A fixed point formula of Lefschetz type in Arakelov geometry II: a residue formula / Une formule du point fixe de type Lefschetz en géométrie d'Arakelov II: une formule des résidus

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Abstract

This is the second of a series of papers dealing with an analog in Arakelov geometry of the holomorphic Lefschetz fixed point formula. We use the main result [KR1, Th. 4.4] of the first paper to prove a residue formula "à la Bott" for arithmetic characteristic classes living on arithmetic varieties acted upon by a diagonalisable torus; recent results of Bismut-Goette on the equivariant (Ray-Singer) analytic torsion play a key role in the proof. / Cet article est le second d'une série d'articles dont l'objet est un analogue en géométrie d'Arakelov de la formule du point fixe de Lefschetz holomorphe. Nous utilisons le résultat principal [KR1, Th. 4.4] du premier article pour prouver une formule des résidus "à la Bott" pour des classes caractéristiques vivant sur des variétés arithmétiques munis d'une action de tore; de récents résultats de Bismut-Goette sur la torsion analytique équivariante (de Ray-Singer) joue un rôle clé dans la preuve.

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1 Introduction

This is the second of a series of four papers on equivariant Arakelov theory and a fixed point formula therein. We give here an application of the main result [KR1, Th. 4.4] of the first paper.

We prove a residue formula "à la Bott" (Theorem 2.11) for the arithmetic Chern numbers of arithmetic varieties endowed with the action of a diagonalisable torus. More precisely, this formula computes arithmetic Chern numbers of equivariant Hermitian vector bundles (in particular, the height relatively to some equivariant projective embedding) as a contribution of arithmetic Chern numbers of bundles living of the fixed scheme and an anomaly term, which depends on the complex points of the variety only. Our determination of the anomaly term relies heavily on recent results by Bismut-Goette ([BGo]). The formula in 2.11 is formally similar to Bott's residue formula [AS, III, Prop. 8.13, p. 598] for the characteristic numbers of vector bundles, up to the anomaly term. Our method of proof is similar to Atiyah-Singer's and is described in more detail in the introduction to section 2. The effective computability of the anomaly term is also discussed there.

Apart from the residue formula itself, this article has the following two side results, which are of independent interest and which we choose to highlight here, lest they remain unnoticed in the body of the proof of Th. 2.11. The first one is a corollary of the residue formula, which shows that the height relatively to equivariant line bundles on torus-equivariant arithmetic varieties depends on less data than on general varieties (see corollary 2.9):

Proposition. *Let Y be an arithmetic variety endowed with a torus action. Write $Y_{\mathcal{T}}$ for the fixed point scheme of Y. Suppose that $\overline{L}, \overline{L}'$ are torus-equivariant hermitian line bundles. If there is an equivariant isometry $\overline{L}_{Y_{\mathcal{T}}} \simeq \overline{L}'_{Y_{\mathcal{T}}}$ over $Y_{\mathcal{T}}$ and an equivariant (holomorphic) isometry $\overline{L}_{\mathbf{C}} \simeq \overline{L}'_{\mathbf{C}}$ over $Y_{\mathbf{C}}$ then the height of Y relatively to $\overline{L}$ is equal to the height of Y relatively to $\overline{L}'$.*

The second one is a conjecture which naturally arises in the course of the proof of the residue formula (see lemma 2.3):

Conjecture. *Let M be a S^1-equivariant projective complex manifold, equipped with an S^1-invariant Kähler metric. Let $\overline{E}$ be a S^1-equivariant complex vector bundle on M, equipped with a S^1-invariant hermitian metric. Let $T_{g_t}(\cdot)$ (resp. $R_{g_t}(\cdot)$ of E) be the equivariant analytic torsion of $\overline{E}$ (resp. the equivariant R-genus), with respect to the automorphism $e^{2i\pi t}$. There is a rational function Q with complex coefficients and a pointed neighborhood $\mathring{U}$ of 0 in $\mathbf{R}$ such that*

$$\frac{1}{2}T_{g_t}(M,\overline{E}) - \frac{1}{2}\int_{M_{g_t}} \mathrm{Td}_{g_t}(TM)\mathrm{ch}_{g_t}(E)R_{g_t}(TM) = Q(e^{2\pi it})$$

if $t \in \mathring{U}$

(here M_{g_t} is the fixed point set of the automorphism $e^{2i\pi t}$, ch_{g_t} is the equivariant Chern character and Td_{g_t} is the equivariant Todd genus - see section 4 of [KR1] for more details).

The lemma 2.3 shows that this conjecture is verified, when the geometric objects appearing in it have certain models over the integers but it seems unlikely that the truth of the conjecture should be dependent on the existence of such models.
The appendix is logically independent of the rest of the article. We formulate a conjectural generalisation of the main result of [KR1].
The notations and conventions of the section 4 of [KR1] (describing the main result) and 6.2 (containing a translation of the fixed point formula into arithmetic Chow theory) will be used without comment. This article is a part of the habilitation thesis of the first author.
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2 An "arithmetic" residue formula

In this subsection, we consider arithmetic varieties endowed with an action of a diagonalisable torus. We shall use the fixed point formula [KR1, Th. 4.4] to obtain a formula computing arithmetic characteristic numbers (like the height relatively to a Hermitian line bundle) in terms of arithmetic characteristic numbers of the fixed point scheme (a "residual" term) and an anomaly term derived from the equivariant and non-equivariant analytic torsion. One can express this term using characteristic currents only, without involving the analytic torsion (see subsection 2). See equation (12) for a first version of the residue formula (where the anomaly term is expressed via the analytic torsion) and 2.11 for the final formula (where the anomaly term is expressed using a characteristic current). One can use the residue formula to compute the height of some flag varieties; there the anomaly term can be computed using the explicit values for the torsion given in [K2]. We shall nevertheless not carry out the details of this application, as the next paper [KK] gives a general formula for the height of flag varieties.
The strategy of proof we follow here is parallel to Atiyah-Singer's in [AS, Section 8]. Notice however that our proof, which involves the γ-operations, works in the algebraic case as well. The fundamental step of the proof is a passage to the limit on both sides of the arithmetic fixed point formula, where the limit is taken on finite group schemes of increasing order inside a given torus. Both sides of the fixed point formula can be seen as rational functions of a circle element near 1 and one can thus identify their constant coefficients. The constant coefficient of the arithmetic Lefschetz trace is the arithmetic Euler characteristic, which can in turn be related with arithmetic characteristic numbers via the (arithmetic) Riemann-Roch formula.

Furthermore, following a remark of J.-M. Bismut, we would like to point out that a direct proof of the formula 2.11 seems tractable. One could proceed as in the proof of the fixed point formula [KR1, Th. 4.4] (by deformation to the normal cone) and replace at each step the anomaly formulae for the equivariant analytic torsion by the anomaly formulae for the integral appearing in 2.11, the latter formulae having much easier proofs (as they do not involve the spectrum of Laplace operators). One would thus avoid mentioning the analytic torsion altogether. If [KR1, Th. 4.4] and the work of Bismut-Goette was not available, this would probably be the most natural way to approach the residue formula. Let $\mathcal{T} :=$ Spec $\mathbf{Z}[X, X^{-1}]$ be the one-dimensional torus over $\mathbf{Z}$. Let $f : Y \to$ Spec $\mathbf{Z}$ be a regular scheme, flat over $\mathbf{Z}$, endowed with a $\mathcal{T}$-projective action and such that the fixed scheme $Y_{\mathcal{T}}$ is flat over $\mathbf{Z}$ (this requirement is only necessary because we choose to work with arithmetic Chow theory). Let $d+1$ be the absolute dimension of Y. This action induces a holomorphic group action of the multiplicative group $\mathbf{C}^*$ on the manifold $Y(\mathbf{C}) =: M$ and thus an action of the circle $S^1 \subseteq \mathbf{C}^*$. We equip $Y(\mathbf{C})$ once and for all with an S^1-invariant Kähler metric $\omega^{TY(\mathbf{C})} = \omega^{TM}$ (such a metric can be obtained explicitly via an embedding into some projective space). Now let $m > 0$ be a strictly positive integer coprime to n. Consider the homomorphism $s_{m,n} : \mathbf{Z} \to \mathbf{Z}/n$, given by the formula $a \mapsto m.(a \bmod n)$. This homomorphism induces an immersion $i_{m,n} : \mu_n \to \mathcal{T}$ of group schemes. Let now E be a $\mathcal{T}$-equivariant bundle on Y. Recall that the equivariant structure of E induces a $\mathbf{Z}$-grading on the restriction $E|_{Y_{\mathcal{T}}}$ of E to the fixed point scheme of the action of $\mathcal{T}$ on Y; the k-th term ($k \in \mathbf{Z}$) of this grading is then denoted by E_k.

