CHROMATIC NUMBER IS NOT TOURNAMENT-LOCAL

ANTÓNIO GIRÃO, KEVIN HENDREY, FREDDIE ILLINGWORTH, FLORIAN LEHNER, LUKAS MICHEL, MICHAEL SAVERY, AND RAPHAEL STEINER

Abstract. Scott and Seymour conjectured the existence of a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, for every graph G and tournament T on the same vertex set, $\chi(G) \geqslant f(k)$ implies that $\chi(G[N^+_T(v)]) \geqslant k$ for some vertex v. In this note we disprove this conjecture even if v is replaced by a vertex set of size $\mathcal{O}(\log |V(G)|)$. As a consequence, we answer in the negative a question of Harutyunyan, Le, Thomassé, and Wu concerning the corresponding statement where the graph G is replaced by another tournament, and disprove a related conjecture of Nguyen, Scott, and Seymour. We also show that the setting where chromatic number is replaced by degeneracy exhibits a quite different behaviour.

1. Introduction

The question of what structures must appear in graphs of large chromatic number is one of the most fundamental in modern graph theory. One obvious reason for a graph to have high chromatic number is the presence of a large clique, but constructions from the 1940s and 50s of, for example, Tutte [Des54] and Zykov [Zyk49] demonstrate the existence of triangle-free graphs of arbitrarily large chromatic number. In particular, there are graphs with arbitrarily large chromatic number in which every neighbourhood is independent (and hence 1-colourable).

Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour, and Thomassé [BCC+13] conjectured that the analogous phenomenon does not occur in tournaments. This was confirmed recently in a beautiful paper of Harutyunyan, Le, Thomassé, and Wu [HLTW19] in which they showed that for every k there exists an $f(k)$ such that every tournament with chromatic number[1] at least $f(k)$ contains a vertex v such that $\chi(T[N^+(v)]) \geqslant k$.

Separately, Scott and Seymour [SS16] (see also [HLTW19, Conj. 7]) conjectured a similar result for a graph and a tournament on the same vertex set.

Conjecture 1 (Scott and Seymour)**.** *For every positive integer k there exists a χ such that, for every graph G with $\chi(G) \geqslant \chi$ and every tournament T on the same vertex set, there is a vertex v such that $\chi(G[N^+_T(v)]) \geqslant k$.*

This conjecture is supported by the observation [SS16] that the statement holds when chromatic number is replaced by fractional chromatic number (see Section 4 for more details). The main result of this note is a disproof of Conjecture 1 for $k \geqslant 3$. In fact, we prove something stronger: G and T may be chosen such that the out-neighbourhood[2] of any set of size at most $\frac{\log |V(T)|}{2\chi^2}$ is bipartite.

Theorem 2. *For every positive integer χ there are arbitrarily large N for which there is a graph G and a tournament T on the same N-vertex set such that $\chi(G) = \chi$ and, for every set U of at most $\frac{\log N}{2\chi^2}$ vertices, $\chi(G[N^+_T(U)]) \leqslant 2$.*

Date: 26 May 2023.

A.G. and F.I. were supported by EPSRC grant EP/V007327/1. R.S. was supported by an ETH Zürich Postdoctoral Fellowship. K.H. was supported by the Institute for Basic Science (IBS-R029-C1).

[1] The *chromatic number*, $\chi(T)$, of a tournament T is the least k for which there is a partition of $V(T)$ into k parts each of which induces a transitive (acyclic) subtournament of T.

[2] The *out-neighbourhood*, $N^+(S)$, of a set S is $\bigcup_{v\in S} N^+(v)$. This might contain vertices of S.

We will show that G can in fact be taken to be triangle-free which will be useful for our proof of Corollary 3. We make two remarks concerning the optimality of Theorem 2.

