

Planar graphs with the maximum number of induced 6-cycles

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Abstract

For large n we determine the maximum number of induced 6-cycles which can be contained in a planar graph on n vertices, and we classify the graphs which achieve this maximum. In particular we show that the maximum is achieved by the graph obtained by blowing up three pairwise non-adjacent vertices in a 6-cycle to sets of as even size as possible, and that every extremal example closely resembles this graph. This extends previous work by the author which solves the problem for 4-cycles and 5-cycles. The 5-cycle problem was also solved independently by Ghosh, Györi, Janzer, Paulos, Salia, and Zamora.

1 Introduction

The problem of determining for a fixed small graph H the maximum number of induced copies of H which can be contained in a graph on n vertices was first considered in 1975 by Pippenger and Golumbic [18], and has since received considerable attention. As summarised in [7], this maximum is now known asymptotically for all graphs H on at most four vertices except the path of length 3. In the case where H is a k -cycle for $k \geq 5$, Pippenger and Golumbic conjectured in their paper that the maximum is asymptotically $\frac{n^k}{k^k - k}$, which they showed is an asymptotic lower bound for all k -vertex graphs H . In 2016 this conjecture was verified in the case $k = 5$ by Balogh, Hu, Lidický, and Pfender in [3]. The current best known upper bound for general k -cycles is $2n^k/k^k$, which is due to Král', Norin, and Volec [16]. Results for other graphs H can be found in [14] and [22].

A related problem on which significant progress has been made recently is to consider what happens when we restrict to planar graphs. There are two interesting problems in this setting, firstly to determine the maximum number of not necessarily induced copies of a small graph H that can be contained in a planar graph on n vertices, which we write

as $f(n, H)$, and secondly to determine the same quantity when we insist that the copies of H are induced. We write $f_I(n, H)$ for this second quantity.

In a moment we shall focus on the case where H is a cycle, but we first note that $f(n, H)$ has also been studied for other graphs H , for example see [2], [4], [6], [9], [10], [20], and [21]. In the general case, Huynh, Joret, and Wood determined in [15] the order of magnitude of $f(n, H)$ for all H in terms of a graph parameter called the ‘flap-number’ of H . Liu then showed in [17] (Corollary 6.1 in that paper) that for all graphs H , $f_I(n, H) = \Theta(f(n, H))$, which in turn determines the order of magnitude of $f_I(n, H)$ for all H .

Focussing now on the case where H is a cycle, in 1979 Hakimi and Schmeichel [13] showed that $f(n, C_3) = 3n - 8$ for $n \geq 3$ and $f(n, C_4) = \frac{1}{2}(n^2 + 3n - 22)$ for $n \geq 4$. Much more recently, in 2019, $f(n, C_5)$ was determined for all $n \geq 5$ by Győri, Paulos, Salia, Tompkins, and Zamora in [11]. Since every copy of C_3 in a graph is an induced copy, Hakimi and Schmeichel’s result also determines $f_I(n, C_3)$ for all $n \geq 3$. The values of $f_I(n, C_4)$ and $f_I(n, C_5)$ were determined for large enough n in [19]. In each case the extremal graphs were classified in the respective paper, with a small correction to the extremal graphs for $f(7, C_4)$ and $f(8, C_4)$ given by Alameddine in [1]. The value of $f_I(n, C_5)$ for large n was also found independently by Ghosh, Győri, Janzer, Paulos, Salia, and Zamora in [8].

No exact results are known for longer cycles but Cox and Martin showed in [4] and [5] that $f(n, C_{2k}) = \binom{n}{k}^k + o(n^k)$ for $k \in \{3, 4, 5, 6\}$, and they conjectured that this remains true for all $k \geq 7$. We can exhibit this lower bound by, for $m \geq 3$, letting $F_{n,m}$ be the n -vertex graph obtained by blowing up $\lfloor \frac{m}{2} \rfloor$ pairwise non-adjacent vertices in an m -cycle to sets of as even size as possible (see Figures 1a and 1b). Then $F_{n,2k}$ is a planar n -vertex graph containing $\binom{n}{k}^k + o(n^k)$ cycles of length $2k$, so $f(n, C_{2k}) \geq \binom{n}{k}^k + o(n^k)$. In fact, all of these $2k$ -cycles are induced, so we have the same lower bound for $f_I(n, C_{2k})$. Thus since $f_I(n, H) \leq f(n, H)$ for all n and H , we have that if $f(n, C_{2k}) = \binom{n}{k}^k + o(n^k)$, then the same is true of $f_I(n, C_{2k})$. The best known upper bounds on $f(n, C_{2k})$ and $f_I(n, C_{2k})$ for $k \geq 7$ are $f_I(n, C_{2k}) \leq f(n, C_{2k}) \leq \frac{n^k}{k!} + O(n^{k-1/5})$ for all $k \geq 7$, again due to Cox and Martin in [4].

Much less is known about the case where H is an odd cycle of length greater than 5. In the same spirit as above, to obtain a lower bound on $f_I(n, C_{2k+1})$ we can observe that $F_{n,2k+1}$ is an n -vertex planar graph containing $\binom{n}{k}^k + o(n^k)$ induced $(2k+1)$ -cycles. To obtain a better lower bound on $f(n, C_{2k+1})$, for each $m \geq 3$ we can let $F'_{n,m}$ be the graph obtained from $F_{n,m}$ by adding a path through each of the blown-up vertex classes (see Figures 1c and 1d). Then $F'_{n,2k}$ is an n -vertex planar graph containing $2k \binom{n}{k}^k + o(n^k)$ (non-induced) $(2k+1)$ -cycles. Hakimi and Schmeichel showed in their paper that $f(n, C_{2k+1}) = O(n^k)$ (and thus the same is true of $f_I(n, C_{2k+1})$) and this seems to remain the best known upper bound.

The constructions of the graphs $F_{n,m}$ and $F'_{n,m}$ were first given in [12] and [9] respectively, and these graphs were later considered in [4], [5], and [8]. In [19] the following conjecture was made.

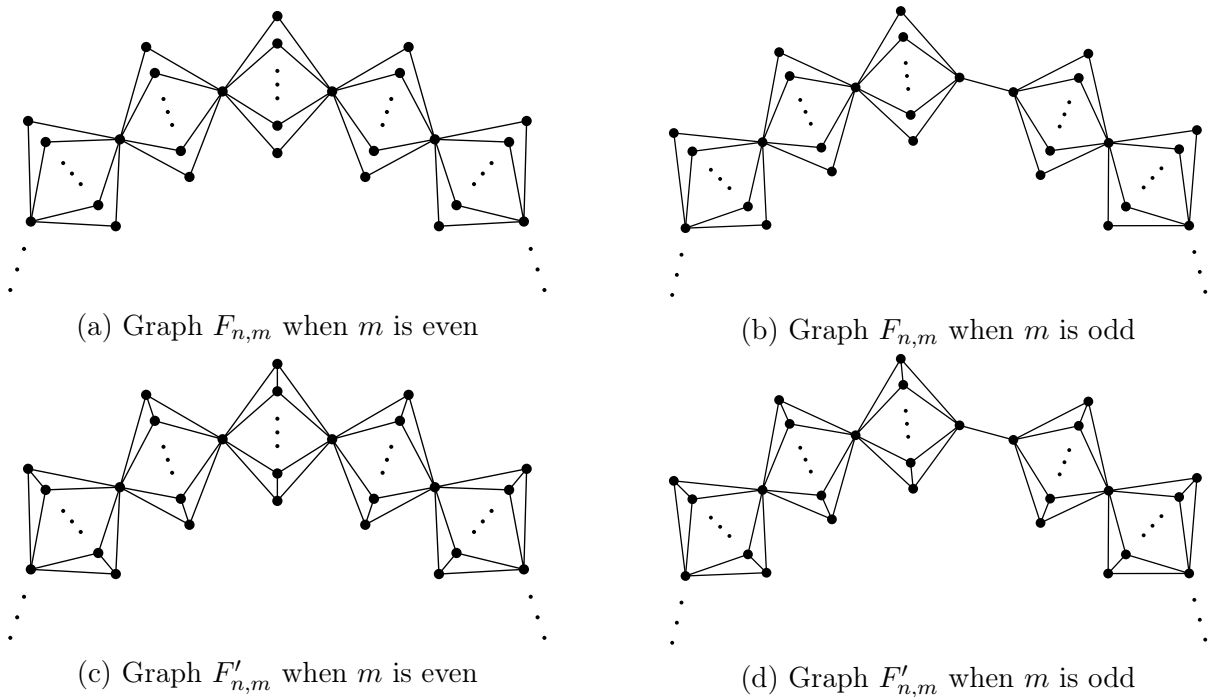


Figure 1: The graphs $F_{n,m}$ and $F'_{n,m}$ in different cases

Conjecture 1 ([19]). *For $k \geq 6$ and n sufficiently large relative to k , the graph $F_{n,k}$ contains $f_I(n, C_k)$ induced k -cycles.*

In this paper we will show that for large enough n , the n -vertex planar graphs containing $f_I(n, C_6)$ induced 6-cycles are exactly those which are subgraphs of $F'_{n,6}$ and which contain $F_{n,6}$ as a subgraph, thus establishing the conjecture when $k = 6$. By counting the number of induced 6-cycles in $F_{n,6}$ we can obtain from this a closed-form expression for $f_I(n, C_6)$ for large n . Before we formally state this result, for each $n \geq 6$ we will define \mathcal{F}_n to be the family consisting of the subgraphs of $F'_{n,6}$ which contain $F_{n,6}$ as a subgraph. Formally, we define \mathcal{F}_n as follows.

Definition 2. For $n \geq 6$ the family \mathcal{F}_n consists of the graphs G such that all of the following hold:

- (i) G has exactly n vertices, with vertex set $\{u_1, u_2, u_3\} \cup A \cup B \cup C$, where $|A| + |B| + |C| = n - 3$ and the sizes of A , B , and C are as equal as possible,
- (ii) for all $a \in A$, $b \in B$, and $c \in C$, G contains the edges au_1 , au_2 , bu_2 , bu_3 , cu_3 , and cu_1 , and
- (iii) there exist labelings of A , B , and C as $\{a_1, \dots, a_{|A|}\}$, $\{b_1, \dots, b_{|B|}\}$, and $\{c_1, \dots, c_{|C|}\}$ respectively such that any remaining edges of G form a subset of $\{a_i a_{i+1} : 1 \leq i \leq |A| - 1\} \cup \{b_i b_{i+1} : 1 \leq i \leq |B| - 1\} \cup \{c_i c_{i+1} : 1 \leq i \leq |C| - 1\}$.