Lemma 2.1 *Write $E^{m,n}$ for E viewed as a μ_n-equivariant bundle via $i_{m,n}$. There exists an $\epsilon > 0$ such that for all $k \in \mathbf{Z}$ the natural injection $E_k \to E^{n,m}_{s_{m,n}(k)}$ is an isomorphism if $1/n < \epsilon$.*

Proof: This natural injection is an isomorphism iff the equality $s_{m,n}(k) = s_{m,n}(k')$ ($k, k' \in \mathbf{Z}$) implies that $k = k'$. Now notice that the kernel of $s_{m,n}$ is generated by n. Thus the implication is realized if we choose ϵ such that $1/\epsilon > 2.\max\{|k|\,|k \in \mathbf{Z}, E_k \neq 0\}$ and we are done. **Q.E.D.**

Corollary 2.2 *Let P be a projective space over $\mathbf{Z}$ endowed with a global action of the torus $\mathcal{T}$. Write $P^{m,n}$ for P viewed as a μ_n-equivariant scheme via $i_{m,n}$. Then there exists $\epsilon > 0$, such that if $1/n < \epsilon$, then the closed immersion $P_{\mathcal{T}} \to P^{m,n}_{\mathcal{T}}$ is an isomorphism.*

Proof: Let M be a free $\mathbf{Z}$-module endowed with a $\mathcal{T}$-action, such that there is an equivariant isomorphism $P \simeq \mathbf{P}(M)$. Let us write $M^{m,n}$ for M viewed as a μ_n-comodule via $i_{m,n}$. By the description of the fixed scheme given in [KR1, Prop. 2.9], we have $P_{\mathcal{T}} = \coprod_{k \in \mathbf{Z}} \mathbf{P}(M_k)$ and $P^{m,n}_{\mathcal{T}} = \coprod_{k \in \mathbf{Z}/n} \mathbf{P}(M^{m,n}_k)$. Furthermore, by construction the immersion $\mathbf{P}(M_k) \to P$ factors through the immersion $\mathbf{P}(M_k) \to \mathbf{P}(M^{m,n}_{s_{m,n}(k)})$ induced by the injection $M_k \to M^{n,m}_{s_{m,n}(k)}$.

By the last lemma, there exists an $\epsilon > 0$ such that for all $k \in \mathbf{Z}$ the natural injection $M_k \to M^{n,m}_{s_{m,n}(k)}$ is an isomorphism if $1/n < \epsilon$. From this, we can conclude. **Q.E.D.**

Let again E be a $\mathcal{T}$-equivariant bundle on Y, such that the cohomology of E vanishes in positive degrees. We equip $E_{\mathbf{C}}$ with an S^1-invariant metric (such a metric can be obtained from an arbitrary metric by integration). Consider E and Y as μ_n-equivariant via $i_{m,n}$. We shall apply [KR1, Th. 6.14] to E. For this application, we fix the primitive root of unity $\mathrm{e}^{2i\pi m/n}$ of $\mu_n(\mathbf{C})$. If $\alpha \in \mathbf{C}^*$, we shall write $g(\alpha)$ for the corresponding automorphism of $Y(\mathbf{C})$ and we let $g_{m,n} := g(\mathrm{e}^{2i\pi m/n})$. Set $M := Y(\mathbf{C})$. By [KR1, Th. 6.14], we get

$$\begin{aligned}\widehat{\deg}_{\mu_n}(R^0 f_*(\overline{E})) &= \frac{1}{2}T_{g_{m,n}}(M,\overline{E}) - \frac{1}{2}\int_{M_{g_{m,n}}} \mathrm{Td}_{g_{m,n}}(TM)\mathrm{ch}_{g_{m,n}}(E)R_{g_{m,n}}(TM) \\ &\quad + \widehat{\deg}\big(f^{\mu_n}_*\big(\sum_{i=0}^{\mathrm{rk}\, N^\vee_{Y/Y_{\mu_n}}}(-1)^i \widehat{\mathrm{ch}}_{\mu_n}(\Lambda^i(\overline{N}^\vee_{Y/Y_{\mu_n}}))\big)^{-1} \\ &\quad .\widehat{\mathrm{Td}}(\overline{Tf^{\mu_n}}).\widehat{\mathrm{ch}}_{\mu_n}(\overline{E})\big) \qquad (1)\end{aligned}$$

Furthermore, using the last lemma and its corollary, we see that there is an $\epsilon > 0$ and a formal Laurent power series $\mathcal{Q}(\cdot)$ with coefficients in $\widehat{\mathrm{CH}}(Y_{\mathcal{T}})_{\mathbf{C}}$ (of the form $P_1(z)/P_2(z)$, where $P_1(z)$ is a polynomial with coefficients in $\widehat{\mathrm{CH}}(Y_{\mathcal{T}})_{\mathbf{C}}$ and $P_2(z)$ is a polynomial with rational coefficients), such that for all n, m coprime with $1/n < \epsilon$, the term

$$\big(\sum_{i=0}^{\mathrm{rk}\, N^\vee_{Y/Y_{\mu_n}}}(-1)^i \widehat{\mathrm{ch}}_{\mu_n}(\Lambda^i(\overline{N}^\vee_{Y/Y_{\mu_n}}))\big)^{-1}.\widehat{\mathrm{Td}}(\overline{Tf^{\mu_n}}).\widehat{\mathrm{ch}}_{\mu_n}(\overline{E}) \qquad (2)$$

equals $\mathcal{Q}(\mathrm{e}^{2i\pi m/n})$. Similarly , there is an $\epsilon > 0$ and a rational function $Q(\cdot)$ with complex coefficients, such that for all n, m coprime with $1/n < \epsilon$, the term

$$\widehat{\deg}\big(f^{\mu_n}_*\big(\sum_{i=0}^{\mathrm{rk}\, N^\vee_{Y/Y_{\mu_n}}}(-1)^i \widehat{\mathrm{ch}}_{\mu_n}(\Lambda^i(\overline{N}^\vee_{Y/Y_{\mu_n}}))\big)^{-1}.\widehat{\mathrm{Td}}(\overline{Tf^{\mu_n}}).\widehat{\mathrm{ch}}_{\mu_n}(\overline{E})\big) \qquad (3)$$

equals $Q(\mathrm{e}^{2i\pi m/n})$. Since the elements of the type $\mathrm{e}^{2i\pi t}$, where $t \in \mathbf{Q}$, form a dense subset of S^1, we see that the function $Q(z)$ is uniquely determined. Let us call $A_{\mathcal{T}}(\overline{E})$ the constant term in the Laurent development of $Q(z)$ around 1. By construction, there is a polynomial $P(z)$ with complex coefficients, such that $\widehat{\deg}_{\mu_n}(R^0 f_*(\overline{E}))$ equals $P(\mathrm{e}^{2i\pi m/n})$. By density again, this polynomial is uniquely determined. The constant term of its Laurent development around 1 (i.e. its value at 1) is the quantity $\widehat{\deg}(R^0 f_* \overline{E})$. Using (1), we thus see that there is a uniquely determined rational function $Q'(z)$ with complex coefficients and an $\epsilon > 0$, such that the quantity

$$\frac{1}{2}T_{g_{m,n}}(M,\overline{E}) - \frac{1}{2}\int_{M_{g_{m,n}}} \mathrm{Td}_{g_{m,n}}(TM)\mathrm{ch}_{g_{m,n}}(E)R_{g_{m,n}}(TM)$$

equals $Q'(\mathrm{e}^{2i\pi m/n})$ if $1/n < \epsilon$.

Now notice the following. Let $I \subset \mathbf{R}$ be an interval such that the fixed point set M_{g_t} does not vary for $t \in I$. Let $g_t := g(e^{2\pi i t})$. Then R_{g_t} varies continuously on I (e.g. using [K2, Remark p. 108]).

Lemma 2.3 *There is a pointed neighborhood $\overset{\circ}{U}$ of 0 in $\mathbf{R}$ such that*

$$\frac{1}{2}T_{g_t}(M,\overline{E}) - \frac{1}{2}\int_{M_{g_t}} \mathrm{Td}_{g_t}(TM)\mathrm{ch}_{g_t}(E)R_{g_t}(TM) = Q'(e^{2\pi i t})$$

if $t \in \overset{\circ}{U}$. Furthermore this equality holds for all up to finitely many values of $e^{2\pi i t} \in S^1$.

Proof: It remains to prove that the analytic torsion $T_{g_t}(M,\overline{E})$ is continuous in t on $\overset{\circ}{U}$ (see also [BGo]). Let $I :=]0,\epsilon[$ be an interval on which the fixed point set M_{g_t} does not vary. Let $M_{g_t} = \bigcup_\mu M_\mu$ be the decomposition of the fixed point set into connected components of dimension $\dim M_\mu =: d_\mu$.