- It is not possible to replace 2 by 1 in the bound on the chromatic number of the out-neighbourhood, even when U consists of a single vertex. Indeed, suppose that $G[N^+_T(v)]$ is independent for every vertex v. Let xy be an edge of G. No out-neighbourhood of a vertex of T can contain both x and y, so $\{x, y\}$ dominates T. But then G is 3-colourable: one colour for each of $N^+_T(x)$ and $N^+_T(y)$, and a final colour for whichever of x and y has not been coloured.
- The bound on the size of U is very close to being best possible. Let S be a dominating set of T of size at most $\lceil \log_2 N \rceil$ (such a set can be constructed greedily). Then $N^+(S)$ contains all vertices of G except perhaps one and so, for any $0 \leqslant \ell \leqslant \chi - 2$, there is some $U \subseteq S$ of size at most $\lceil \log_2(N)/\lfloor \frac{\chi-2}{\ell} \rfloor \rceil$ with $\chi(G[N^+_T(U)]) > \ell$.

Theorem 2 has the following corollary, which resolves in a strong sense a question of Harutyunyan, Le, Thomassé, and Wu [HLTW19] concerning the analogous problem for two tournaments on the same vertex set.

Corollary 3. *For every positive integer χ there are arbitrarily large N for which there are tournaments T_1 and T_2 on the same N-vertex set such that $\chi(T_1) = \chi$ and, for every set U of at most $\frac{\log N}{8\chi^2}$ vertices, $\chi(T_1[N^+_{T_2}(U)]) \leqslant 2$.*

In turn, Corollary 3 has the following immediate consequence which disproves a conjecture of Nguyen, Scott, and Seymour [NSS23].

Corollary 4. *For every positive integer χ there are arbitrarily large N for which there is an N-vertex tournament T and disjoint subsets $A, B \subseteq V(T)$ such that $\chi(T[A]), \chi(T[B]) \geqslant \chi$ and the following holds. For all $A' \subseteq A$ and $B' \subseteq B$ of size at most $\frac{\log N}{32\chi^2}$, both $\chi(A \cap N^+(B'))$ and $\chi(B \cap N^+(A'))$ are at most 2.*

Finally, we include two results for the setting where chromatic number is replaced by degeneracy (or equivalently maximum average degree). Since every graph of high chromatic number has high degeneracy, Theorem 2 shows that for every positive integer d there is a graph G and a tournament T on the same vertex set such that the degeneracy of G is at least d, but the subgraph of G induced on each out-neighbourhood of T is bipartite. Our next result strengthens this statement by ensuring that the graph induced on the out-neighbourhood is 1-degenerate.

Proposition 5. *For every positive integer k, there is a k-regular graph G and a tournament T on the same vertex set such that $G[N^+_T(v)]$ is a forest for every vertex v.*

Despite this result, and in contrast to Theorem 2, if G has high degeneracy and T is a tournament on the same vertex set, then there is a two-vertex set whose out-neighbourhood has high degeneracy.

Theorem 6. *For every positive integer k, every graph G with degeneracy at least $12k$, and every tournament T on the same vertex set, there exist vertices x, y such that $G[N^+(\{x, y\})]$ has degeneracy at least $k - 1$.*

2. Proofs of the main theorems

In this section we present the proof of Theorem 2. Our construction is based on the classical *Schrijver graphs* [Sch78].

Definition 7. *Let $k \geqslant 1$ and $n \geqslant 2k$ be integers. The Kneser graph $\mathsf{KG}(n, k)$ is the graph whose vertex set is $\binom{[n]}{k}$ and in which two distinct sets $S_1, S_2 \in \binom{[n]}{k}$ are adjacent if and only if $S_1 \cap S_2 = \varnothing$. The Schrijver graph $\mathsf{SG}(n, k)$ is the induced subgraph of $\mathsf{KG}(n, k)$ whose vertex* *set consists of all stable sets in* $\binom{[n]}{k}$*. Here, a set* $S \in \binom{[n]}{k}$ *is called* stable *if it does not include two cyclically consecutive*[3] *elements of* $[n]$*.*

Kneser [Kne55] conjectured that the chromatic number of $\mathsf{KG}(n,k)$ is $n-2k+2$. This conjecture remained open for two decades and was first proved by Lovász [Lov78] using homotopy theory (see also Bárány [Bár78] and Greene [Gre02] for very short proofs). Shortly afterwards, Schrijver [Sch78] introduced the graphs $\mathsf{SG}(n,k)$ and proved that $\mathsf{SG}(n,k)$ is vertex-critical with chromatic number $\chi(\mathsf{SG}(n,k)) = \chi(\mathsf{KG}(n,k)) = n-2k+2$.