An arbitrary graph in \mathcal{F}_n is illustrated in Figure 2. In this illustration, red lines indicate edges each of whose addition or omission does not affect whether the graph is in \mathcal{F}_n . We are now ready to state the main result of the paper.

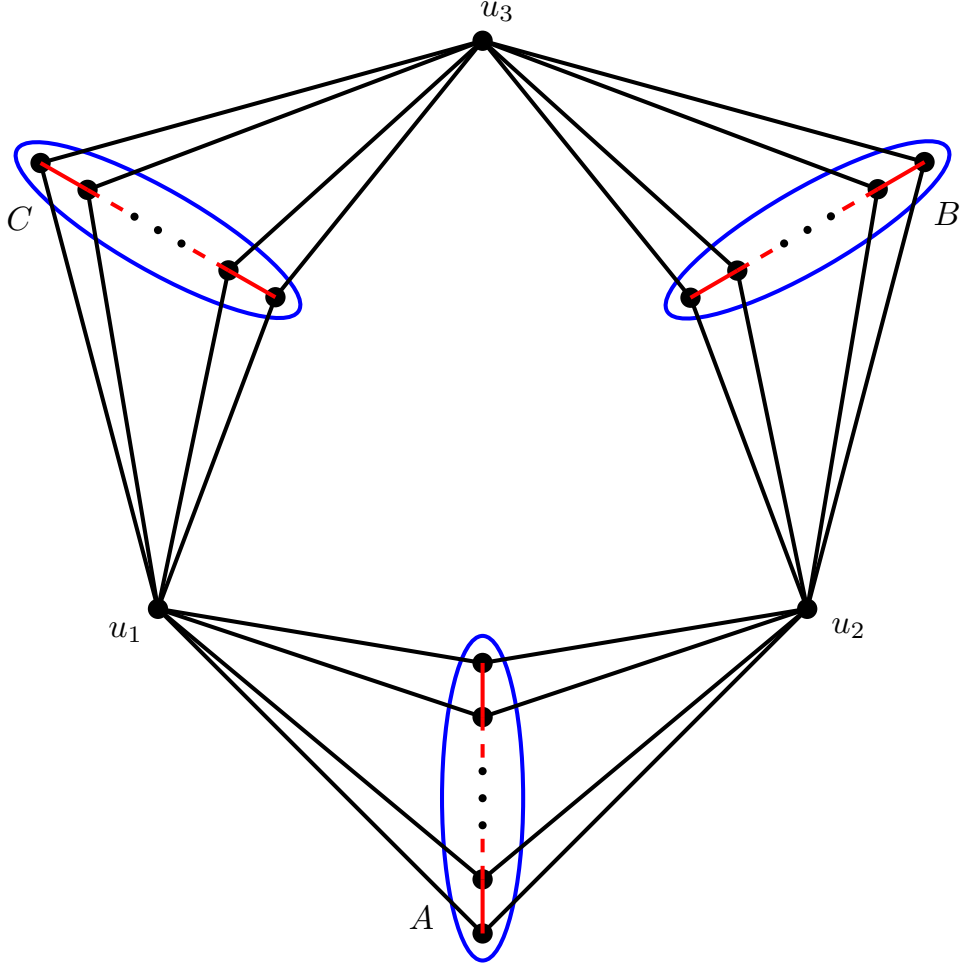


Figure 2: A graph in \mathcal{F}_n

Theorem 3. For large n , G is an n -vertex planar graph containing $f_I(n, C_6)$ induced 6-cycles if and only if $G \in \mathcal{F}_n$. Hence for large n ,

$$f_I(n, C_6) = \begin{cases} \left(\frac{n}{3} - 1\right)^3 = \frac{1}{27}(n^3 - 9n^2 + 27n - 27), & \text{if } n \equiv 0 \pmod{3} \\ \left(\frac{n-4}{3}\right)^2 \left(\frac{n-1}{3}\right) = \frac{1}{27}(n^3 - 9n^2 + 24n - 16), & \text{if } n \equiv 1 \pmod{3} \\ \left(\frac{n-2}{3}\right)^2 \left(\frac{n-5}{3}\right) = \frac{1}{27}(n^3 - 9n^2 + 24n - 20), & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

1.1 Notation and outline of the paper

We use standard notation from graph theory in this paper. In particular, C_k is the k -cycle graph, $K_{a,b}$ is the complete bipartite graph with parts of size a and b , and for a graph G and a subset S of its vertex set, $G - S$ is the subgraph of G obtained by deleting the vertices in S . For a vertex v of a graph G we write $N(v)$ for the (open) neighbourhood of v in G , that is, the set of vertices of G which are adjacent to v .

In the next section we first introduce some of the framework we will use to prove Theorem 3, and then we find some structure in large planar graphs in which no vertex

is in ‘few’ induced 6-cycles. In Section 3 we use this to show that every large planar graph contains a vertex in ‘not too many’ induced 6-cycles, and that if every vertex is in ‘many’ induced 6-cycles and the number of vertices in the graph is a multiple of 3, then the graph is in \mathcal{F}_n for the appropriate n . In Section 4 we use these results to show that for large n every vertex in a planar n -vertex graph containing $f_I(n, C_6)$ induced 6-cycles is in ‘many’ induced 6-cycles, from which we can deduce that all such graphs are in \mathcal{F}_n , which is the substance of Theorem 3. Finally in Section 5 we consider to what extent the arguments used are applicable to the problem of determining $f_I(n, C_k)$ for $k > 6$.

2 Preliminaries

We will use the following notation throughout this paper. Let v be a vertex of a planar graph G with distinct neighbours u and w . Define X^{uvw} to be the set of vertices of $G - \{u, v, w\}$ which are in an induced 6-cycle in G containing the path uvw . Then let $X_1^{uvw} = X^{uvw} \cap N(u)$, $X_3^{uvw} = X^{uvw} \cap N(w)$, and $X_2^{uvw} = X^{uvw} \setminus (X_1^{uvw} \cup X_3^{uvw})$, so that every induced 6-cycle in G containing the path uvw is of the form $uvwx_3x_2x_1$ for some $x_1 \in X_1$, $x_2 \in X_2$, and $x_3 \in X_3$.

Clearly the sets X_1^{uvw} , X_2^{uvw} , and X_3^{uvw} are pairwise disjoint. If there exists an induced 6-cycle in G containing the path uvw , then they are all non-empty and every vertex in X_1^{uvw} has a neighbour in X_2^{uvw} , every vertex in X_2^{uvw} has a neighbour in X_1^{uvw} and a neighbour in X_3^{uvw} , and every vertex in X_3^{uvw} has a neighbour in X_2^{uvw} . The following fundamental lemma gives further properties of these sets and introduces a framework that we will use often. It is inspired by a lemma in [8].

Lemma 4. *Let G , v , u , and w be as above, and let $X_1 = X_1^{uvw}$, $X_2 = X_2^{uvw}$, and $X_3 = X_3^{uvw}$. Assume that there exists an induced 6-cycle in G containing the path uvw . Then G_1 , the bipartite subgraph of G induced between X_1 and X_2 , is acyclic and hence the number of induced 6-cycles in G containing the path uvw is at most $|X_3|(|X_1| + |X_2| - 1)$. Similarly G_2 , the bipartite subgraph of G induced between X_2 and X_3 , is acyclic and the number of induced 6-cycles in G containing the path uvw is at most $|X_1|(|X_2| + |X_3| - 1)$.*

Proof. Suppose for a contradiction that G_1 contains the cycle $x_1y_1 \dots x_ky_k$ for some $k \geq 2$, where $x_i \in X_1$ and $y_i \in X_2$ for all i . Let $z_1 \in X_3$ be a common neighbour of w and y_1 , and let $z_2 \in X_3$ be a common neighbour of w and y_2 . If $z_1 \neq z_2$, then $\{x_1, x_2, w\}$ and $\{y_1, y_2, u\}$ form the partite sets of a subdivision of $K_{3,3}$ in G , which contradicts Kuratowski’s theorem. Otherwise $\{x_1, x_2, z_1\}$ and $\{y_1, y_2, u\}$ form such partite sets, again giving a contradiction. Hence G_1 is acyclic, as is G_2 , similarly.

Every induced 6-cycle in G containing the path uvw contains an edge of G_1 and a vertex in X_3 , and this edge and vertex pair determines the 6-cycle. Since G_1 is acyclic and X_1 and X_2 are non-empty, there are at most $|X_1| + |X_2| - 1$ edges in G_1 . Hence there are at most $|X_3|(|X_1| + |X_2| - 1)$ induced 6-cycles in G which contain the path uvw . Similarly, considering G_2 in place of G_1 and X_1 in place of X_3 , G contains at most $|X_1|(|X_2| + |X_3| - 1)$ such 6-cycles. \square

When we apply this result in Section 3 we will use the following basic facts.

Fact 5. (a) If $a, b, c \geq 0$ and $ab \geq c^2$, then $a + b \geq 2c$ with equality if and only if $a = b = c$.

(b) If $a, b, c, d \geq 0$, $ab \geq cd + 1$, and $|c - d| \leq 1$, then $a + b > c + d$.

Let G be a plane graph with a (not necessarily induced) $K_{2,7}$ subgraph. The drawing of this $K_{2,7}$ subgraph induced by the drawing of G splits the plane into seven regions, six bounded and one unbounded. Following the authors of [8], we say that this $K_{2,7}$ subgraph of G is an *empty* $K_{2,7}$ in G if the interiors of all six of the bounded regions formed by the $K_{2,7}$ subgraph contain no vertices of G . If G contains such a copy of $K_{2,7}$ then we say that G *contains an empty* $K_{2,7}$.

Inspired by a definition in [19], we say that two vertices in a planar graph G are *principal neighbours* if they are adjacent and there is an induced 6-cycle in G containing both of them. Adapting an observation by the authors of [8], we note that given an empty $K_{2,7}$ in a plane graph G , one of the vertices in the part of size 7 (the ‘central’ one, in the natural sense) has only two principal neighbours in G , namely the vertices in the part of the $K_{2,7}$ of size 2.