Let $P^\perp$ denote the projection of $\Gamma^\infty(\Lambda^q T^{*0,1}M \otimes E)$ on the orthogonal complement of the kernel of the Kodaira-Laplace operator $\square_q$ for $0 \leq q \leq d$. As shown in Donnelly [Do, Th. 5.1], Donnelly and Patodi [DoP, Th. 3.1] (see also [BeGeV, Th. 6.11]) the trace of the equivariant heat kernel of $\square$ for $u \to 0$ has an asymptotic expansion providing the formula

$$\sum_q (-1)^{q+1} q \mathrm{Tr}\, g_t^* e^{-u\square_q} P^\perp \sim \sum_\mu \sum_{k=-d_\mu}^{\infty} u^k \int_{M_\mu} b_k^\mu(t,x)\, d\mathrm{vol}_x$$

where the $b_k(t,x)$ are rational functions in t which are non-singular on I. Thus the analytic torsion is given by

$$\begin{aligned} T_{g_t}(M,\overline{E}) &= \int_1^\infty \sum_q (-1)^{q+1} q \mathrm{Tr}\, g_t^* e^{-u\square_q} P^\perp \frac{du}{u} \\ &+ \int_0^1 \left(\sum_q (-1)^{q+1} q \mathrm{Tr}\, g_t^* e^{-u\square_q} P^\perp - \sum_\mu \sum_{k=-d_\mu}^{0} u^k \int_{M_\mu} b_k^\mu(t,x)\, d\mathrm{vol}_x \right) \frac{du}{u} \\ &+ \sum_\mu \sum_{k=-d_\mu}^{-1} \frac{1}{k} \int_{M_\mu} b_k^\mu(t,x)\, d\mathrm{vol}_x - \Gamma'(1) \sum_\mu \int_{M_\mu} b_0^\mu(t,x)\, d\mathrm{vol}_x \;. \end{aligned}$$

The integrand of the first term is uniformly bounded (in t) by the non-equivariant heat kernel. Hence we see in particular that $T_{g_t}(M,\overline{E})$ is continuous in $t \in I$. As the equation in the lemma holds on a dense subset of I, it holds in I and by symmetry for a pointed neighborhood of 0. **Q.E.D.**

Recall that $d+1$ is the (absolute) dimension of Y. Consider the vector field $K \in \Gamma(TM)$ such that $e^{tK} = g_t$ on M. In [K1] the function R^{rot} on $\mathbf{R} \setminus 2\pi\mathbf{Z}$

has been defined as

$$R^{\rm rot}(\phi) := \lim_{s\to 0^+}\frac{\partial}{\partial s}\sum_{k=1}^{\infty}\frac{\sin k\phi}{k^s}$$

(according to Abel's Lemma the series in this definition converges for Re $s > 0$).

Corollary 2.4 *Let $D_\mu \subset \mathbf{Z}$ denote the set of all non-zero eigenvalues of the action of $K/2\pi$ on $TM_{|M_\mu}$ at the fixed point component M_μ. There are rational functions $Q'', Q_{\mu,k,l}$ for $k \in D_\mu$, $0 \le l \le d_\mu$ such that for all but finiteley many values of $e^{2\pi it}$*

$$T_{g_t}(M,\overline{E}) = Q''(e^{2\pi it}) + \sum_\mu \sum_{k\in D_\mu}\sum_{l=0}^{d_\mu} Q_{\mu,k,l}(e^{2\pi it})\cdot(\frac{\partial}{\partial t})^l R^{\rm rot}(2\pi ikt) \ .$$

The functions $Q_{\mu,k,l}$ depend only on the holomorphic structure of E and the complex structure on M.

Proof: For $\zeta = e^{i\phi} \in S^1$, $\zeta \ne 1$, let $L(\zeta, s)$ denote the zeta function defined in [KR1, section 3.3] with $L(\zeta, s) = \sum_{k=1}^\infty k^{-s}\zeta^k$ for Re $s > 0$. In [K2, equation (77)] it is shown that $L(\zeta,-l)$ is a rational function in ζ for $l \in \mathbf{N}_0$. Also by [K2, equation (80)],

$$\frac{\partial}{\partial s}_{|s=-l}(L(e^{i\phi},s)-(-1)^l L(e^{-i\phi},s)) = \left(\frac{-i\partial}{\partial\phi}\right)^l 2iR^{\rm rot}(\phi) \ .$$

The corollary follows by the definition of the Bismut R_g-class (see [KR1, Def. 3.6]) and lemma 2.3. **Q.E.D.**

Remark. One might reasonably conjecture that the Lemma 2.3 is valid on any compact Kähler manifold endowed with a holomorphic action of S^1.

Let us call $L_{\mathcal{T}}(\overline{E})$ the constant term in the Laurent development of $Q'(z)$ around $z = 1$. By lemma 2.3 we obtain

$$\widehat{\deg}\,(R^0 f_*(\overline{E})) = L_{\mathcal{T}}(\overline{E}) + A_{\mathcal{T}}(\overline{E}).$$

Since for any $\mathcal{T}$-equivariant bundle, one can find a resolution by acyclic (i.e. whose cohomology vanishes in positive degrees) $\mathcal{T}$-equivariant bundles, one can drop the acyclicity statement in the last equation. More explicitly, one obtains

$$\sum_{q\ge 0}(-1)^q\big(\widehat{\deg}\,((R^q f_*(\overline{E}))_{\rm free}) + \log(\#(R^q f_*(\overline{E}))_{\rm Tors})\big) = L_{\mathcal{T}}(\overline{E}) + A_{\mathcal{T}}(\overline{E}). \quad (4)$$

Notice that $Q'(z)$ and thus $L_{\mathcal{T}}(\overline{E})$ depends on the Kähler form ω^{TM} and $\overline{E_{\mathbf{C}}}$ only and can thus be computed without reference to the finite part of Y.
In the next subsection, we shall apply the last equation to a specific virtual vector bundle, which has the property that its Chern character has only a top

degree term and compute $A_{\mathcal{T}}(\cdot)$ in this case. We then obtain a first version of the residue formula, which arises from the fact that the left hand-side of the last equation is also computed by the (non-equivariant) arithmetic Riemann-Roch. The following subsection then shows how $L_{\mathcal{T}}(\cdot)$ can be computed using the results of Bismut-Goette [BGo]; combining the results of that subsection with the first version of the residue formula gives our final version 2.11.

2.1 Determination of the residual term

Let $\overline{F}$ be an $\mathcal{T}$-equivariant Hermitian vector bundle on Y.

Definition 2.5 *The polynomial equivariant arithmetic total Chern class $\widehat{c}_t(\overline{F}) \in \widehat{\mathrm{CH}}_{\mathbf{C}}(Y_{\mathcal{T}})[t]$ is defined by the formula*

$$\widehat{c}_t(\overline{F}) := \prod_{n\in\mathbf{Z}} \sum_{p=0}^{\mathrm{rk}\, F_n} \sum_{j=0}^{p} \binom{\mathrm{rk}\, F_n - j}{p-j} \widehat{c}_j(\overline{F}_n)(2\pi int)^{p-j}$$

where i is the imaginary constant.

We can accordingly define the k-th polynomial (equivariant, arithmetic) Chern class $\widehat{c}_{k,t}(\overline{F})$ of $\overline{F}$ as the part of $\widehat{c}_t(\overline{F})$ lying in $(\widehat{\mathrm{CH}}(Y_{\mathcal{T}})_{\mathbf{C}}[t])^{(k)}$, where $(\widehat{\mathrm{CH}}(Y_{\mathcal{T}})_{\mathbf{C}}[t])^{(k)}$ are the homogeneous polynomials of weighted degree k (with respect to the grading of $\widehat{\mathrm{CH}}(Y_{\mathcal{T}})_{\mathbf{C}}$). Define now $\Lambda_t(\overline{F})$ as the formal power series $\sum_{i\geq 0} \Lambda^i(\overline{F}).t^i$. Let $\gamma^q(\overline{F})$ be the q-th coefficient in the formal power series $\Lambda_{t/(1-t)}(\overline{F})$; this is a $\mathbf{Z}$-linear combination of equivariant Hermitian bundles. We denote by $\widehat{\mathrm{ch}}_t(\overline{F})$ the polynomial equivariant Chern character and by $\widehat{\mathrm{ch}}_t^q(\overline{F})$ the component of $\widehat{\mathrm{ch}}_t(\overline{F})$ lying in $(\widehat{\mathrm{CH}}(Y_{\mathcal{T}})_{\mathbf{C}}[t])^{(q)}$. We recall its definition. Let $N_j(x_1, \ldots, x_r)$ be the j-th Newton polynomial $\frac{1}{j!}(x_1^j + \ldots + x_r^j)$ in the variables $x_1, \ldots, x_r$. For $l \geq 0$, let $\sigma_l(x_1, \ldots, x_r)$ be the l-th symmetric function in the variables $x_1, \ldots x_r$. By the fundamental theorem on symmetric functions, there is a polynomial in r variables N'_j, such that $N'_j(\sigma_1(x_1, \ldots x_r), \ldots, \sigma_r(x_1, \ldots x_r)) = N_j(x_1, \ldots x_r)$. We let $\widehat{\mathrm{ch}}_t(\overline{F}) := \sum_{j\geq 0} N'_j(\widehat{c}_{1,t}(\overline{F}), \widehat{c}_{2,t}(\overline{F}), \ldots \widehat{c}_{\mathrm{rk}\, F,t}(\overline{F}))$.