To prove Theorem 2, we will show that for every integer $\chi \geqslant 3$ and every sufficiently large integer k there exists a tournament T on the same vertex set as $\mathsf{SG}(2k+\chi-2,k)$ such that for every $U \subseteq V(T)$ which is sufficiently small, the out-neighbourhood of U in T induces a bipartite subgraph of $\mathsf{SG}(2k+\chi-2,k)$. As $\chi(\mathsf{SG}(2k+\chi-2,k)) = \chi$, this will prove Theorem 2.

In constructing our tournament, we rely on the following combinatorial statement which follows directly from the existence of tournaments with high domination number.

Lemma 8. *For every positive integer* t *there is some* n_0 *such that for all integers* $n \geqslant n_0$ *there exists a function* $f\colon \binom{[n]}{t} \to 2^{[n]}$ *with the following two properties:*

- *for every* $A, B \in \binom{[n]}{t}$*, at least one of* $A \cap f(B)$ *and* $B \cap f(A)$ *is empty, and*
- *for every collection* $(A_i)_{i\in I}$ *of at most* $\frac{\log n}{2t}$ *sets from* $\binom{[n]}{t}$*,*

$$\bigcap_{i\in I} f(A_i) \neq \varnothing.$$

Proof. By a classical result of Erdős [Erd63] (see [GS71] for an explicit construction), for every sufficiently large n there is an n-vertex tournament in which every set of at most $\log(n)/2$ vertices is dominated by a vertex outside the set. Let n be large enough that this result holds and that $\log(n)/2 \geqslant t$, and let T be the corresponding tournament. Identify $V(T)$ with $[n]$ and, for $A \in \binom{[n]}{t}$, define $f(A)$ as

$$f(A) := \{v \in [n] \setminus A \colon v \text{ dominates } A\}.$$

We claim f satisfies the two properties of the lemma statement. Firstly, let $A, B \in \binom{[n]}{t}$ and suppose for a contradiction that $A \cap f(B)$ and $B \cap f(A)$ are both non-empty. Then there is some $a \in A \setminus B$ that dominates B and some $b \in B \setminus A$ that dominates A. This implies that a and b are distinct, and the edge between them is oriented in both directions, which is a contradiction. Next, let $(A_i)_{i\in I}$ be a collection of at most $\frac{\log n}{2t}$ sets from $\binom{[n]}{t}$. Let $A = \bigcup_{i\in I} A_i$ which is a set of size at most $\log(n)/2$. By the definition of T some vertex $x \notin A$ dominates A, but then $x \in \bigcap_{i\in I} f(A_i)$, as required. □

Before giving the proof of Theorem 2, let us fix the following notation: for a set $S \in \binom{[n]}{k}$, we denote by $\mathsf{gap}(S)$ the set of "left-elements" of cyclically consecutive pairs of $[n]$ that are disjoint from S. Concretely, $r \in \mathsf{gap}(S)$ if and only if $\{r, r+1\} \cap S = \varnothing$, where addition is to be understood modulo n (that is, $n+1$ is identified with 1). Pause to note that every stable set $S \subseteq [n]$ of size k (that is, every vertex of the Schrijver graph $\mathsf{SG}(n,k)$) satisfies $|\mathsf{gap}(S)| = n-2k$. Every $S \in \binom{[n]}{k}$ can be recovered from $\mathsf{gap}(S)$ and so $|V(\mathsf{SG}(n,k))| \leqslant \binom{n}{n-2k}$.

Proof of Theorem 2. The result is trivial for $\chi \leqslant 2$, so let $\chi \geqslant 3$ be an integer, $t := \chi - 2$, and n_0 be as given by Lemma 8. Pick some positive integer $k > t$ such that $2k + t \geqslant n_0$, set $n := 2k+t$, and set $G := \mathsf{SG}(n,k)$. Note that G is triangle-free, has chromatic number χ and, for any $S \in V(\mathsf{SG}(n,k))$, $\mathsf{gap}(S) \in \binom{[n]}{t}$. Hence, $N := |V(\mathsf{SG}(n,k))| \leqslant \binom{n}{t} \leqslant n^t$.