Indeed, if u and w are the vertices in the part of size 2, and v_1, \dots, v_7 are the vertices in the part of size 7 (labeled in a natural order so that v_4 is the central one), then v_4 takes its neighbours from among u, w, v_3 , and v_5 . Clearly no induced 6-cycle contains the path uv_4v_3 since u and v_3 are neighbours, and the same applies for the paths uv_4v_5 , wv_4v_3 , and wv_4v_5 . If there is an induced 6-cycle containing the path $v_3v_4v_5$ then there is a path of length 4 from v_3 to v_5 in G which avoids v_4, u , and w . This is impossible due to the other common neighbours of u and w and thus the only principal neighbours of v_4 are u and w .

The next lemma allows us to find empty $K_{2,7}$ ’s in plane graphs under certain conditions. The argument is based on an argument in [8].

Lemma 6. *Let $0 < c < 1$ be a real constant and let n be large relative to c . Let G be an n -vertex plane graph and suppose that every vertex of G is in at least $n^2/10$ induced 6-cycles. Suppose that u and w are distinct vertices of G with a common neighbourhood of size at least cn . Then G contains an empty $K_{2,7}$ whose part of size 2 is $\{u, w\}$ and hence there is a vertex in G whose only principal neighbours are u and w .*

Proof. Let $t = |N(u) \cap N(w)|$, so $t \geq cn$. Label the vertices in $N(u) \cap N(w)$ as v_1, \dots, v_t in a natural order, that is, such that the drawing of the complete bipartite graph with parts $\{u, w\}$ and $\{v_1, \dots, v_t\}$ induced by the drawing of G splits the plane into t regions R_1, \dots, R_t where the boundary of R_i is formed of the cycle $v_iuv_{i+1}w$ for all i (with addition in the subscript taken modulo t), and where R_t is unbounded.

Let $3 \leq i \leq t - 3$ and consider the region R_i . Suppose for a contradiction that there is at least one vertex of G in the interior of R_i , but not more than $n^{1/3}$. Note that none of these vertices are in the common neighbourhood of u and w . Let z be a vertex in the interior of R_i . Then z is in at least $n^2/10$ induced 6-cycles in G by assumption.

The number of induced 6-cycles containing z which only use vertices in the interior of R_i or on the boundary of R_i is at most $(n^{1/3} + 3)^5 \leq n^{7/4}$ since there are at most $n^{1/3} + 3$ options for each of the vertices in the 6-cycle besides z . We now upper bound the number of induced 6-cycles which contain z and a vertex which is in the exterior of R_i .

Any such cycle contains at least two vertices on the boundary of R_i . Clearly it cannot contain all four vertices on the boundary since these form a 4-cycle. Moreover, it cannot contain three vertices on the boundary. Indeed, any such three vertices form a path in G , so since the 6-cycle is induced it must contain this path. Of the remaining three vertices in the 6-cycle, one is z and another, x , is in the exterior of R_i . Since z and x are not adjacent, the final vertex in the 6-cycle is one of their common neighbours. This common neighbour must be the fourth vertex on the boundary of R_i , so the 6-cycle is not induced.

Hence any induced 6-cycle containing z and a vertex in the exterior of R_i contains exactly two vertices on the boundary of R_i , and these must be non-adjacent since they must appear non-consecutively in the 6-cycle. However, they cannot be v_i and v_{i+1} since there is no path of length at most 4 from v_i to v_{i+1} contained in the exterior of R_i because of the $t - 2$ other common neighbours of u and w . Hence any induced 6-cycle containing z and a vertex in the exterior of R_i contains u and w .

Since z is not a common neighbour of u and w , of the three other vertices in the cycle exactly one or two must be in the interior of R_i . The number of such cycles in which exactly two of the other vertices are in the interior of R_i is at most $n^{5/3}$ since there are at most $n^{1/3}$ options for each of the two other vertices in the interior of R_i , and at most n for the vertex in the exterior.

The number of such cycles with exactly one other vertex in the interior of R_i is at most $3n^{4/3}$, since there are at most $n^{1/3}$ options for the other vertex in the interior of R_i , and there are at most $3n - 6$ edges in the (planar) bipartite graph induced between $N(u)$ and $N(w)$, so the number of options for the edge contained in the exterior of R_i is less than $3n$.

Therefore z is contained at most $n^{7/4} + n^{5/3} + 3n^{4/3} < n^2/10$ induced 6-cycles, which contradicts our assumption. So for $3 \leq i \leq t - 3$, if R_i has a vertex in its interior, then there are least $n^{1/3}$ such vertices. Let $l = |\{3 \leq i \leq t - 3 : R_i \text{ has a vertex in its interior}\}|$. Then $ln^{1/3} \leq n$, so $l \leq n^{2/3}$. Hence since $t \geq cn$, there exist six consecutive values of i in $3, \dots, t - 3$ such that R_i has no vertices in its interior for each of these values of i . Thus G contains an empty $K_{2,7}$ in which the part of size 2 is $\{u, w\}$. The final part of the lemma follows from the discussion above. \square

In Lemma 7 and its subsequent corollary, we use Lemma 6 to show that if a large planar graph has no vertices in ‘few’ induced 6-cycles, then it contains an induced 6-cycle reminiscent of those in the graphs in the families $(\mathcal{F}_n)_n$.

Lemma 7. *Let n be large and let G be an n -vertex planar graph. Suppose that every vertex of G is in at least $n^2/10$ induced 6-cycles. Suppose v_1 is a vertex of G with at most five principal neighbours. Then G contains an induced 6-cycle $u_1v_1u_2v_2u_3v_3$ such that the only principal neighbours of v_2 are u_2 and u_3 , and the only principal neighbours of v_3 are u_1 and u_3 .*

Proof. Let u_1 and u_2 be neighbours of v_1 such that there are at least $n^2/100$ induced 6-cycles in G containing the path $u_1v_1u_2$. Let $X_1 = X_1^{u_1v_1u_2}$, $X_2 = X_2^{u_1v_1u_2}$, and $X_3 = X_3^{u_1v_1u_2}$. Let G_1 and G_2 be the bipartite subgraphs of G induced between partite classes X_1 and X_2 , and X_2 and X_3 respectively. By Lemma 4, G_1 and G_2 are acyclic.

Every induced 6-cycle in G containing $u_1v_1u_2$ contains an edge of G_2 , and every edge in G_2 is in at most n such cycles, since these cycles are determined by the vertex in X_1 that they contain. There are at most n edges in G_2 since it is acyclic. Let l be the number of edges in G_2 which are in at least $n/1000$ induced 6-cycles in G containing the path $u_1v_1u_2$. Then

$$\frac{n^2}{100} \leq nl + \frac{(n-l)n}{1000},$$

so $l \geq n/111$.

If $x_2 \in X_2$ and $x_3 \in X_3$ are neighbours and there are at least $n/1000$ induced 6-cycles in G containing the path $u_1v_1u_2x_3x_2$, then x_2 has at least $n/1000$ neighbours in X_1 . Hence each of the l edges above contains a vertex in X_2 which has degree at least $n/1000$ in G_1 . Since G_1 is acyclic it contains at most n edges, so at most 1000 vertices in X_2 can have degree at least $n/1000$ in G_1 . Thus there exists $u_3 \in X_2$ which is an endpoint of at least $n/111000$ of the l edges, and this u_3 satisfies $|N(u_1) \cap N(u_3)| \geq n/1000$ and $|N(u_2) \cap N(u_3)| \geq n/111000$.

By Lemma 6, there exist vertices $v_3 \in N(u_1) \cap N(u_3)$ and $v_2 \in N(u_2) \cap N(u_3)$ whose only principal neighbours are u_1 and u_3 , and u_2 and u_3 respectively. Moreover, that lemma implies that we can pick v_2 so that its only neighbours are u_2 , u_3 , and some vertices in the neighbourhood of u_2 . Similarly, we can pick v_3 such that its only neighbours are u_1 , u_3 , and some vertices in the neighbourhood of u_1 . Since u_1 and u_2 are not neighbours, we have $v_3 \neq u_2, u_3$, and $v_3 \notin N(u_2)$. Hence v_2 and v_3 are not neighbours. The cycle $u_1v_1u_2v_2u_3v_3$ is therefore an induced 6-cycle in G , which completes the proof of the lemma. \square

Corollary 8. *Let n be large and let G be an n -vertex planar graph. Suppose that every vertex of G is in at least $n^2/10$ induced 6-cycles. Then G contains an induced 6-cycle $u_1v_1u_2v_2u_3v_3$ such that for each $i \in \{1, 2, 3\}$ the only principal neighbours of v_i are u_i and u_{i+1} , where addition in the subscript is taken modulo 3.*

Proof. Every planar graph contains a vertex of degree at most 5. Applying Lemma 7 to such a vertex of G shows that there exists a vertex v_1 in G with only two principal neighbours. Applying Lemma 7 to v_1 completes the proof of the corollary. \square

3 Large planar graphs contain a vertex in few induced 6-cycles

In this section we will show that every large planar graph contains a vertex in ‘few’ induced 6-cycles, and that if the number of vertices in the graph is a multiple of 3 and no vertex is in ‘very few’ induced 6-cycles, then the graph is in the family \mathcal{F}_n for the appropriate n . The next two lemmas deal with the case where n is a multiple of 3.

Lemma 9. *Let $n \equiv 0 \pmod{3}$ be large, and let G be an n -vertex planar graph. Suppose that every vertex of G is contained in at least $(\frac{n}{3} - 1)^2$ induced 6-cycles. If G contains a (not necessarily induced) copy of $K_{2, n/3-1}$, then $G \in \mathcal{F}_n$.*

Proof. Suppose that G contains a copy of $K_{2, n/3-1}$. Fix such a copy and let u_1 and u_2 be the vertices in the part of size 2. By Lemma 6, G contains a vertex v_1 whose only principal neighbours are u_1 and u_2 , so by Lemma 7, G contains an induced 6-cycle $u_1v_1u_2v_2u_3v_3$ such that for each $i \in \{1, 2, 3\}$ the only principal neighbours of v_i are u_i and u_{i+1} , with subscript addition taken modulo 3.