Lemma 2.6 *The element $\widehat{\mathrm{ch}}_t^p(\gamma^q(\overline{F} - \mathrm{rk}\, F))$ is equal to $\widehat{c}_{q,t}(\overline{F})$ if $p = q$ and vanishes if $p < q$.*

Proof: It is proved in [GS3, II, Th. 7.3.4] that $\widehat{\mathrm{ch}}$, as a map from the arithmetic Grothendieck group $\widehat{K}_0(Y)$ to the arithmetic Chow theory $\widehat{\mathrm{CH}}(Y)$ is a map of λ-rings, where the second ring is endowed with the λ-ring structure arising from its grading. Thus $\widehat{\mathrm{ch}}_t^p(\gamma^q(\overline{F} - \mathrm{rk}\, F))$ is a polynomial in the Chern classes $\widehat{c}_1(\overline{F}), \widehat{c}_2(\overline{F}), \ldots$ and the variable t. By construction, its coefficients only depend on the equivariant structure of F restricted to $Y_{\mathcal{T}}$. We can thus suppose for the time of this proof that the action of $\mathcal{T}$ on Y is trivial. To identify these coefficients, we consider the analogous expression $\mathrm{ch}_t^p(\gamma^q(F-\mathrm{rk}\,F))$ with values in the polynomial ring $\mathrm{CH}(Y)[t]$, where $\mathrm{CH}(Y)$ is the algebraic Chow ring. By the same token this is a polynomial in the classical Chern classes $c_1(F), c_2(F), \dots$ and the variable t. As the forgetful map $\widehat{\mathrm{CH}}(Y) \to \mathrm{CH}(Y)$ is a map of λ-rings, the coefficients of these polynomials are the same. Thus we can apply the algebraic equivariant splitting principle [Thom, Th. 3.1] and suppose that $F = \oplus_{i=1}^{j} L_j$, where the L_i are equivariant line bundles. We compute $\mathrm{ch}_t^p(\gamma^q(F-\mathrm{rk}\,F)) = \mathrm{ch}_t^p(\sigma_q(L_1-1,\dots,L_j-1)) = (\sigma_q(\mathrm{ch}_t(L_1)-1,\dots,\mathrm{ch}_t(L_j)-1))^{(p)}$. As the term of lowest degree in $ch_t(L_i-1)$ is $c_{1,t}(L_i)$, which is of (total!) degree 1, the term of lowest degree in the expression after the last equality is $\sigma_q(c_{1,t}(L_1),\dots,c_{1,t}(L_j))$ which is of degree q and is equal to $c_{q,t}(F)$ and so we are done. **Q.E.D.**

Remark. An equivariant holomorphic vector bundle E splits at every component M_μ of the fixed point set into a sum of vector bundles $\bigoplus E_\theta$ such that K acts on E_θ as $i\theta \in i\mathbf{R}$. The E_θ are those $E_{n,\mathbf{C}}$ which do not vanish on M_μ. Equip E with an invariant Hermitian metric. Then the polynomial equivariant total Chern form $c_{tK}(\cdot)$ is given by the formula

$$c_{tK}(\overline{E})_{|M_\mu} = \det(\frac{-\Omega^E}{2\pi i} + it\Theta^E + \mathrm{Id}) = \prod_{\theta\in\mathbf{R}} \sum_{q=0}^{\mathrm{rk}\,E_\theta} c_q(\overline{E}_\theta)(1+it\theta)^{\mathrm{rk}\,E_\theta - q} \qquad (5)$$

where Θ^E denotes the action of K on E restricted to M_μ. Let N be the normal bundle to the fixed point set. Set

$$(c_{tK}^{\mathrm{top}}(\overline{N})^{-1})' := \frac{\partial}{\partial b}_{|b=0} c_{\mathrm{rk}\,N}(\frac{-\Omega^N}{2\pi i} + it\Theta + b\,\mathrm{Id})^{-1}$$

where Θ is the action of K on N. Furthermore, let r denote the additive characteristic class which is given by

$$\begin{aligned} r_K(L)_{|M_\mu} \quad &:= \quad -\frac{1}{c_1(L)+i\phi_\mu}\left(-2\Gamma'(1) + 2\log|\phi_\mu| + \log(1+\frac{c_1(L)}{i\phi_\mu})\right) \\ &= \quad -\sum_{j\geq 0} \frac{(-c_1(L))^j}{(i\phi_\mu)^{j+1}}\left(-2\Gamma'(1) + 2\log|\phi_\mu| - \sum_{k=1}^{j}\frac{1}{k}\right) \end{aligned}$$

for L a line bundle acted upon by K with an angle $\phi_\mu \in \mathbf{R}$ at M_μ (i.e. the Lie derivative by K acts as multiplication by ϕ_μ).

In the next proposition, if $\overline{E}$ is a Hermitian equivariant bundle, we write $\widehat{\mathrm{Td}}_{g_t}(\overline{Tf})\widehat{\mathrm{ch}}_{g_t}(\overline{E})$ for the formal Laurent power series development in t of the function $\mathcal{Q}(\mathrm{e}^{2\pi it})$, where $\mathcal{Q}(\cdot)$ is the function defined in (2). Set $\widehat{c}_t^{\mathrm{top}}(\overline{E}) := \widehat{c}_{\mathrm{rk}\,E,t}(\overline{E})$ for any equivariant Hermitian vector bundle $\overline{E}$. Note that this class is invertible in the ring of Laurent polynomials $\widehat{\mathrm{CH}}(Y_{\mathcal{T}})_{\mathbf{C}}[t,1/t]$ if $E_{Y_{\mathcal{T}}}$ has no invariant subbundle (for an explicit expression see equation (8)).

Proposition 2.7 *Let $q_1, \dots, q_k$ be natural numbers such that $\sum_j q_j = d+1$. Let $\overline{E}^1, \dots, \overline{E}^k$ be $\mathcal{T}$-equivariant Hermitian bundles. Set $x := \prod_j \gamma^{q_j}(\overline{E}^j - \operatorname{rk} E^j)$. The expression*

$$\widehat{\mathrm{Td}}_{g_t}(\overline{Tf})\widehat{\mathrm{ch}}_{g_t}(x) \tag{6}$$

has a formal Taylor series expansion in t. Its constant term is given by

$$\widehat{c}_t^{\,\mathrm{top}}(\overline{N})^{-1} \prod_j \widehat{c}_{q_j,t}(\overline{E}^j) \tag{7}$$

which is independent of t. Also for $t \to 0$

$$\begin{aligned} &\mathrm{Td}_{g_t}(TM) R_{g_t}(TM) \mathrm{ch}_{g_t}(x) \\ &= \log(t^2) \cdot (c_K^{\mathrm{top}}(N)^{-1})' \prod_j c_{q_j,K}(E^j) \\ &\quad + \frac{r_K(N)}{c_K^{\mathrm{top}}(N)} \prod_j c_{q_j,K}(E^j) + O(t) \ . \end{aligned}$$

Note that the first statement implies that $\widehat{\deg} f_*^{\mathcal{T}}(\widehat{c}_t^{\,\mathrm{top}}(\overline{N})^{-1} \prod_j \widehat{c}_{q_j,t}(\overline{E}^j)) = A_{\mathcal{T}}(x)$.