Let $f\colon \binom{[n]}{t} \to 2^{[n]}$ be the function from Lemma 8. Define a directed graph D on the same vertex set as G that has a directed edge from a vertex S_1 to a vertex S_2 if and only if $f(\mathsf{gap}(S_1)) \cap \mathsf{gap}(S_2) = \varnothing$. Note, by the first property of f guaranteed by Lemma 8, that

[3] By this we mean a pair $i, i+1$ where $1 \leqslant i < n$ or the pair $n, 1$.

any two distinct vertices of D are connected by an arc in at least one of the two possible directions. Hence, there exists a spanning subdigraph T of D which is a tournament.

Let U be any set of at most $\frac{\log N}{2\chi^2} \leqslant \frac{\log N}{2t^2} \leqslant \frac{\log n}{2t}$ vertices. To finish the proof we will show that the out-neighbourhood $N_D^+(U)$ induces a bipartite subgraph of G (and hence the same is true for the out-neighbourhood $N_T^+(U) \subseteq N_D^+(U)$ in T). Write $U = \{S_1, \ldots, S_{|U|}\}$. By the second property of f guaranteed by Lemma 8, there is some $r \in [n]$ common to all the $f(\mathsf{gap}(S_i))$. By the definition of D, any $S \in N_D^+(U)$ satisfies $r \notin \mathsf{gap}(S)$ and so $S \cap \{r, r+1\} \neq \varnothing$. Colouring all the vertices in the out-neighbourhood that include the element r with one colour and all the remaining vertices (which necessarily contain $r+1$) with another colour provides a proper 2-colouring of $G[N_D^+(S)]$. This concludes the proof of the theorem. □

We can convert the graph G from Theorem 2 to a tournament: pick any linear order on the vertices of G and construct a tournament T_1 whose back-edge graph is G. We will show that $\chi(G)$ and $\chi(T_1)$ are closely related, and thus prove Corollary 3.

Proof of Corollary 3. Let $K := 2\chi$ and n be sufficiently large. By Theorem 2 there is a triangle-free graph G with chromatic number K and a tournament T on the same N-vertex set such that, for every set U of at most $\frac{\log N}{8\chi^2}$ vertices, $\chi(G[N_T^+(U)]) \leqslant 2$. Let $(V(G), \prec)$ be a linear order and define a tournament T_1 with vertex set $V(G)$ as follows: there is an arc from vertex u to vertex v in T_1 if either $v \prec u$ and $uv \in E(G)$ or $u \prec v$ and $uv \notin E(G)$. We further set $T_2 := T$ and claim that the pair (T_1, T_2) of tournaments satisfies the statement of the corollary.

Let $W \subseteq V(G)$ be any set of vertices where $T_1[W]$ is transitive. Note that if $v_1v_2v_3$ is a path in G (so $v_1v_3 \notin E(G)$ by triangle-freeness) and $v_1 \prec v_2 \prec v_3$, then $v_1v_2v_3$ is a cyclic triangle in T_1 and so v_1, v_2, v_3 are not all in W. In particular, the partition $W = W_1 \cup W_2$ where

$$W_1 := \{w \in W\colon \text{there is } w' \in W \text{ such that } w' \prec w \text{ and } w'w \in E(G)\},$$
$$W_2 := \{w \in W\colon \text{there is no } w' \in W \text{ such that } w' \prec w \text{ and } w'w \in E(G)\},$$

gives a proper 2-colouring of the vertices of $G[W]$. Since this holds for any W where $T_1[W]$ is transitive, we have $\chi(T_1) \geqslant \chi(G)/2 = \chi$.

To finish the proof, consider any set U of at most $\frac{\log N}{8\chi^2} = \frac{\log N}{2K^2}$ vertices. Note that $G[N_T^+(U)] = G[N_{T_2}^+(U)]$ is bipartite. Let I_1, I_2 be two disjoint independent sets in G such that $I_1 \cup I_2 = N_{T_2}^+(U)$. Now consider any two vertices $u, v \in I_j$ for some $j \in \{1, 2\}$ and note that since $uv \notin E(G)$, there is an arc from u to v in T_1 if and only if $u \prec v$. Hence $T_1[I_1]$ and $T_1[I_2]$ are transitive tournaments and so $\chi(T_1[N_{T_2}^+(U)]) \leqslant 2$. □

To prove Corollary 4, we can now take the two tournaments T_1 and T_2 from Corollary 3 and combine them appropriately: we simply orient the edges within A and B according to T_1, and the edges between A and B according to T_2.

