu_n}(\mathbf{Z})$ are mapped on 0 by $\widehat{\deg}_{\mu_n}$, which proves the compatibility.
Denote by I the map $\widehat{K}_0^{\mu_n}(\mathbf{Z}) \to R(\mu_n) \oplus \mathbf{C}$. To see that I is an isomorphism, consider that by construction it is surjective. To see that it is injective, suppose that $I(x) = 0$ for some $x \in \widehat{K}_0^{\mu_n}(\mathbf{Z})$. Then x can be represented by $\eta \in \widetilde{\mathfrak{A}}(\mathbf{Z}) \simeq \mathbf{C}$. As the degree 1 part of $I(x)$ vanishes, we see that $\eta = 0$ and thus $x = 0$. It follows immediately from the definitions that the map I is a map of $R(\mu_n)$-modules. This proves the claim. **Q.E.D.**

The degree 1 part of the isomorphism described in the Lemma 7.11 (which has values in $\mathbf{C}$) will be denoted by $\widehat{\deg}_{\mu_n}(\cdot)$.
Let now X be a regular scheme which is projective and flat over $\mathbf{Z}$. Let ω_X be a Kähler form on $X(\mathbf{C})$. To such a scheme, Gillet-Soulé associate an arithmetic Chow ring $\widehat{\mathrm{CH}}(X)$ (see [GS2]), which carries a natural grading analogous to the grading of the classical Chow group. If X' is another variety with the same properties and $f : X' \to X$ is any morphism, there is a pull-back map $f^* : \widehat{\mathrm{CH}}(X') \to \widehat{\mathrm{CH}}(X)$; if f is smooth over $\mathbf{Q}$ and projective, there is a push-forward map $f_* : \widehat{\mathrm{CH}}(X') \to \widehat{\mathrm{CH}}(X)$ which satisfies the projection formula $f_*(f^*(x)x') = x.f_*(x')$ for all $x \in \widehat{\mathrm{CH}}(X)$ and for all $x' \in \widehat{\mathrm{CH}}(X')$. For a hermitian bundle $\overline{E}$ on X, Gillet-Soulé also define an **arithmetic Chern character** $\widehat{\mathrm{ch}}(\overline{E}) \in \widehat{\mathrm{CH}}(X)_{\mathbf{Q}}$ (resp. an **arithmetic Todd class** $\widehat{\mathrm{Td}}(\overline{E}) \in \widehat{\mathrm{CH}}(X)_{\mathbf{Q}}$). If f is projective and smooth over $\mathbf{Q}$, they associate an element $\widehat{\mathrm{Td}}(\overline{Tf}) \in \widehat{\mathrm{CH}}(X)_{\mathbf{Q}}$ to the map f and the Kähler form ω_X; if f is everywhere smooth, this element corresponds to the arithmetic Todd class of the relative tangent bundle equipped with the restriction of the Kähler metric. They also show that there is a natural isomorphism $\widehat{\deg} : \widehat{\mathrm{CH}}^1(\mathbf{Z}) \to \mathbf{R}$, called the **arithmetic degree**; if we denote by $\widehat{c}_1$ the degree one part of $\widehat{\mathrm{ch}}$, then $\widehat{\deg}(\widehat{c}_1(\overline{V})) = -\log(\mathrm{covol}(\overline{V}))$ for every finitely generated free hermitian $\mathbf{Z}$-module $\overline{V}$.
Gillet-Soulé prove in [GS8] a Riemann-Roch theorem for the arithmetic Chern character and the push-forward map in arithmetic Chow theory. To formulate it, let us denote by $R(\cdot)$ the class $R_g(\cdot)$ associated to the action of the identity on the base space and on the bundle; let also $T(\cdot)$ denote the equivariant analytic torsion associated to the action of the identity on the bundle and the base space (this is the Ray-Singer analytic torsion). Let $\#S$ denote the cardinality of a set S and denote by A_{Tors} the torsion subgroup of an abelian group A. The following theorem is proved in [GS8, 4.2.3].

Theorem 7.12 *Let $h : X \to \mathbf{Z}$ be a regular scheme, projective and flat over $\mathbf{Z}$. Let $\overline{E}$ be a hermitian bundle over X. The equality*

$$-\sum_{q\geq 0}(-1)^q\big(\log(\mathrm{covol}(\overline{H^q(X,E)})) - \log(\#H^q(X,E)_{\mathrm{Tors}})\big)$$
$$= \frac{1}{2}T(X(\mathbf{C}),\overline{E}) - \frac{1}{2}\int_{X(\mathbf{C})}\mathrm{Td}(TX_{\mathbf{C}})R(TX_{\mathbf{C}})\mathrm{ch}(E_{\mathbf{C}}) + \widehat{\deg}\big(h_*(\widehat{\mathrm{Td}}(\overline{Th})\widehat{\mathrm{ch}}(\overline{E}))\big)$$

holds.

For another approach to the preceding theorem, see [Fal]. We shall now combine this theorem with the formula Th. 4.4. Let again $f : Y \to \mathbf{Z}$ be a regular μ_n-projective scheme. Suppose that Y_{μ_n} is flat over $\mathbf{Z}$.
N.B. The last hypothesis is only necessary because arithmetic Chow groups are defined under the assumption of flatness; if one wishes to drop this hypothesis,

one might use the groups $\mathrm{Gr}\widehat{K}_0(\cdot)_{\mathbf{Q}}$ defined in [R1, Sec. 8] instead of the groups $\widehat{\mathrm{CH}}(\cdot)_{\mathbf{Q}}$.