Proof: To prove that the first statement holds, we consider that by construction, both the expression (7) and the constant term of (6) (as a formal Laurent power series) are universal polynomials in the Chern classes of the terms of the grading of Tf and the terms of the grading of x. By using Grassmannians (more precisely, products of Grassmannians) as in the proof of 2.6, we can reduce the problem of the determination of these coefficients to the algebraic case and then suppose that all the relevant bundles split. Thus, without loss of generality, we consider a vector bundle $\overline{E} := \bigoplus_\nu \overline{L}_\nu$ which splits into a direct sum of line bundles $\overline{L}_\nu$ on which $\mathcal{T}$ acts with multiplicity m_ν. Assume now $m_\nu \neq 0$ for all ν. Set $x_\nu := \hat{c}_1(\overline{L}_\nu)$. Then

$$\widehat{\mathrm{Td}}_{g_t}(\overline{E}) = \prod \left(1 - e^{-2\pi i t m_\nu - x_\nu}\right)^{-1} .$$

Now

$$\begin{aligned} \left(1 - e^{-2\pi i t m_\nu - x_\nu}\right)^{-1} &= \frac{1}{2\pi i t m_\nu + x_\nu} + O(1) \\ &= \sum_{j=0}^{d - \operatorname{rk} N + 1} \frac{(-x_\nu)^j}{(2\pi i t m_\nu)^{j+1}} + O(1) \end{aligned}$$

as $t \to 0$. By definition,

$$\widehat{c}_t^{\,\mathrm{top}}(\overline{E})^{-1} = \prod_\nu \frac{1}{2\pi i t m_\nu + x_\nu} = \prod_\nu \sum_{j=0}^{d - \operatorname{rk} N + 1} \frac{(-x_\nu)^j}{(2\pi i t m_\nu)^{j+1}} \ . \tag{8}$$

Thus, $\widehat{\mathrm{Td}}_{g_t}(\overline{E})$ has a Laurent expansion of the form

$$\widehat{\mathrm{Td}}_{g_t}(\overline{E}) = \widehat{c}_t^{\mathrm{top}}(\overline{E})^{-1} + \sum_{j=0}^{d-\mathrm{rk}\, N+1} t^{-\mathrm{rk}\, N-j}\hat{q}_j(t)$$

with classes $\hat{q}_j$ of degree j which have a Taylor expansion in t and $\hat{q}_j(0) = 0$. As $\widehat{\mathrm{Td}}_{g_t}(\overline{Tf}^{\mathcal{T}}) = 1+$ (terms of higher degree), we get in particular for the relative tangent bundle (assumed w.l.o.g. to be not only a virtual bundle, but a vector bundle)

$$\widehat{\mathrm{Td}}_{g_t}(\overline{Tf}) = \widehat{c}_t^{\mathrm{top}}(\overline{N})^{-1} + \sum_{j=0}^{d-\mathrm{rk}\, N+1} t^{-\mathrm{rk}\, N-j}\hat{p}_j(t) \tag{9}$$

with classes $\hat{p}_j$ of degree j which have a Taylor expansion in t and $\hat{p}_j(0) = 0$. Let $\deg_Y\alpha$ denote the degree of a Chow class α and define $\deg_t t^k := k$ for $k \in \mathbf{Z}$. Then any component α_t of the power series $\widehat{\mathrm{Td}}_{g_t}(\overline{Tf})$ satisfies

$$(\deg_Y + \deg_t)\alpha_t \geq -\mathrm{rk}\, N$$

and equality is achieved precisely for the summand $\widehat{c}_t^{\mathrm{top}}(\overline{N})^{-1}$. Furthermore, by Lemma 2.6

$$\widehat{\mathrm{ch}}_{g_t}(x) = \prod_{j=1}^{k}\left(\widehat{c}_{q_j,t}(\overline{E}^j) + \widehat{s}_j(t)\right) \tag{10}$$

with classes $\widehat{s}_j(t)$ of degree larger than q_j which have a Taylor expansion in t and $\widehat{s}_j(0) = 0$. Hence, any component α_t of the power series $\widehat{\mathrm{ch}}_{g_t}(x)$ satisfies

$$(\deg_Y + \deg_t)\alpha_t \geq d + 1$$

and equality holds iff α_t is in the $\prod_{j=1}^{k}\widehat{c}_{q_j,t}(\overline{E}^j)$-part. Hence any component α_t of $\widehat{\mathrm{Td}}_{g_t}(\overline{Tf})\widehat{\mathrm{ch}}_{g_t}(x)$ satisfies

$$(\deg_Y + \deg_t)\alpha_t \geq d - \mathrm{rk}\, N + 1\ . \tag{11}$$

In particular the product has no singular terms in t, as $\deg_Y\beta \leq d - \mathrm{rk}\, N + 1$ for any Chow class β on the fixed point scheme. In other words, by multiplying formulae (9) and (10) one obtains

$$\widehat{\mathrm{Td}}_{g_t}(\overline{Tf})\widehat{\mathrm{ch}}_{g_t}(x) = \widehat{c}_t^{\mathrm{top}}(\overline{N})^{-1}\prod_{j=1}^{k}\widehat{c}_{q_j,t}(\overline{E}^j) + O(t)\ ,$$

and the first summand on the right hand side has $\deg_Y = d - \mathrm{rk}\, N + 1$, thus it is constant in t. Hence we get formula (7). Now choose $\epsilon > 0$ such that the fixed point set of g_t does not vary on $t \in]0, \epsilon[$. To prove the second formula, we proceed similarly and we formally split TM as a topological vector bundle into line bundles with first Chern class x_ν, acted upon by K with an angle θ_ν. The formulae for the Lerch zeta function in [K2, p. 108] or in [B2, Th. 7.10] show that the R-class is given by

$$R_{g_t}(TM) = -\sum_{\theta\neq 0} \frac{1}{x_\nu + it\theta_\nu}\left(-2\Gamma'(1) + 2\log|t\theta_\nu| + \log(1+\frac{x_\nu}{it\theta_\nu})\right) + O(1)$$

for $t \to 0$. Note that the singular term is of the form $\alpha_1(t)\log|t| + \alpha_2(t)$ with

$$(\deg_Y + \deg_t)\alpha_\mu(t) \geq -1$$

for $\mu = 1, 2$ (in fact, equality holds). As $(c_{tK}^{\text{top}}(N)^{-1})' = c_{tK}^{\text{top}}(N)^{-1}\sum_{\theta\neq 0}\frac{-1}{x_\nu+it\theta_\nu}$ by definition, one obtains

$$\begin{aligned}
&\mathrm{Td}_{g_t}(TM)R_{g_t}(TM)\mathrm{ch}_{g_t}(x)\\
&= \frac{r_{tK}(N)}{c_{tK}^{\text{top}}(N)}\prod_j c_{q_j,tK}(E^j) + O(t)\\
&= \left((c_K^{\text{top}}(N)^{-1})'\log(t^2) + \frac{r_K(N)}{c_K^{\text{top}}(N)}\right)\prod_j c_{q_j,K}(E^j) + O(t)
\end{aligned}$$

because the first term on the right hand side is again independent of $t \in \mathbf{R}$ (except the $\log(t^2)$). **Q.E.D.**

Note that the arithmetic Euler characteristic has a Taylor expansion in t. Thus we get using Proposition 2.7 and Lemma 2.3

Corollary 2.8 *The equivariant analytic torsion of x on the d-dimensional Kähler manifold M has an asymptotic expansion for $t \to 0$*

$$T_{g_t}(M, \prod_j \gamma^{q_j}(\overline{E}^j - \mathrm{rk}\, E^j)) = \log(t^2)\int_{M_K}\prod_j c_{q_j,K}(\overline{E}^j)(c_K^{\text{top}}(\overline{N})^{-1})' + C_0 + O(t)$$

with $C_0 \in \mathbf{C}$.

A more general version of this corollary is a consequence of [BGo] (see the next section). We now combine our results with the (non-equivariant) arithmetic Riemann-Roch theorem. We compute

$$\begin{aligned}
&\widehat{\deg}\,(f_*(\prod_j \hat{c}_{q_j}(\overline{E}^j)))\\
&= \widehat{\deg}\,(f_*(\widehat{\mathrm{Td}}(\overline{Tf})\prod_j \hat{c}_{q_j}(\overline{E}^j))) = \widehat{\deg}\,(f_*(\widehat{\mathrm{Td}}(\overline{Tf})\widehat{\mathrm{ch}}(\prod_j \gamma^{q_j}(\overline{E}^j - \mathrm{rk}\, E^j))))\\
&= \widehat{\deg}\,(f_*(\prod_j \gamma^{q_j}(\overline{E}^j - \mathrm{rk}\, E^j))) - \frac{1}{2}T(\prod_j \gamma^{q_j}(\overline{E}^j - \mathrm{rk}\, E^j))\\
&= L_{\mathcal{T}}(\prod_j \gamma^{q_j}(\overline{E}^j - \mathrm{rk}\, E^j)) - \frac{1}{2}T(\prod_j \gamma^{q_j}(\overline{E}^j - \mathrm{rk}\, E^j))
\end{aligned}$$

$$+A_{\mathcal{T}}(\prod_j \gamma^{q_j}(\overline{E}^j - \mathrm{rk}\, E^j))$$

$$= L_{\mathcal{T}}(\prod_j \gamma^{q_j}(\overline{E}^j - \mathrm{rk}\, E^j)) - \frac{1}{2}T(\prod_j \gamma^{q_j}(\overline{E}^j - \mathrm{rk}\, E^j))$$

$$+\widehat{\deg}\,(f_*^{\mathcal{T}}(\frac{\prod_j \widehat{c}_{q_j,t}(\overline{E}^j)}{\widehat{c}_t^{\mathrm{top}}(\overline{N})}))$$