Let $A = N(u_1) \cap N(u_2)$, so $|A| \geq \frac{n}{3} - 1$. Now let $X = X^{u_1v_1u_2}$, $X_1 = X_1^{u_1v_1u_2}$, $X_2 = X_2^{u_1v_1u_2}$, and $X_3 = X_3^{u_1v_1u_2}$. Note that $|X| \leq \frac{2n}{3} - 1$ since A , X , and $\{u_1, u_2\}$ are pairwise disjoint. By Lemma 4 we have

$$|X_3|(|X_1| + |X_2| - 1) \geq \left(\frac{n}{3} - 1\right)^2 \quad (1)$$

and

$$|X_1|(|X_2| + |X_3| - 1) \geq \left(\frac{n}{3} - 1\right)^2. \quad (2)$$

Applying Fact 5(a) gives $|X| \geq \frac{2n}{3} - 1$, so in fact $|X| = \frac{2n}{3} - 1$. This implies that $|A| = \frac{n}{3} - 1$. Applying the equality statement of Fact 5(a) to inequality (1) gives $|X_3| = \frac{n}{3} - 1$, and applying it to inequality (2) gives $|X_1| = \frac{n}{3} - 1$. In turn, this implies $|X_2| = 1$, so $X_2 = \{u_3\}$. Let $B = X_3$ and $C = X_1$. Then $\{u_1, u_2, u_3\}$, A , B , and C form a partition of the vertex set of G , and for each $a \in A$, $b \in B$, and $c \in C$ the edges au_1 , au_2 , bu_2 , bu_3 , cu_3 and cu_1 are present in G .

Since $u_1v_1u_2v_2u_3v_3$ is an induced 6-cycle in G , there are no edges between any of u_1 , u_2 , and u_3 . Inspecting the argument of Lemma 4 and noting that inequalities (1) and (2) are in fact equalities, we deduce that there are no edges between B and C . By the definition of X there are no edges between u_1 and B or between u_2 and C . Repeating the arguments above with each of v_2 and v_3 in place of v_1 shows that there are no edges between u_3 and A , between A and B , or between A and C .

The only other possible edges of G are edges within each of the vertex classes A , B , and C . By the planarity of G , for each of these vertex classes any edges within that vertex class clearly form a subset of the edge set of some path through the vertex set. Hence $G \in \mathcal{F}_n$ as required. \square

Lemma 10. *Let $n \equiv 0 \pmod{3}$ be large. Let G be an n -vertex planar graph and suppose that every vertex of G is in at least $(\frac{n}{3} - 1)^2$ induced 6-cycles. Then $G \in \mathcal{F}_n$.*

Proof. By Corollary 8, G contains an induced 6-cycle $u_1v_1u_2v_2u_3v_3$ such that for each $i \in \{1, 2, 3\}$ the only principal neighbours of v_i are u_i and u_{i+1} , where addition in the subscript is taken modulo 3. Let $X = X^{u_1v_1u_2}$, $X_1 = X_1^{u_1v_1u_2}$, $X_2 = X_2^{u_1v_1u_2}$, $X_3 = X_3^{u_1v_1u_2}$, $Y = X^{u_2v_2u_3}$, and $Z = X^{u_3v_3u_1}$.

By Lemma 4 and Fact 5(a) we have $|X| \geq \frac{2n}{3} - 1$, and similarly $|Y|, |Z| \geq \frac{2n}{3} - 1$. Suppose $|X|, |Y|, |Z| \geq \frac{2n}{3} + 2$. Then $|X| + |Y| + |Z| \geq 2n + 6$. Each of u_1 , u_2 , and u_3

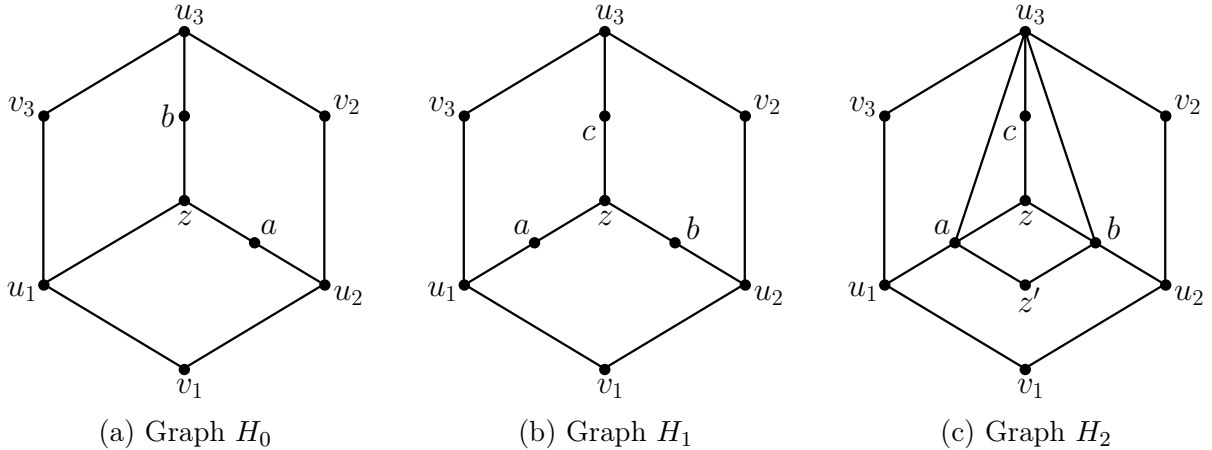


Figure 3: Some graphs used in the proof of Claim 1

appears in exactly one of X , Y , and Z , so $|X \setminus \{u_1, u_2, u_3\}| + |Y \setminus \{u_1, u_2, u_3\}| + |Z \setminus \{u_1, u_2, u_3\}| \geq 2n + 3 = 2(n - 3) + 9$. So there are at least nine vertices in G which appear in all three of X , Y , and Z . The following claim asserts that this is impossible.

Claim 1. *There are at most eight vertices in $X \cap Y \cap Z$.*

Proof. If $z \in X \cap Y \cap Z$, then since z is in an induced 6-cycle in G containing v_1 , z is not in the common neighbourhood of u_1 and u_2 . Similarly z is not in the common neighbourhood of u_1 and u_3 or u_2 and u_3 . So every vertex in $X \cap Y \cap Z$ is adjacent to at most one of u_1 , u_2 , and u_3 .

Let Γ be the induced 6-cycle $u_1v_1u_2v_2u_3v_3$. Fix a drawing of G and note that in this drawing Γ splits the plane into two regions. We will show that in the interiors of each of these regions there can be at most four vertices in $X \cap Y \cap Z$. Since none of the vertices in Γ are in $X \cap Y \cap Z$ this will be sufficient to prove the claim.

First suppose that $z, z' \in X \cap Y \cap Z \cap N(u_1)$ are distinct vertices in one of the regions formed by Γ . Since z is in an induced 6-cycle containing v_2 , it has neighbours a and b such that $zau_2v_2u_3b$ is an induced 6-cycle. Note that a and b are distinct from u_1 since they're adjacent to u_2 and u_3 respectively, and they are distinct from each of v_1 and v_3 because they are in an induced 6-cycle which does not contain u_1 .

Let H_0 be the subgraph of G consisting of the cycle Γ and the edges u_1z , za , au_2 , zb , and bu_3 , as shown in Figure 3a. The drawing of H_0 induced by the drawing of G splits the plane into four regions, whose boundaries are formed of the cycles Γ , $u_1zau_2v_1$, $u_1zbu_3v_3$, and $zau_2v_2u_3b$. Since $a \in N(u_2)$ and $b \in N(u_3)$, z' is distinct from both of these vertices. Hence we may assume that z' is in the region with boundary $u_1zau_2v_1$. Since z' is in an induced 6-cycle containing v_2 , it has a common neighbour with u_3 , and this common neighbour must be in the cycle $u_1zau_2v_1$. However, none of these vertices are adjacent to u_3 , so we have a contradiction. Hence there are at most two vertices in $X \cap Y \cap Z \cap N(u_1)$, with at most one in each of the regions formed by Γ . Similarly there are at most four vertices in $X \cap Y \cap Z$ which are adjacent to u_2 or u_3 .

Now suppose that $z, z' \in X \cap Y \cap Z$ are distinct vertices in one of the regions formed

by Γ which have no neighbours among u_1 , u_2 , and u_3 . Note that z and z' are not adjacent to any of v_1 , v_2 , or v_3 since these vertices only have two principal neighbours, but they each share an induced 6-cycle with z and z' . Since z is in an induced 6-cycle containing v_1 , it has neighbours x_1 and x_2 such that $v_1u_1x_1zx_2u_2$ is an induced 6-cycle.

Suppose first that $x_1 \in N(u_3)$. Then since z is in an induced 6-cycle containing v_3 it has neighbours x_3 and x_4 such that $v_3u_3x_3zx_4u_1$ is an induced 6-cycle. Both x_3 and x_4 are distinct from x_1 , and at most one of them can be equal to x_2 , so there exist distinct $a, b, c \in N(z)$ such that $a \in N(u_1)$, $b \in N(u_2)$, and $c \in N(u_3)$. If instead $x_1 \notin N(u_3)$, then since z is in an induced 6-cycle containing v_2 , there exist x_5 and x_6 such that $v_2u_2x_5zx_6u_3$ is an induced 6-cycle. Neither x_5 nor x_6 can be equal to x_1 since $x_1 \notin N(u_2) \cup N(u_3)$, so again there exist distinct a, b , and c as above.

Fix such a, b , and c , and let H_1 be the subgraph of G consisting of the cycle Γ and the edges za, zb, zc, au_1, bu_2 , and cu_3 , as shown in Figure 3b. The drawing of H_1 induced by the drawing of G splits the plane into four regions with boundaries formed of the cycles Γ , $zau_1v_1u_2b$, $zbu_2v_2u_3c$, and $zcu_3v_3u_1a$.

Since z' is not in H_1 , we may assume that z' is in the region with boundary $zau_1v_1u_2b$. As z' is in an induced 6-cycle containing v_2 , z' must have a common neighbour with u_3 . This common neighbour must be in the cycle $zau_1v_1u_2b$ so we may assume that it is a . There is an induced 6-cycle in G containing z' and v_3 , so z' has a neighbour which is adjacent to u_3 but not to u_1 . This neighbour must be b . Let H_2 be the subgraph of G obtained by adding the edges $z'a, au_3, z'b$, and bu_3 to H_1 , as shown in Figure 3c.