Definition 7.13 *Let $\overline{E}$ be an equivariant hermitian bundle on Y. The equivariant arithmetic Chern character $\widehat{\mathrm{ch}}_{\mu_n}(\overline{E})$ of $\overline{E}$ is the element $\sum_{k\in\mathbf{Z}/(n)}\widehat{\mathrm{ch}}(\overline{E}_k)\otimes\zeta_n^k$ of $\widehat{\mathrm{CH}}(Y_{\mu_n})\otimes_{\mathbf{Z}}\mathbf{C}$.*

The following theorem is an equivariant refinement of the arithmetic Riemann-Roch theorem. In this form, it has been conjectured by J.-M. Bismut (see [B2, Par. (l), p. 353] and also Soulé's question in [SABK, 1.5, p. 162]). Let $\widehat{\mathrm{Td}}_{\mu_n}(\overline{Tf})$ stand for

$$\Big(\sum_{i=0}^{\mathrm{rk}(N^{\vee}_{Y/Y_{\mu_n}})}(-1)^i\widehat{\mathrm{ch}}_{\mu_n}(\Lambda^i(\overline{N}^{\vee}_{Y/Y_{\mu_n}}))\Big)^{-1}\cdot\widehat{\mathrm{Td}}(\overline{Tf^{\mu_n}})\ .$$

Theorem 7.14 *Let $\overline{E}$ be an equivariant hermitian vector bundle on Y. The equality*

$$\begin{aligned}-\sum_{q\geq 0}(-1)^q\Big(\sum_{k\in\mathbf{Z}/(n)}\zeta_n^k.\big(\log(\mathrm{covol}(\overline{H^q(Y,E)}_k))-\log(\#H^q(Y,E)_{k,\mathrm{Tors}})\big)\Big)\\ =\frac{1}{2}T_g(Y(\mathbf{C}),\overline{E})-\frac{1}{2}\int_{Y_{\mu_n}(\mathbf{C})}\mathrm{Td}_g(TY_{\mathbf{C}})\mathrm{ch}_g(E_{\mathbf{C}})R_g(TY_{\mathbf{C}})\\ +\widehat{\deg}\big(f_*(\widehat{\mathrm{Td}}_{\mu_n}(\overline{Tf})\widehat{\mathrm{ch}}_{\mu_n}(\overline{E}))\big)\end{aligned}$$

holds.

Proof: To obtain the left hand side of the equality minus the term $\frac{1}{2}T_g(Y(\mathbf{C}),\overline{E})$, compose the left arrow in the diagram of Th. 4.4 with the map $\widehat{\deg}_{\mu_n}$. To obtain its right hand side minus the term $\frac{1}{2}T_g(Y(\mathbf{C}),\overline{E})$ (i.e. the expression $\widehat{\deg}\big(f_*(\widehat{\mathrm{Td}}_{\mu_n}(\overline{Tf})\widehat{\mathrm{ch}}_{\mu_n}(\overline{E}))\big)-\frac{1}{2}\int_{Y_{\mu_n}(\mathbf{C})}\mathrm{Td}_g(TY_{\mathbf{C}})\mathrm{ch}_g(E_{\mathbf{C}})R_g(TY_{\mathbf{C}})$), compose the right arrow in the diagram of Th. 4.4 with $\widehat{\deg}_{\mu_n}$ and compute the resulting expression using Th. 7.12. **Q.E.D.**

Notice that in the last formula, the map $\widehat{\deg}\circ f_*$ has been implicitly extended to $\widehat{\mathrm{CH}}(Y_{\mu_n})\otimes_{\mathbf{Z}}\mathbf{C}$ by linearity. Notice also that the non-equivariant analytic torsion, which is implicitly present on the right side of the diagram of Th. 4.4, has disappeared. We notice that there exists an immersion of group schemes $\mu_{n/(n,m)}\to\mu_n$ for all $m\in\mathbf{Z}/(n)$ (recall that (n,m) is the greatest common divisor of m and n), corresponding to the surjection $\mathbf{Z}/(n)\to\mathbf{Z}/(n/(n,m))$ of ordinary groups which maps 1 on m. The scheme Y as well as the bundle E

are thus naturally $\mu_{n/(n,m)}$-equivariant. For each m, we shall choose ζ_n^m as a generator of $R_{n/(m,n)}$, when we apply Th. 4.4. For simplicity's sake, let us now suppose that f is flat and that $H^i(Y,E)=0$ for $i>0$. Let us write $\overline{H^0(Y,E)}^m$ for the hermitian module $\overline{H^0(Y,E)}$ viewed as a hermitian $\mu_{n/(n,m)}$-comodule. Using the Fourier transform on finite abelian groups, we can compute that for $k\in\mathbf{Z}/(n)$

$$-\log(\mathrm{covol}(\overline{H^0(Y,E)}_k))=\frac{1}{n}\sum_{k'\in R_n}\widehat{\deg}_{\mu_{n/(n,k')}}(\overline{H^0(Y,E)}^{k'})\zeta_n^{-k.k'}.$$

We can thus apply the formula in Th. 7.14 to compute $\log(\mathrm{covol}(\overline{H^0(Y,E)}_k))$.

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