The first equality is justified by the fact that the 0-degree part of the Todd class is 1; the second one is 2.6; the third one is justified by the arithmetic Riemann-Roch theorem ([GS8, eq. (1)]); the fourth one is justified by (4) and the last one by the last proposition. Finally, we get the following residue formula:

$$\widehat{\deg}\, f_*(\prod_j \widehat{c}_{q_j}(\overline{E}^j)) = L_{\mathcal{T}}(\prod_j \gamma^{q_j}(\overline{E}^j - \mathrm{rk}\, E^j))$$

$$-\frac{1}{2}T(Y(\mathbf{C}), \prod_j \gamma^{q_j}(\overline{E}^j - \mathrm{rk}\, E^j)) + \widehat{\deg}\, f_*^{\mathcal{T}}(\frac{\prod_j \widehat{c}_{q_j,t}(\overline{E}^j)}{\widehat{c}_t^{\mathrm{top}}(\overline{N})}). \qquad (12)$$

In particular, if $\overline{L}$ is a $\mathcal{T}$-equivariant line bundle on Y, one obtains the following formula for the height $h_Y(\overline{L})$ of Y relatively to $\overline{L}$:

$$h_{\overline{L}}(Y) := \widehat{\deg}\,(\widehat{c}_1(\overline{L})^{d+1}) = L_{\mathcal{T}}((\overline{L}-1)^{d+1})$$

$$-\frac{1}{2}T(Y(\mathbf{C}), (\overline{L}-1)^{d+1}) + \widehat{\deg}\, f_*^{\mathcal{T}}(\frac{\widehat{c}_{1,t}(\overline{L})^{d+1}}{\widehat{c}_t^{\mathrm{top}}(\overline{N})}).$$

In our final residue formula, we shall use results of Bismut-Goette to give a formula for the term $L_{\mathcal{T}}(\cdot) - \frac{1}{2}T(Y(\mathbf{C}), \cdot)$. Notice however that the last identity already implies the following corollary:

Corollary 2.9 *Let Y be an arithmetic variety endowed with a $\mathcal{T}$-action. Suppose that $\overline{L}, \overline{L}'$ are $\mathcal{T}$-equivariant hermitian line bundles. If there is an equivariant isometry $\overline{L}_{Y_{\mathcal{T}}} \simeq \overline{L}'_{Y_{\mathcal{T}}}$ over $Y_{\mathcal{T}}$ and an equivariant (holomorphic) isometry $\overline{L}_{\mathbf{C}} \simeq \overline{L}'_{\mathbf{C}}$ over $Y_{\mathbf{C}}$ then $h_{\overline{L}}(Y) = h_{\overline{L}'}(Y)$.*

2.2 The limit of the equivariant torsion

Let K' denote any nonzero multiple of K. The vector field K' is Hamiltonian with respect to the Kähler form as the action on M factors through a projective space. Let $M_{K'} = M_K$ denote the fixed point set with respect to the action of K'. For any equivariant holomorphic Hermitian vector bundle $\overline{F}$ we denote by $\mu^F(K') \in \Gamma(M, \mathrm{End}(F))$ the section given by the action of the difference of the Lie derivative and the covariant derivative $L^F_{K'} - \nabla^F_{K'}$ on F. Set as in [BeGeV, ch. 7]

$$\mathrm{Td}_{K'}(\overline{TM}) := \mathrm{Td}(-\frac{\Omega^{TM}}{2\pi i} + \mu^{TM}(K')) \in \mathfrak{A}(M)$$

and

$$\mathrm{ch}_{K'}(\overline{F}) := \mathrm{Tr}\,\exp(-\frac{\Omega^F}{2\pi i} + \mu^F(K')) \in \mathfrak{A}(M)\ .$$

The Chern class $c_{q,K'}(\overline{F})$ for $0 \le q \le \mathrm{rk}\,F$ is defined as the part of total degree $\deg_Y + \deg_t = q$ of

$$\det(\frac{-\Omega^F}{2\pi i} + t\mu^F(K') + \mathrm{Id})$$

at $t = 1$, thus $c_{q,K'}(\overline{F}) = c_q(-\Omega^F/2\pi i + \mu^F(K'))$. Let $K'^* \in T^*_{\mathbf{R}}M$ denote the 1-form dual to K' via the metric on $T_{\mathbf{R}}M$, hence $\iota_{K'}K'^* = \|K'\|^2$ is the norm square in $T_{\mathbf{R}}M$. Set $d_{K'}K'^* := (d - 2\pi i\iota_{K'})K'^*$ and define

$$s_{K'}(u) := \frac{-\omega^{TM}}{2\pi u}\exp(\frac{d_{K'}K'^*}{4\pi i u}) = \sum_{\nu=0}^{d-1}\frac{-\omega^{TM}(dK'^*)^\nu}{2\pi u(4\pi i u)^\nu \nu!}e^{-\|K'\|^2/2u}\ .$$

For a smooth differential form η it is shown in [BGo] (see also [B1, section C,D]) that the following integrals are well-defined:

$$A_{K'}(\eta)(s) := \frac{1}{\Gamma(s)}\int_0^1\int_M \eta s_{K'}(u)u^{s-1}du$$

for Re $s > 1$ and

$$B_{K'}(\eta)(s) := \frac{1}{\Gamma(s)}\int_1^\infty\int_M \eta s_{K'}(u)u^{s-1}du$$

for Re $s < 1$. Also it is shown in [BGo] (compare [B1, Proof of theorem 7]) that $s \mapsto A_{K'}(\eta)(s)$ has a meromorphic extension to $\mathbf{C}$ which is holomorphic at $s = 0$ and that

$$\begin{aligned} A_{K'}(\eta)'(0) + B_{K'}(\eta)'(0) &= \int_1^\infty\int_M \eta s_{K'}(u)\frac{du}{u} \\ &+ \int_0^1\int_M \eta\left(s_{K'}(u) + \left(\frac{\omega^{TM}}{2\pi u}c_{K'}^{\mathrm{top}}(\overline{N})^{-1} - (c_{K'}^{\mathrm{top}}(\overline{N})^{-1})'\right)\delta_{M_{K'}}\right)\frac{du}{u} \\ &+ \int_{M_{K'}}\eta\left(\frac{\omega^{TM}}{2\pi}c_{K'}^{\mathrm{top}}(\overline{N})^{-1} - \Gamma'(1)(c_{K'}^{\mathrm{top}}(\overline{N})^{-1})'\right) \end{aligned}$$

for the derivatives $A_{K'}(\eta)'$, $B_{K'}(\eta)'$ of $A_{K'}(\eta)$, $B_{K'}(\eta)$ with respect to s; also

$$A_{K'}(\eta)(0) + B_{K'}(\eta)(0) = \int_{M_{K'}}\eta(c_{K'}^{\mathrm{top}}(\overline{N})^{-1})'\ .$$

Define the **Bismut S-current** $S_{K'}(M, \omega^{TM})$ by the relation

$$\int_M \eta S_{K'}(M, \omega^{TM}) := A_{K'}(\eta)'(0) + B_{K'}(\eta)'(0)\ .$$

In particular, one notices

$$\begin{aligned}\int_M \eta S_{K'}(M,\omega^{TM}) = \lim_{a\to 0^+} \Bigg[\int_a^\infty \int_M \eta s_{K'}(u)\frac{du}{u} \\ + \int_a^1 \int_M \eta \left(\frac{\omega^{TM}}{2\pi u} c_{K'}^{\mathrm{top}}(\overline{N})^{-1} - (c_{K'}^{\mathrm{top}}(\overline{N})^{-1})' \right) \delta_{M_{K'}} \frac{du}{u} \\ + \int_{M_{K'}} \eta \left(\frac{\omega^{TM}}{2\pi} c_{K'}^{\mathrm{top}}(\overline{N})^{-1} - \Gamma'(1)(c_{K'}^{\mathrm{top}}(\overline{N})^{-1})' \right) \Bigg] \\ = \lim_{a\to 0^+} \Bigg[\int_M \eta \cdot 2i\omega^{TM} \frac{1-\exp(\frac{d_{K'}K'^*}{4\pi i a})}{d_{K'}K'^*} \\ + \int_{M_{K'}} \eta \left(\frac{\omega^{TM}}{2\pi a} c_{K'}^{\mathrm{top}}(\overline{N})^{-1} - (\Gamma'(1)+\log a)(c_{K'}^{\mathrm{top}}(\overline{N})^{-1})' \right) \Bigg] .\end{aligned}$$

By Lemma 2.3 and Proposition 2.7, we already know that

$$\lim_{t\to 0} \left(T_{g_t}(M,x) - \log(t^2) \int_{M_K} \prod_j c_{q_j,K}(\overline{E}^j) \cdot (c_K^{\mathrm{top}}(\overline{N})^{-1})' \right)$$

exists and

$$\begin{aligned}2L_{\mathcal{T}}(x) &= \lim_{t\to 0} \left(T_{g_t}(M,x) - \log(t^2) \int_{M_K} \prod_j c_{q_j,K}(\overline{E}^j) \cdot (c_K^{\mathrm{top}}(\overline{N})^{-1})' \right) \\ &\quad -\frac{1}{2} \int_{Y_{\mathcal{T}}(\mathbf{C})} \prod_j c_{q_j,K}(E^j) \cdot \frac{r_K(N)}{c_K^{\mathrm{top}}(N)} . \qquad (13)\end{aligned}$$

Now we shall compute this limit.