Since z is in an induced 6-cycle containing v_3 , it has a common neighbour a' with u_1 which is not a neighbour of u_3 . The cycle $az'bu_3$ separates z and u_1 in G , so a' must be a vertex in this cycle. However a and b are both adjacent to u_3 and neither u_3 nor z' are adjacent to u_1 , so no such a' exists. This gives the desired contradiction, and hence there are at most two vertices in $X \cap Y \cap Z$ which are adjacent to none of u_1, u_2 , and u_3 . This completes the proof of the claim. \square

Hence we may assume that $|X| \in \{\frac{2n}{3} - 1, \frac{2n}{3}, \frac{2n}{3} + 1\}$. We will consider each of these cases in turn. Note that if a vertex in X_2 has $\frac{n}{3} - 1$ neighbours in X_3 , then these are all common neighbours with u_2 , so by Lemma 9, $G \in \mathcal{F}_n$. Recall also that by the definitions of X_2 and X_3 , every vertex in X_2 has a neighbour in X_3 and vice versa. Suppose first that $|X| = \frac{2n}{3} - 1$. Then by Lemma 4 and Fact 5(a), we have $|X_1| = |X_3| = \frac{n}{3} - 1$. Hence $|X_2| = 1$, so the lone vertex in X_2 has at least $\frac{n}{3} - 1$ neighbours in X_3 which implies that $G \in \mathcal{F}_n$.

For the remaining two possible values of $|X|$ we will make use of the following claim. As in Lemmas 4 and 7, let G_1 be the bipartite subgraph of G induced between X_1 and X_2 , and let G_2 be that induced between X_2 and X_3 .

Claim 2. *Let $s \leq 3$ be a positive integer. Suppose that there exists a set T of at most four vertices in X_2 such that after excluding any one of them, those that remain have total degree at least s in G_2 . Then $s \leq |X| - \frac{2n}{3} + 1$, and if $s = |X| - \frac{2n}{3} + 1$, then G_2 is connected.*

Proof. Both G_1 and G_2 are acyclic by Lemma 4, and by the proof of that lemma an

induced 6-cycle in G containing v_1 is determined by the vertex in X_1 and the edge of G_2 that it contains. If $v \in X_1$ has at most one neighbour in T , then there are at least s edges in G_2 which are not in an induced 6-cycle in G containing v and v_1 . Since G_1 is acyclic, any pair of distinct vertices in X_2 has at most one common neighbour in X_1 , so all but at most six vertices in X_1 , have at most one neighbour in T .

Thus the number of induced 6-cycles in G containing v_1 is at most $|X_1|(|X_2| + |X_3| - 1) - s(|X_1| - 6) = |X_1|(|X_2| + |X_3| - (s + 1)) + 6s$. So $|X_1|(|X_2| + |X_3| - (s + 1)) \geq (\frac{n}{3} - 1)^2 - 6s \geq (\frac{n-4}{3})^2$ since n is large. Noting that G_2 has at least $s + 1$ edges so $|X_2| + |X_3| - (s + 1) \geq 1$, we can apply Fact 5(a) to obtain $|X| - (s + 1) \geq \frac{2n-8}{3}$. Since the left-hand side is an integer, this implies that $|X| - (s + 1) \geq \frac{2n}{3} - 2$, or equivalently $s \leq |X| - \frac{2n}{3} + 1$. If moreover G_2 is not connected, then the number of induced 6-cycles containing v_1 is at most $|X_1|(|X_2| + |X_3| - 2) - s(|X_1| - 6)$ which gives $s \leq |X| - \frac{2n}{3}$ by a similar argument. \square

Suppose that $|X| = \frac{2n}{3}$. Then $|X| - \frac{2n}{3} + 1 = 1$ so by Claim 2, $|X_2| \leq 2$. If $|X_2| = 1$, then we may assume that $|X_3| \geq \frac{n}{3}$, so the lone vertex in X_2 has at least $\frac{n}{3}$ neighbours in X_3 which implies $G \in \mathcal{F}_n$. If $|X_2| = 2$, then we may assume that $|X_3| \geq \frac{n}{3} - 1$. By Claim 2 one vertex in X_2 has only one neighbour in X_3 and G_2 is connected, so the other vertex in X_2 has at least $\frac{n}{3} - 1$ neighbours in X_3 and $G \in \mathcal{F}_n$.

Now suppose that $|X| = \frac{2n}{3} + 1$. Then $|X| - \frac{2n}{3} + 1 = 2$, so by Claim 2, $|X_2| \leq 3$. If $|X_2| = 1$, then we may assume that $|X_3| \geq \frac{n}{3}$ so the lone vertex in X_2 has at least $\frac{n}{3}$ neighbours in X_3 and $G \in \mathcal{F}_n$. If $|X_2| = 2$, then we may assume that $|X_3| \geq \frac{n}{3}$. By the claim, one vertex in X_2 has degree at most 2 in G_2 . Moreover, if one vertex in X_2 has degree exactly 2 in G_2 , then G_2 is connected so the other has at least $\frac{n}{3} - 1$ neighbours in X_3 and $G \in \mathcal{F}_n$. If instead one vertex in X_2 has exactly one neighbour in X_3 , then again the other has at least $\frac{n}{3} - 1$, and $G \in \mathcal{F}_n$. Finally if $|X_2| = 3$, then we may assume that $|X_3| \geq \frac{n}{3} - 1$. By the claim, some pair of vertices in X_2 each have only one neighbour in X_3 and G_2 is connected, so the remaining vertex has at least $\frac{n}{3} - 1$ neighbours in X_3 . Thus $G \in \mathcal{F}_n$. This completes the proof of the lemma. \square

The next two lemmas concern the cases where $n \equiv 1 \pmod{3}$ or $n \equiv 2 \pmod{3}$. Their proofs make heavy use of some of the ideas in the previous two proofs in this section.

Lemma 11. *Let n be large and assume $n \equiv 1 \pmod{3}$. Then every n -vertex planar graph contains a vertex in at most $(\frac{n-4}{3})^2$ induced 6-cycles.*

Proof. Let G be an n -vertex planar graph and suppose for a contradiction that every vertex in G is in at least $(\frac{n-4}{3})^2 + 1$ induced 6-cycles. Then by Corollary 8, G contains a vertex with only two principal neighbours. Let v_1 be such a vertex and apply Lemma 7 to obtain an induced 6-cycle $u_1v_1u_2v_2u_3v_3$ such that for each $i \in \{1, 2, 3\}$ the only principal neighbours of v_i are u_i and u_{i+1} , where addition in the subscript is taken modulo 3.

Let $X, X_1, X_2, X_3, Y, Z, G_1$, and G_2 be as in the proof of Lemma 10. By Lemma 4 and Fact 5(b) we have $|X| \geq \frac{2n-2}{3}$, and similarly $|Y|, |Z| \geq \frac{2n-2}{3}$. Claim 1 still holds in this setting by the same proof, and applying this in the same way as in the proof of

Lemma 10, we deduce that at least one of X , Y , and Z has size less than $\frac{2n+7}{3}$. Hence we may assume that $|X| \in \{\frac{2n-2}{3}, \frac{2n+1}{3}, \frac{2n+4}{3}\}$. We will shortly consider each of these possible values of $|X|$ in turn and show that each leads to a contradiction.

First, a straightforward modification of Claim 2 states that if $s \leq 4$ is a positive integer and there exists a set of at most five vertices in X_2 such that after excluding any one of them those that remain have total degree at least s in G_2 , then $s \leq |X| - \frac{2n-5}{3}$, and if $s = |X| - \frac{2n-5}{3}$, then G_2 is connected. We also state and prove the following claim.

Claim 3. *Suppose that for every triple of distinct vertices u , v , and w in G such that u and w are the only principal neighbours of v we have $|X^{uvw}| \geq \frac{2n+t}{3}$ where $t \in \{-2, 1, 4\}$. Then G contains no copy of $K_{2,(n-t)/3-1}$.*

Proof. Suppose for a contradiction that G contains a copy of $K_{2,(n-t)/3-1}$. Let u and w be the vertices in the part of this bipartite graph of size 2. By Lemma 6, G contains a vertex v whose only principal neighbours are u and w , and by assumption $|X^{uvw}| \geq \frac{2n+t}{3}$. However, X^{uvw} is disjoint from the vertex set of the copy of $K_{2,(n-t)/3-1}$, which contradicts the fact that G contains only n vertices. \square

Thus G contains no copy of $K_{2,(n-1)/3}$. We now consider each of the three possible values of $|X|$. Suppose first that $|X| = \frac{2n-2}{3}$, then $|X| - \frac{2n-5}{3} = 1$ so by the modified Claim 2, $|X_2| \leq 2$. If $|X_2| = 1$, then we may assume that $|X_3| \geq \frac{n-1}{3}$ so the single vertex in X_2 has at least $\frac{n-1}{3}$ common neighbours with u_2 , which is a contradiction.

If $|X_2| = 2$, then G_2 is connected and one vertex in X_2 has degree 1 in G_2 . By symmetry, the same holds if we replace G_2 by G_1 . If one vertex in X_2 has degree 1 in G_1 and the other has degree 1 in G_2 , then the number of induced 6-cycles containing v_1 is clearly at most linear in n , so in fact one vertex in X_2 has degree 1 in G_1 and in G_2 .

Thus one vertex in X_2 is in at most one induced 6-cycle containing v_1 , and the other is in at most $|X_1||X_3|$. Hence

$$|X_1||X_3| + 1 \geq \left(\frac{n-4}{3}\right)^2 + 1, \quad (3)$$

and $|X_1| + |X_3| = \frac{2n-8}{3}$ so by Fact 5(a), $|X_1| = |X_3| = \frac{n-4}{3}$. Moreover, since we have equality in (3), there are no edges between X_1 and X_3 . Note that $v_3 \in X_1$ and $u_3 \in X_2$. If u_3 is the vertex in X_2 of degree 1 in G_1 , then v_3 is its sole neighbour in X_1 . But then v_3 is in an induced 6-cycle in G containing the other vertex in X_2 , which contradicts the fact that u_1 and u_3 are its only principal neighbours. Hence u_3 is the vertex in X_2 of degree $\frac{n-4}{3}$ in G_1 and G_2 . Let the other vertex in X_2 be z , and let $B = X_3$ and $C = X_1$. This is shown in Figure 4a.

Note that B and Y are disjoint, so $|Y| \leq \frac{2n-2}{3}$. Hence in fact $|Y| = \frac{2n-2}{3}$. Applying the arguments above with Y in place of X we deduce that u_1 and u_2 have at least $\frac{n-4}{3}$ common neighbours. Since G has only n vertices and $X \cap N(u_1) \cap N(u_2) = \emptyset$, u_1 and u_2 must in fact have exactly $\frac{n-4}{3}$ common neighbours. Let A be this set of common neighbours.

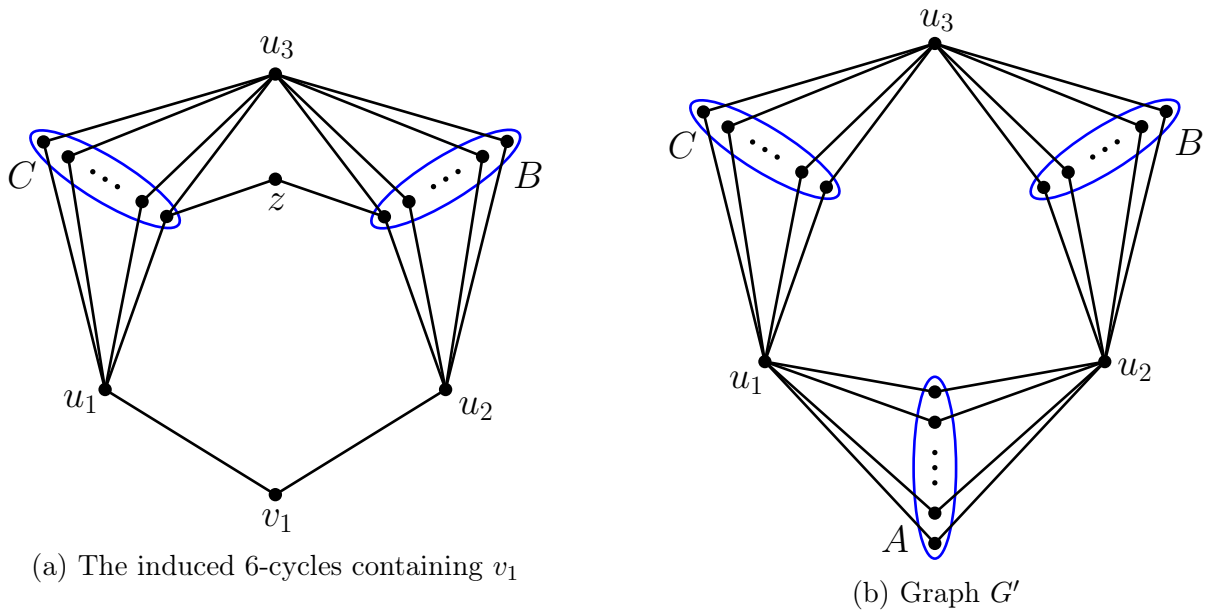


Figure 4: Two graphs used in the proof of Lemma 11

Let G' be the subgraph of G consisting of the edges $au_1, au_2, bu_2, bu_3, cu_3$, and cu_1 for each $a \in A, b \in B$, and $c \in C$, as shown in Figure 4b. Fix a drawing of G , and consider the induced drawing of G' . No face in the drawing of G' has more than six vertices on its boundary, so z has degree at most 6 in G . By Lemma 4, in order for z to be in at least $(\frac{n-4}{3})^2 + 1$ induced 6-cycles, it must have two distinct neighbours both of whose degrees in G are linear in n . By considering the faces in the drawing of G' again, we see that the only vertices in $G - \{z\}$ of degree greater than 6 are u_1, u_2 , and u_3 , but z is adjacent to neither u_1 nor u_2 , so we have a contradiction.

Therefore $|X_2| \neq 2$ and hence $|X| \neq \frac{2n-2}{3}$. Applying Claim 3 shows that G therefore contains no copy of $K_{2,(n-4)/3}$. Next suppose that $|X| = \frac{2n+1}{3}$, so $|X| - \frac{2n-5}{3} = 2$. The modified Claim 2 implies that $|X_2| \leq 3$. If $|X_2| = 1$, then we may assume that $|X_3| \geq \frac{n-1}{3}$ which gives rise to a copy of $K_{2,(n-1)/3}$ in G , and hence a contradiction. If $|X_2| = 2$, then we may again assume that $|X_3| \geq \frac{n-1}{3}$. By the modified Claim 2, one vertex in X_2 has degree at most 2 in G_2 , and if one has degree exactly 2, then G_2 is connected. In this case the other vertex has degree at least $\frac{n-4}{3}$ in G_2 which is a contradiction. If instead one vertex in X_2 has degree 1 in G_2 , then again the other has degree at least $\frac{n-4}{3}$, so again we arrive at a contradiction. If $|X_2| = 3$, then we may assume that $|X_3| \geq \frac{n-4}{3}$. By the modified claim, some two vertices in X_2 each have degree 1 in G_2 and G_2 is connected, so the remaining vertex in X_2 has at least $\frac{n-4}{3}$ neighbours in X_3 which is a contradiction.

Therefore $|X| \neq \frac{2n+1}{3}$, so G contains no copy of $K_{2,(n-7)/3}$ by Claim 3. Suppose finally that $|X| = \frac{2n+4}{3}$, in which case $|X| - \frac{2n-5}{3} = 3$ and the modified claim implies that $|X_2| \leq 4$. If $|X_2| = 1$, then we may assume that $|X_3| \geq \frac{n+2}{3}$ and hence G has a copy of $K_{2,(n+2)/3}$ which is a contradiction. If $|X_2| = 2$, then we may assume that $|X_3| \geq \frac{n-1}{3}$. By the modified Claim 2, one vertex in X_2 has degree at most 3 in G_2 , and if one has degree exactly 3, then G_2 is connected. In this case, the other vertex in X_2 has at least

$\frac{n-7}{3}$ neighbours in X_3 , which is a contradiction. So one vertex in X_2 has at most two neighbours in X_3 , and hence the other has at least $\frac{n-7}{3}$, which is another contradiction.

If $|X_2| = 3$, then we may again assume that $|X_3| \geq \frac{n-1}{3}$. By the modified claim, one vertex in X_2 has degree 1 in G_2 , and another has degree at most 2. Moreover if this second vertex has degree exactly 2 in G_2 , then G_2 is connected, so the remaining vertex in X_2 has at least $\frac{n-4}{3}$ neighbours in X_3 , a contradiction. Otherwise some two vertices in X_2 each have only one neighbour in X_3 , so the final vertex has at least $\frac{n-7}{3}$ such neighbours, a contradiction. If $|X_2| = 4$, then we may assume that $|X_3| \geq \frac{n-4}{3}$. By the modified claim, some three vertices in X_2 each have only one neighbour in X_3 and G_2 is connected, so the remaining vertex in X_2 has at least $\frac{n-4}{3}$ neighbours in X_3 , which is a contradiction.

Thus $|X| \neq \frac{2n+4}{3}$, so $|X| \notin \{\frac{2n-2}{3}, \frac{2n+1}{3}, \frac{2n+4}{3}\}$ which gives the required contradiction and completes the proof of the lemma. \square

The proof of the next lemma is very similar to that of Lemma 11, but is more straightforward. We include it nonetheless, for completeness.

Lemma 12. *Let n be large and assume $n \equiv 2 \pmod{3}$. Then every n -vertex planar graph contains a vertex in at most $\binom{n-5}{3} \binom{n-2}{3}$ induced 6-cycles.*

Proof. Let G be an n -vertex planar graph and suppose for a contradiction that every vertex in G is in at least $\binom{n-5}{3} \binom{n-2}{3} + 1$ induced 6-cycles. Then by Corollary 8, G contains a vertex with only two principal neighbours. Let v_1 be such a vertex and apply Lemma 7 to obtain an induced 6-cycle $u_1v_1u_2v_2u_3v_3$ such that for each $i \in \{1, 2, 3\}$ the only principal neighbours of v_i are u_i and u_{i+1} where addition in the subscript is taken modulo 3.

Let $X, X_1, X_2, X_3, Y, Z, G_1$, and G_2 be as in the proofs of Lemmas 10 and 11. By Lemma 4 and Fact 5(b) we have $|X| \geq \frac{2n-1}{3}$, and similarly for Y and Z . Again, Claim 1 holds in this setting by the same argument, so applying this in the same way as before shows that at least one of X, Y , and Z has size at most $\frac{2n+5}{3}$. Hence we may assume that $|X| \in \{\frac{2n-1}{3}, \frac{2n+2}{3}, \frac{2n+5}{3}\}$. As before, we will consider each of these cases in turn and will obtain a contradiction in each case.

The relevant straightforward adaptation of Claim 2 to this setting (using Fact 5(b) in the proof in place of Fact 5(a)) is that if $s \leq 4$ is a positive integer and there exists a set of at most five vertices in X_2 such that after excluding any one of them those that remain have total degree at least s in G_2 , then $s \leq |X| - \frac{2n-4}{3}$. Moreover if $s = |X| - \frac{2n-4}{3}$, then G_2 is connected. A modified version of Claim 3 also holds in this setting, where we consider $t \in \{-1, 2, 5\}$ instead of $t \in \{-2, 1, 4\}$. The proof of this is exactly the same as that of the original version. By this modified Claim 3, G contains no copy of $K_{2, (n-2)/3}$.

Suppose that $|X| = \frac{2n-1}{3}$. Then $|X| - \frac{2n-4}{3} = 1$ so by the modified Claim 2, $|X_2| \leq 2$. If $|X_2| = 1$, then we may assume that $|X_3| \geq \frac{n-2}{3}$ which gives a copy of $K_{2, (n-2)/3}$ and hence a contradiction. If $|X_2| = 2$, then we may again assume that $|X_3| \geq \frac{n-2}{3}$. By the modified Claim 2, one vertex in X_2 has degree 1 in G_2 and G_2 is connected, so the other vertex in X_2 has at least $\frac{n-2}{3}$ neighbours in X_3 , which is a contradiction.

Hence $|X| \neq \frac{2n-1}{3}$, so by the modified Claim 3, G contains no copy of $K_{2,(n-5)/3}$. Suppose that $|X| = \frac{2n+2}{3}$, so $|X| - \frac{2n-4}{3} = 2$ and hence by the modified Claim 2, $|X_2| \leq 3$. If $|X_2| = 1$, then we may assume that $|X_3| \geq \frac{n+1}{3}$ which gives a copy of $K_{2,(n+1)/3}$ and hence a contradiction. If $|X_2| = 2$, then we may assume that $|X_3| \geq \frac{n-2}{3}$. One vertex in X_2 has degree at most 2 in G_2 , and if one has degree exactly 2 then G_2 is connected, so the other has degree at least $\frac{n-5}{3}$ which is a contradiction. Similarly if one vertex in X_2 has degree 1 in G_2 , then the other has degree at least $\frac{n-5}{3}$. If $|X_2| = 3$, then again we may assume that $|X_3| \geq \frac{n-2}{3}$. By the modified Claim 2 some two vertices in X_2 have degree 1 in G_2 and G_2 is connected, so the other vertex has degree at least $\frac{n-2}{3}$ in G_2 , which is a contradiction.