Theorem 2.10 *The limit of the equivariant analytic torsion of* $x = \prod_j \gamma^{q_j}(\overline{E}^j - \mathrm{rk}\, E^j)$ *associated to the action of* g_t *for* $t \to 0$ *is given by*

$$\begin{aligned}\lim_{t\to 0} \left(T_{g_t}(M,x) - \log(t^2) \int_{M_K} \prod_j c_{q_j,K}(\overline{E}^j) \cdot (c_K^{\mathrm{top}}(\overline{N})^{-1})' \right) \\ = T(M,x) + \int_M \prod_j c_{q_j,K}(\overline{E}^j) \cdot S_K(M,\omega^{TM})\end{aligned}$$

Proof: Let $I_{K'}$ denote the additive equivariant characteristic class which is given for a line bundle L as follows: If K' acts at the fixed point p by an angle $\theta \in \mathbf{R}$ on L, then

$$I_{K'}(L)_{|p} := \sum_{k\neq 0} \frac{\log(1+\frac{\theta}{2\pi k})}{c_1(L)+i\theta+2k\pi i} .$$

The main result of [BGo] implies that for $t \in \mathbf{R} \setminus \{0\}$, t sufficiently small, there is a power series T_t in t with $T_0 = T(M,x)$ such that

$$\begin{aligned} T_{g_t}(M,x) - T_t &= \int_M \mathrm{Td}_{tK}(\overline{TM})\mathrm{ch}_{tK}(x) S_{tK}(M,\omega^{TM}) \\ &\quad - \int_{M_g} \mathrm{Td}_{g_t}(TM)\mathrm{ch}_{g_t}(x) I_{tK}(N_{M_g/M}) \; . \end{aligned}$$

For $t \to 0$, both $I_{tK}(N_{M_g/M}) \to 0$ and $\mathrm{Td}_{g_t}(\overline{TM})\mathrm{ch}_{g_t}(x) \to 0$ (by eq. (11)), thus the last summand vanishes.

As in equation (10) $\mathrm{ch}_{tK}(x) = \prod_j c_{q_j,tK}(\overline{E}^j) + \tilde{\eta}(t)$ with a form $\tilde{\eta}$ such that $(\deg_Y + \deg_t)\tilde{\eta}(t) > d+1$. Thus $\mathrm{Td}_{tK}(\overline{TM})\mathrm{ch}_{tK}(x) = \prod_j c_{q_j,tK}(\overline{E}^j) + \eta(t)$ with $(\deg_Y + \deg_t)\eta(t) > d+1$. Also $(\deg_Y + \deg_t) s_{tK}(t^2u) = -1$, hence we observe that

$$\int_M \mathrm{Td}_{tK}(\overline{TM})\mathrm{ch}_{tK}(x) s_{tK}(t^2u) = \int_M \left(\prod_j c_{q_j,K}(\overline{E}^j) s_K(u) + \eta(t) s_{tK}(t^2u) \right) \; .$$

Let $\tilde{\eta}(t)$ denote the form obtained from $\eta(t)$ by multiplying the degree $\deg_Y = j$ part with t^{-j-1} for $0 \le j \le d$. By making the change of variable from u to t^2u we get

$$(A_{tK} + B_{tK})(\mathrm{Td}_{tK}(\overline{TM})\mathrm{ch}_{tK}(x))(s) = t^{2s}(A_K + B_K)(\prod_j c_{q_j,K}(\overline{E}^j) + \tilde{\eta}(t))(s) \; .$$

Thus we find

$$\begin{aligned} &\int_M \mathrm{Td}_{tK}(\overline{TM})\mathrm{ch}_{tK}(x) S_{tK}(M,\omega^{TM}) \\ &= \log(t^2) \cdot (A_K + B_K)(\prod_j c_{q_j,K}(\overline{E}^j))(0) \\ &\quad + (A_K + B_K)(\prod_j c_{q_j,K}(\overline{E}^j))'(0) + O(t \log(t^2)) \end{aligned}$$

which implies the statement of the theorem. **Q.E.D.**

2.3 The residue formula

By combining equation (12) and Theorem 2.10, we obtain the following formula. Recall that $\mathcal{T}$ is the one-dimensional diagonalisable torus over Spec $\mathbf{Z}$, that $f : Y \to \mathrm{Spec}\,\mathbf{Z}$ is a flat, $\mathcal{T}$-projective morphism and that the fixed scheme $f^{\mathcal{T}} : Y_{\mathcal{T}} \to \mathrm{Spec}\,\mathbf{Z}$ is assumed to be flat over Spec $\mathbf{Z}$. We let $d+1$ be the absolute dimension of Y. We choose $\mathcal{T}$-equivariant Hermitian bundles $\overline{E}^j$ on Y and positive integers q_j such that $\sum_j q_j = d+1$. We deduce by combining equations (12), (13) and Theorem 2.10

Theorem 2.11

$$\widehat{\deg}\left(f_*(\prod_j \widehat{c}_{q_j}(\overline{E}^j))\right) = \widehat{\deg}\left(f_*^{\mathcal{T}}(\frac{\prod_j \widehat{c}_{q_j,t}(\overline{E}^j)}{\widehat{c}_t^{\mathrm{top}}(\overline{N})})\right)$$
$$+\frac{1}{2}\int_{Y(\mathbf{C})}\prod_j c_{q_j,K}(\overline{E}^j)\cdot S_K(Y(\mathbf{C}),\omega^{TY(\mathbf{C})}) - \frac{1}{2}\int_{Y_{\mathcal{T}}(\mathbf{C})}\prod_j c_{q_j,K}(E^j)\cdot\frac{r_K(N)}{c_K^{\mathrm{top}}(N)}\ .$$

Example. Assume that the fixed point scheme is flat of Krull dimension 1. The normal bundle to $Y_{\mathcal{T}}$ splits as $N = \bigoplus_{n\in\mathbf{Z}} N_n$. Thus

$$\hat{c}_t^{\mathrm{top}}(\overline{N})^{-1} = \frac{1}{\prod_n (2\pi itn)^{\mathrm{rk}\, N_n}}\left(1 - \sum_{n\in\mathbf{Z}}\frac{\hat{c}_1(\overline{N}_n)}{2\pi itn}\right)$$

by equation (8). Also, at a given point $p \in M_K$ the tangent space decomposes as $TM_{|p} = \bigoplus TM_{\theta_\nu}$, where K acts with angle θ_ν on TM_{θ_ν}. Then

$$\frac{r_K(N)}{c_K^{\mathrm{top}}(N)}_{|p} = \frac{1}{\prod_\theta i\theta}\sum_\theta \frac{2\Gamma'(1) - 2\log|\theta|}{i\theta}$$

where the θ are counted with their multiplicity. Furthermore, in this case

$$\begin{aligned}\int_M \eta S_K(M,\omega^{TM}) = {}& \lim_{a\to 0^+}\left[\int_M \eta\cdot 2i\omega^{TM}\frac{1-\exp(\frac{d_K K^*}{4\pi ia})}{d_K K^*}\right.\\ &\left.-(\Gamma'(1)+\log a)\int_{M_K}\eta\cdot(c_K^{\mathrm{top}}(\overline{N})^{-1})'\right]\\ = {}& \int_M \eta\cdot 2i\omega^{TM}\left(\frac{1}{d_K K^*} + \frac{(dK^*)^{d-1}}{(2\pi i\|K\|)^d}\right)\\ &-\lim_{a\to 0^+}\left[\int_M \eta\cdot\frac{2i\omega^{TM}(dK^*)^{d-1}}{(2\pi i\|K\|)^d}(1-e^{-\frac{1}{2a}\|K\|^2})\right.\\ &\left.+(\Gamma'(1)+\log a)\int_{M_K}\eta\cdot(c_K^{\mathrm{top}}(\overline{N})^{-1})'\right]\ .\end{aligned}$$