Hence $|X| \neq \frac{2n+2}{3}$, so there is no copy of $K_{2,(n-8)/3}$ in G by the modified Claim 3. Suppose finally that $|X| = \frac{2n+5}{3}$, so $|X| - \frac{2n-4}{3} = 3$ and hence $|X_2| \leq 4$. If $|X_2| = 1$, then we may assume that $|X_3| \geq \frac{n+1}{3}$ which gives a copy of $K_{2,(n+1)/3}$ in G , and hence a contradiction. If $|X_2| = 2$, then we may again assume that $|X_3| \geq \frac{n+1}{3}$. One vertex in X_2 has degree at most 3 in G_2 , so the other has degree at least $\frac{n-8}{3}$ in G_2 , which is a contradiction.

If $|X_2| = 3$, then we may assume that $|X_3| \geq \frac{n-2}{3}$. By the modified Claim 2, one vertex in X_2 has degree 1 in G_2 and another has degree at most 2. If this second vertex has degree exactly 2 in G_2 , then G_2 is connected so the remaining vertex has degree at least $\frac{n-5}{3}$ in G_2 , a contradiction. If instead two vertices in X_2 have degree 1 in G_2 , then the remaining vertex has degree at least $\frac{n-8}{3}$, which is another contradiction. Finally, if $|X_2| = 4$, then we may assume that $|X_3| \geq \frac{n-2}{3}$. Some three vertices in X_2 have degree 1 in G_2 and G_2 is connected, so the remaining vertex has degree at least $\frac{n-2}{3}$, which is a contradiction. Thus $|X| \neq \frac{2n+8}{3}$, so $|X| \notin \{\frac{2n-1}{3}, \frac{2n+2}{3}, \frac{2n+5}{3}\}$ which gives the required contradiction and completes the proof of the lemma. \square

4 Proof of Theorem 3

We are now ready to prove Theorem 3.

Proof of Theorem 3. First, clearly any graph in \mathcal{F}_n is planar. For $n \geq 6$, let

$$h_0(n) = \begin{cases} \left(\frac{n}{3} - 1\right)^3, & \text{if } n \equiv 0 \pmod{3} \\ \left(\frac{n-4}{3}\right)^2 \left(\frac{n-1}{3}\right), & \text{if } n \equiv 1 \pmod{3} \\ \left(\frac{n-2}{3}\right)^2 \left(\frac{n-5}{3}\right), & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

Let $G \in \mathcal{F}_n$ and label its vertices as in Definition 2. For each $a \in A$, $b \in B$, and $c \in C$, the cycle $u_1 a u_2 b u_3 c$ is an induced 6-cycle in G , so G contains at least $h_0(n)$ induced 6-cycles.

Suppose there exists another induced 6-cycle, Γ , in G . We may assume that Γ contains some $a \in A$. There is clearly no 6-cycle in G consisting entirely of vertices in A , so we may also assume that u_2 is one of the neighbours of a in Γ . If the other neighbour of u_2 in Γ is some $b \in B$, then Γ contains no further vertices in B since they are all neighbours

of u_2 . Hence the other neighbour of b in Γ is u_3 , and the other neighbour of u_3 in Γ is some $c \in C$. The remaining vertex in Γ is a common neighbour of a and c , so it must be u_1 . Hence Γ is one of the induced 6-cycles described above, which is a contradiction. In the other case Γ contains the path au_2a' for some $a' \in A$. It contains no further vertices in A , so the other neighbour of a' in Γ is u_1 , but this is also a neighbour of a , which contradicts Γ being an induced 6-cycle. Therefore any graph in \mathcal{F}_n contains exactly $h_0(n)$ induced 6-cycles.

For $n \geq 6$, define \mathcal{G}_n to be the family of n -vertex planar graphs containing $f_I(n, C_6)$ induced 6-cycles. To prove the theorem it is sufficient to show that $\mathcal{G}_n \subseteq \mathcal{F}_n$ for large enough n . For $n \geq 6$ let

$$h_1(n) = \begin{cases} \left(\frac{n}{3} - 1\right)^2, & \text{if } n \equiv 0 \pmod{3} \\ \left(\frac{n-4}{3}\right)^2, & \text{if } n \equiv 1 \pmod{3} \\ \left(\frac{n-2}{3}\right)\left(\frac{n-5}{3}\right), & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

so that $h_0(n) - h_0(n-1) = h_1(n)$ for all $n \geq 7$. By Lemmas 11 and 12, when n is large and not divisible by 3, every graph in \mathcal{G}_n has a vertex in at most $h_1(n)$ induced 6-cycles. By taking such a graph and deleting a vertex in $h_1(n)$ induced 6-cycles, we obtain an $(n-1)$ -vertex planar graph containing at least $f_I(n, C_6) - h_1(n)$ induced 6-cycles. Thus if n is large and not divisible by 3, then $f_I(n, C_6) - f_I(n-1, C_6) \leq h_1(n)$.

By Lemma 10, when n is large and divisible by 3, every graph in \mathcal{G}_n is either in \mathcal{F}_n or has a vertex in fewer than $h_1(n)$ induced 6-cycles. By the count above, every graph in \mathcal{F}_n contains a vertex in exactly $h_1(n)$ induced 6-cycles, so by the same argument as in the case where n is not a multiple of 3, if n is large and divisible by 3, then $f_I(n, C_6) - f_I(n-1, C_6) \leq h_1(n)$.

If $f_I(n, C_6) - f_I(n-1, C_6) < h_1(n)$ for infinitely many values of n , then for large enough n we have $h_0(n) > f_I(n, C_6)$, which is a contradiction, since every graph in \mathcal{F}_n is planar and contains $h_0(n)$ induced 6-cycles. Therefore for large enough n , we have

$$f_I(n, C_6) - f_I(n-1, C_6) = h_1(n). \quad (4)$$

If n is large enough that this equality holds and $G \in \mathcal{G}_n$, then every vertex in G is in at least $h_1(n)$ induced 6-cycles, otherwise we could delete the vertex in the fewest such cycles to obtain an $(n-1)$ -vertex planar graph containing more than $f_I(n-1, C_6)$ induced 6-cycles. Hence if n is large and divisible by 3, and $G \in \mathcal{G}_n$, then $G \in \mathcal{F}_n$ by Lemma 10.

Now let n be large with $n \equiv 1 \pmod{3}$, and let $G \in \mathcal{G}_n$. By Lemma 11 and equation (4) there exists a vertex in G in exactly $h_1(n)$ induced 6-cycles. Deleting such a vertex, z , yields $G' \in \mathcal{G}_{n-1}$. Since $n-1 \equiv 0 \pmod{3}$, we have $G' \in \mathcal{F}_{n-1}$. Label the vertices of G' according to Definition 2. If z has more than two neighbours in any of A , B , or C in G , then this clearly gives rise to a subdivision of $K_{3,3}$, which contradicts the fact that G is planar. Hence z has degree at most 9 in G .

It follows that z has two neighbours with which it is in at least $n^2/360$ induced 6-cycles. By Lemma 4, these two neighbours must each have degree at least $n/360$ in G , so without loss of generality z is adjacent to u_1 and u_2 in G . Since z is adjacent to at

most two vertices in A , there exists $a \in A$ which has no path of length at most 4 to z which avoids u_1 and u_2 . Any induced 6-cycle in G containing a and z must contain u_1 and u_2 , but au_1zu_2 is a 4-cycle in G , so in fact a and z do not share an induced 6-cycle in G . Thus a is in exactly $h_1(n-1) = h_1(n)$ induced 6-cycles in G .

Let G'' be the graph obtained by deleting a from G , then as before we have $G'' \in \mathcal{F}_{n-1}$. Since z is adjacent to u_1 and u_2 , it follows that z is not adjacent to u_3 , nor to any vertices in B or C . So we obtain G from G'' by adding a vertex whose neighbours are u_1, u_2 , and some vertices in A . Since G is planar, it is now clear that $G \in \mathcal{F}_n$.

Finally suppose $n \equiv 2 \pmod{3}$ and let $G \in \mathcal{G}_n$. By Lemma 12 and equation (4), there exists a vertex, z , in G in exactly $h_1(n)$ induced 6-cycles. Deleting this vertex yields $G' \in \mathcal{F}_{n-1}$. Label the vertices of G' according to Definition 2, with A the larger of the three vertex classes. As before, z is adjacent to at least two of u_1, u_2 and u_3 .

If z is adjacent to u_1 and u_2 , then as in the case $n \equiv 1 \pmod{3}$ there exists some $a \in A$ which is not in an induced 6-cycle containing z , and hence is in exactly $h_1(n-1)$ induced 6-cycles in G . But $h_1(n-1) < h_1(n)$, and every vertex of G is in at least $h_1(n)$ induced 6-cycles by equation (4), so we have a contradiction. Hence without loss of generality z is adjacent to u_2 and u_3 in G .

Again, there is some $b \in B$ which is not in an induced 6-cycle in G containing z , so b is in exactly $h_1(n)$ induced 6-cycles in G . Deleting b from G yields another graph in \mathcal{F}_{n-1} , so it follows that z has no neighbours in $A \cup C \cup \{u_1\}$, and thus we obtain G from G' by adding a vertex whose neighbours are u_2, u_3 , and some vertices in B . By the planarity of G , it is clear that $G \in \mathcal{F}_n$. This completes the proof of the theorem. \square

5 Longer even cycles

In this section we discuss to what extent the ideas used above to prove Theorem 3 might be of use in tackling Conjecture 1 for cycles of length greater than 6. In the case of odd cycles it seems that very little of the argument can be straightforwardly adapted, so we focus on the case of even cycles. Recall that we determined the value of $f_I(n, C_6)$ for large n in roughly the following steps:

1. For large n , we found some structure in those n -vertex planar graphs in which every vertex is in at least $n^2/10$ induced 6-cycles.
2. For large n , we showed that if G is an n -vertex planar graph, then it contains a vertex in at most $h_1(n)$ induced 6-cycles, where $h_1(n)$ is the minimum number of induced 6-cycles that a vertex in a graph in \mathcal{F}_n is contained in.
3. For n large and divisible by 3 we showed that if G is an n -vertex planar graph in which every vertex is in at least $h_1(n)$ induced 6-cycles, then $G \in \mathcal{F}_n$.
4. We used the fact that in an n -vertex planar graph containing $f_I(n, C_6)$ induced 6-cycles every vertex is in at least $f_I(n, C_6) - f_I(n-1, C_6)$ induced 6-cycles, combined with step 2, to deduce that $f_I(n, C_6) = f_I(n-1, C_6) + h_1(n)$ for large enough n . We then used step 3 to determine the value of $f_I(n, C_6)$ exactly for large n .

In attempting to adapt this to $2k$ -cycles for $k \geq 4$, we first note that the minimum number of induced $2k$ -cycles that a vertex in $F_{n,2k}$ is contained in is $(n/k)^{k-1} + O(n^{k-2})$. In step 1 it would therefore be appropriate to consider n -vertex planar graphs in which every vertex is in at least cn^{k-1} induced $2k$ -cycles, where $c > 0$ is a constant and n is large. In Lemma 14 below we show that such graphs contain structure analogous to that in Corollary 8, and similar to that found in the graph $F_{n,2k}$.

Given the analogues of steps 2 and 3 in the case of $2k$ -cycles, an argument similar to that summarised in step 4 would yield Conjecture 1 for even cycles. Unfortunately, it does not seem to be straightforward to generalise steps 2 and 3. The main obstacle is that given a path uvw in a graph G , for $k \geq 4$ it is possible that a vertex may appear in different ‘positions’ in different induced $2k$ -cycles containing the path uvw . For example there might exist one induced $2k$ -cycle in which vertex x is at distance 2 from u in the cycle and another in which it is at distance 3.

Nevertheless, below we give two results which roughly correspond to generalisations of Lemma 6 and Corollary 8 to the case of $2k$ -cycles for $k \geq 4$ in the hope that they might be of some use in a proof of Conjecture 1 in the case of even cycles, and at the very least to lend some credibility to Conjecture 1 in this case. For the lemmas to be relevant, there needs to exist for each $k \geq 4$ a constant $c > 0$ such that if n is sufficiently large relative to k , then every vertex in an n -vertex planar graph containing $f_I(n, C_{2k})$ induced $2k$ -cycles is in at least cn^{k-1} induced $2k$ -cycles. In light of the fact that $f_I(n, C_{2k}) \geq (n/k)^k + O(n^{k-1})$ (as exhibited by the graphs $F_{n,2k}$), this does not seem an unreasonable assumption.

Lemma 13. *Let $k \geq 4$ be an integer, and let $c_0, c_1 > 0$ be constants. Let n be large, and let G be an n -vertex planar graph. Suppose that every vertex of G is in at least $c_0 n^{k-1}$ induced $2k$ -cycles, and that there exist distinct vertices, u and w , of G with a common neighbourhood of size at least $c_1 n$. Then there exists $v \in N(u) \cap N(w)$ such that every induced $2k$ -cycle containing v also contains u and w .*

The proof of Lemma 13 is a straightforward adaptation of that of Lemma 6, and we omit it. The only additional observation required for the proof is that in an n -vertex planar graph G containing distinct vertices u and w , the number of paths of length i from u to w is at most $n(6n-12)^{i/2-1}$ if i is even, and at most $(6n-12)^{(i-1)/2}$ if i is odd.

Lemma 14. *Let $k \geq 4$ be an integer and let $c > 0$ be a constant. Then there exists a constant $c' > 0$ such that if n is large and G is an n -vertex planar graph each of whose vertices is in at least cn^{k-1} induced $2k$ -cycles, then G contains distinct vertices $x_0, x_2, x_4, \dots, x_{2k-2}$ such that for each $i \in \{0, 1, \dots, k-1\}$ we have $|N(x_{2i}) \cap N(x_{2i+2})| \geq c'n$, where addition in the subscript is taken modulo $2k$.*

Proof. Let n and G be as in the statement of the lemma, and let v be a vertex of G which has at most five neighbours with which it is in an induced $2k$ -cycle in G . Such a vertex exists since G is planar and thus contains a vertex of degree at most 5. Let x_0 and x_{2k-2} be distinct neighbours of v such that there are at least $cn^{k-1}/10$ induced $2k$ -cycles in G which contain the path x_0vx_{2k-2} .

Claim 4. For each $i \in \{0, 1, 2, \dots, k-2\}$ and constant $\lambda > 0$, there exists a constant $\lambda' = \lambda'(\lambda, i) > 0$ such that if $x_1, x_2, x_3, \dots, x_{2i}$ are distinct vertices such that the path $x_{2k-2}vx_0x_1 \dots x_{2i}$ is contained in at least λn^{k-i-1} induced $2k$ -cycles in G , then there exist vertices $x_{2i+2}, x_{2i+4}, x_{2i+6}, \dots, x_{2k-4}$ all distinct from each other and from x_0, x_1, \dots, x_{2i} , and x_{2k-2} such that for all $j \in \{i, i+1, \dots, k-2\}$ we have $|N(x_{2j}) \cap N(x_{2j+2})| \geq \lambda'n$.

Proof. We will prove the claim by reverse induction on i . If $i = k-2$, then let $\lambda > 0$ and suppose that $x_1, x_2, \dots, x_{2k-4}$ are distinct vertices of G such that the path $x_{2k-2}vx_0x_1 \dots x_{2k-4}$ is contained in at least λn induced $2k$ -cycles. This path has length $2k-2$, so x_{2k-4} and x_{2k-2} have at least λn common neighbours. Thus we can take $\lambda'(\lambda, k-2) = \lambda$.

Now let $i \in \{0, 1, 2, \dots, k-3\}$ and suppose that $\lambda'(\lambda, i+1) > 0$ exists for all $\lambda > 0$. Let $\lambda > 0$, and let x_1, x_2, \dots, x_{2i} be distinct vertices of G such that the path $x_{2k-2}vx_0x_1 \dots x_{2i}$ is contained in at least λn^{k-i-1} induced $2k$ -cycles. Let L be the set of ordered pairs (a, b) of vertices of G such that there is an induced $2k$ -cycle in G of the form $x_{2k-2}vx_0x_1 \dots x_{2i}zaby_{2i+4} \dots y_{2k-3}$, and let L' be the subset of L consisting of the pairs which appear in at least $\lambda_0 n^{k-i-2}$ cycles of this form, where $\lambda_0 > 0$ is a small constant to be determined later. Every induced $2k$ -cycle containing the path $x_{2k-2}vx_0x_1 \dots x_{2i}$ is of this form for some pair $(a, b) \in L$, and each pair appears in at most $n(6n)^{k-i-3}$ of these induced $2k$ -cycles, because there are at most n options for the vertex z , and at most $6n-12 < 6n$ options for each of the edges $y_{2i+4}y_{2i+5}, \dots, y_{2k-4}y_{2k-3}$. We also have $|L| \leq 6n-12 < 6n$, so if $l = |L'|$, then the total number of induced $2k$ -cycles containing the path $x_{2k-2}vx_0x_1 \dots x_{2i}$ is at most

$$ln(6n)^{k-i-3} + (6n-l)\lambda_0 n^{k-i-2}.$$

This expression is therefore at least λn^{k-i-1} , so rearranging we obtain

$$l \geq \frac{\lambda - 6\lambda_0}{6^{k-i-3} - \lambda_0} n.$$

Let $\lambda_1 = (\lambda - 6\lambda_0)/(6^{k-i-3} - \lambda_0)$ and choose $\lambda_0 > 0$ small enough that $\lambda_1 > 0$. For each pair $(a, b) \in L'$ there are at least $\lambda_0 n^{k-i-2}$ induced $2k$ -cycles in G of the form $x_{2k-2}vx_0x_1 \dots x_{2i}zaby_{2i+4} \dots y_{2k-3}$. There are at most $6n-12 < 6n$ options for each of the pairs $(y_{2i+4}, y_{2i+5}), \dots, (y_{2k-4}, y_{2k-3})$, and there are $k-i-3$ of these pairs, so the number of common neighbours of x_{2i} and a is at least $\lambda_0 n/6^{k-i-3}$.

Let $\lambda_2 = \lambda_0/6^{k-i-3}$ so that for each pair in L' , the vertex in the first position has degree at least $\lambda_2 n$. Since the total number of edges in G is at most $3n-6 < 3n$, there are at most $6/\lambda_2$ vertices which appear as the first vertex in a pair in L' . Let $\lambda_3 = 6/\lambda_2$. Since $|L'| \geq \lambda_1 n$, there is a vertex, x_{2i+2} , of G which is the first vertex in at least $\lambda_1 n/\lambda_3$ pairs in L' . We have shown that x_{2i} and x_{2i+2} have at least $\lambda_2 n$ common neighbours.

Moreover, there are at least $(\lambda_1 \lambda_0/\lambda_3)n^{k-i-1}$ induced $2k$ -cycles in G of the form $x_{2k-2}vx_0x_1 \dots x_{2i}zx_{2i+2}y_{2i+3}y_{2i+4} \dots y_{2k-3}$, so since G contains only n vertices, there exists $x_{2i+1} \in N(x_{2i}) \cap N(x_{2i+2})$ such that there are at least $(\lambda_1 \lambda_0/\lambda_3)n^{k-i-2}$ induced $2k$ -cycles in G containing the path $x_{2k-2}vx_0x_1 \dots x_{2i}x_{2i+1}x_{2i+2}$. Applying the induction hypothesis, we see that we may take $\lambda'(\lambda, i) = \min(\lambda_2, \lambda'(\lambda_1 \lambda_0/\lambda_3, i+1))$, and the claim is proved. \square

Applying the claim with $i = 0$ and $\lambda = c/10$ shows that there exists a constant $c'' > 0$ and two distinct vertices u and w in G which have a common neighbourhood of size at least $c''n$. By Lemma 13, there exists a vertex in $N(u) \cap N(w)$ such that every induced $2k$ -cycle containing that vertex also contains u and w . We may therefore take v to be such a vertex, then take $x_0 = u$ and $x_{2k-2} = w$ and apply the claim again with $i = 0$ and $\lambda = c$ to complete the proof of the lemma. \square

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