Now consider a line bundle $\overline{\mathcal{L}}$, splitting as $\bigoplus_k \overline{\mathcal{L}}_k$ on the fixed point scheme (where the $\mathcal{L}_k$ are locally free of rank ≤ 1). We find

$$\begin{aligned}\widehat{c}_{1,t}(\overline{\mathcal{L}})^{d+1} &= \sum_k (\widehat{c}_1(\overline{\mathcal{L}}_k) + 2\pi itk\,\mathrm{rk}\,\mathcal{L}_k)^{d+1}\\ &= \sum_k \left((2\pi itk)^{d+1}\mathrm{rk}\,\mathcal{L}_k + (d+1)(2\pi itk)^d\hat{c}_1(\overline{\mathcal{L}}_k)\right)\ ,\end{aligned}$$

thus

$$
\begin{aligned}
&\widehat{\deg}\, f_*^{\mathcal{T}}\left(\frac{\widehat{c}_{1,t}(\overline{\mathcal{L}})^{d+1}}{\widehat{c}_t^{\mathrm{top}}(\overline{N})}\right)\\
&= \quad \widehat{\deg}\, f_*^{\mathcal{T}} \sum_k \frac{k^d}{\prod_n n^{\mathrm{rk}\, N_n}} \left((d+1)\widehat{c}_1(\overline{\mathcal{L}}_k) - k\, \mathrm{rk}\, \mathcal{L}_k \cdot \sum_{n\in\mathbf{Z}} \frac{\widehat{c}_1(\overline{N}_n)}{n}\right).
\end{aligned}
$$

Now notice that at a given fixed point p over $\mathbf{C}$ all but one $\mathcal{L}_{k,\mathbf{C}}$ vanish and set $\phi_p := 2\pi k$. We compute

$$
-\frac{1}{2}\sum_{p\in M_K} \frac{c_{1,K}(L)^{d+1}}{c_K^{\mathrm{top}}(N)} r_K(N) = \sum_{p\in M_K} \frac{\phi_p^{d+1}}{\prod_\theta \theta} \sum_\theta \frac{-\Gamma'(1)+\log|\theta|}{\theta}
$$

and

$$
-\frac{1}{2}(\Gamma'(1)+\log a) \sum_{p\in M_K} c_{1,K}(L)^{d+1}(c_K^{\mathrm{top}}(\overline{N})^{-1})' = \sum_{p\in M_K} \frac{\phi_p^{d+1}}{\prod_\theta \theta} \sum_\theta \frac{\Gamma'(1)+\log a}{2\theta}\ .
$$

Hence we finally get

$$
\begin{aligned}
\widehat{\deg}\, f_* \widehat{c}_1(\overline{\mathcal{L}})^{d+1} \quad = \quad & \widehat{\deg}\, f_*^{\mathcal{T}} \sum_k \frac{k^d}{\prod_n n^{\mathrm{rk}\, N_n}} \left((d+1)\widehat{c}_1(\overline{\mathcal{L}}_k) - k\, \mathrm{rk}\, \mathcal{L}_k \cdot \sum_{n\in\mathbf{Z}} \frac{\widehat{c}_1(\overline{N}_n)}{n}\right)\\
& + \sum_{p\in M_K} \frac{\phi_p^{d+1}}{\prod_\theta \theta} \sum_\theta \frac{-\Gamma'(1)+\log(\theta^2)}{2\theta}\\
& + \int_M c_{1,K}(L)^{d+1} \cdot i\omega^{TM} \left(\frac{1}{d_K K^*} + \frac{(dK^*)^{d-1}}{(2\pi i\|K\|)^d}\right)\\
& - \lim_{a\to 0^+} \left[\int_M \mu^L(K)^{d+1} \cdot \frac{i\omega^{TM}(dK^*)^{d-1}}{(2\pi i\|K\|)^d}(1-e^{-\frac{1}{2a}\|K\|^2})\right.\\
& \left. - \log a \sum_{p\in M_K} \frac{\phi_p^{d+1}}{\prod_\theta \theta} \sum_\theta \frac{1}{2\theta}\right].
\end{aligned}
$$

3 Appendix: a conjectural relative fixed point formula in Arakelov theory

Since the first part of this series of articles was written, Xiaonan Ma defined in [Ma] higher analogs of the equivariant analytic torsion and proved curvature and anomaly formulae for it (in the case of fibrations by tori, this had already been done in [K4]). Once such formulae for torsion forms are available, one can formulate a conjectural fixed point formula, which fully generalizes [KR1, Th. 4.4] to the relative setting. Let G be a compact Lie group and let M and M'

be complex manifolds on which G acts by holomorphic automorphisms. Let $f: M \to M'$ be a smooth G-equivariant morphism of complex manifolds. Let ω^{TM} be a G-invariant Kähler metric on M (a Kähler fibration structure would in fact be sufficient). Let $\overline{E}$ be a G-equivariant Hermitian holomorphic vector bundle on M and suppose for simplicity that $R^k f_* E = 0$ for $k > 0$. Now let g be the automorphism corresponding to some element of G. The **equivariant higher analytic torsion** $T_g(f, \overline{E})$ is a certain element of $\widetilde{\mathfrak{A}}(M'_g)$, which satisfies the curvature formula

$$dd^c T_g(f, \overline{E}) = \mathrm{ch}_g(f_* \overline{E}) - \int_{M_g/M'_g} \mathrm{Td}_g(\overline{Tf}) \mathrm{ch}_g(\overline{E}).$$

where $\overline{Tf}$ is endowed with the metric induced by ω^{TM}. The term in degree zero of $T_g(f, \overline{E})$ is the equivariant analytic torsion $T_g(f^{-1}(x), \overline{E}|_{f^{-1}(x)})$ of the restriction of $\overline{E}$ to the fiber of f over $x \in M'_g$.
Now let Y, B be μ_n-equivariant arithmetic varieties over some fixed arithmetic ring D and let $f: Y \to B$ be a map over D, which is flat, μ_n-projective and smooth over the complex numbers. Fix an $\mu_n(\mathbf{C})$-invariant Kähler metric on $Y(\mathbf{C})$. If $\overline{E}$ is an f-acyclic (meaning that $R^k f_* E = 0$ if $k > 0$) μ_n-equivariant Hermitian bundle on Y, let $f_* \overline{E}$ be the direct image sheaf (which is locally free), endowed with its natural equivariant structure and L_2-metric. Consider the rule which associates the element $f_* \overline{E} - T_g(f, \overline{E})$ of $\widehat{K}_0^{\mu_n}(B)$ to every f-acyclic equivariant Hermitian bundle $\overline{E}$ and the element $\int_{Y(\mathbf{C})_g/B(\mathbf{C})_g} \mathrm{Td}_g(\overline{Tf})\eta \in \widetilde{\mathfrak{A}}(B_{\mu_n})$ to every $\eta \in \widetilde{\mathfrak{A}}(Y_{\mu_n})$. The proof of the following proposition is then similar to the proof of [KR1, Prop. 4.3].

Proposition 3.1 *The above rule induces a group homomorphism* $f_*: \widehat{K}_0^{\mu_n}(Y) \to \widehat{K}_0^{\mu_n}(B)$.

Let $\mathcal{R}$ be the ring appearing in the statement of [KR1, Th. 4.4]. We are ready to formulate the following conjecture.

Conjecture 3.2 *Let*

$$M(f) = (\lambda_{-1}(\overline{N}^{\vee}_{Y/Y_{\mu_n}}))^{-1} \lambda_{-1}(f^* \overline{N}^{\vee}_{B/B_{\mu_n}})(1 - R_g(N_{Y/Y_{\mu_n}}) + R_g(f^* N_{B/B_{\mu_n}})).$$

The diagram

$$\begin{array}{ccc} \widehat{K}_0^{\mu_n}(Y) & \xrightarrow{M(f)\cdot\rho(\cdot)} & \widehat{K}_0^{\mu_n}(Y_{\mu_n}) \otimes_{R(\mu_n)} \mathcal{R} \\ \downarrow f_* & & \downarrow f_*^{\mu_n} \\ \widehat{K}_0^{\mu_n}(B) & \xrightarrow{\rho(\cdot)} & \widehat{K}_0^{\mu_n}(B_{\mu_n}) \otimes_{R(\mu_n)} \mathcal{R} \end{array}$$

commutes.

About this conjecture, we make the following comments:

(a) One can carry over the principle of the proof of [KR1, Th. 4.4] to prove this conjecture, provided a generalization of the immersion formula [B3, Th. 0.1] is available (which is not the case at the moment). We shall however not go into the details of this argument.

(b) Without formal proof again, we notice that the conjecture holds, if $B_{\mu_n}(\mathbf{C})$ has dimension 0. In that case the torsion forms are not necessary to define the direct image and the proof of [KR1, Th. 4.4] pulls through altogether.

The conjecture described in this appendix can be used to obtain explicit formulae for arithmetic (Arakelov-) characteristic classes of bundles of modular forms on Shimura varieties; see [K5] and [MR].

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