Proof of Corollary 4. Let χ be a positive integer. By Corollary 3, for arbitrarily large N there exist tournaments T_1 and T_2 on the same N-vertex set V with $\chi(T_1) = 2\chi$ and $\chi(T_1[N_{T_2}^+(U)]) \leqslant 2$ for every $U \subseteq V$ of size at most $\frac{\log N}{32\chi^2}$. Partition V into sets A and B such that $\chi(T_1[A]), \chi(T_1[B]) \geqslant \chi$, then construct a new tournament T on V by orienting the edge between $u, v \in V$ to agree with T_1 if $u, v \in A$ or $u, v \in B$, and orienting it to agree with T_2 otherwise. It is not difficult to see that T satisfies the conditions of the corollary. □

3. Degeneracy

In this section we consider the setting in which degeneracy replaces chromatic number. We first show that there is a tournament on the vertex set of the k-dimensional hypercube such that each out-neighbourhood induces a forest in the hypercube, proving Proposition 5.

Therefore, having high degeneracy does not imply that some out-neighbourhood has high degeneracy.

Proof of Proposition 5. For each k, let G_k be the hypercube on 2^k vertices. We will actually prove something stronger than Proposition 5, namely that the *closed* in- and out-neighbourhoods[4] $G_k[N_T^-[v]]$ and $G_k[N_T^+[v]]$ are both forests for every vertex $v \in V(G_k)$. We proceed by induction on k. For $k = 1$ the result is immediate, so given $k \geqslant 1$ let T_k be a tournament on $V(G_k)$ with the desired property. We will view G_{k+1} as the union of two copies of G_k, say G_k^1 and G_k^2, connected via the matching consisting of all edges of the form x^1x^2, where $x^1 \in V(G_k^1)$ and $x^2 \in V(G_k^2)$ denote the copies of a vertex $x \in V(G_k)$. For each $S \subseteq V(G_k)$, we will write $S^{(1)}$ and $S^{(2)}$ for the corresponding sets of vertices in G_k^1 and G_k^2 respectively.

Now define a tournament T_{k+1} on vertex set $V(G_{k+1})$ as follows. First orient the edges within each of $V(G_k^1)$ and $V(G_k^2)$ according to T_k, in the canonical way. Then for each $x \in V(G_k)$, orient every edge between x^1 and $N_{T_k}^-[x]^{(2)}$ away from x^1 and every edge between x^1 and $N_{T_k}^+(x)^{(2)}$ towards x^1. This completes the construction of T_{k+1}. Observe that for each $x \in V(G_k)$, the edges between x^2 and $N_{T_k}^-(x)^{(2)}$ are oriented away from x^2 and the edges between x^2 and $N_{T_k}^+[x]^{(2)}$ are oriented towards x^2.

Let $x \in V(G_k)$ and note that $N_{T_{k+1}}^+[x^1] = N_{T_k}^+[x]^{(1)} \cup N_{T_k}^-[x]^{(2)}$. By the induction hypothesis, $N_{T_k}^+[x]$ and $N_{T_k}^-[x]$ both induce forests in G_k, so $N_{T_k}^+[x]^{(1)}$ and $N_{T_k}^-[x]^{(2)}$ do the same in G_{k+1}. Since there is exactly one edge in G_{k+1} between these two sets, namely x^1x^2, the graph $G_{k+1}[N_{T_{k+1}}^+[x^1]]$ is acyclic. Analogous arguments show that $G_{k+1}[N_{T_{k+1}}^-[x^1]]$, $G_{k+1}[N_{T_{k+1}}^+[x^2]]$, and $G_{k+1}[N_{T_{k+1}}^-[x^2]]$ are all acyclic too. Since every vertex of G_{k+1} is of the form x^1 or x^2 for some $x \in V(G_k)$, this completes the proof. □

However, we will now show that, unlike with chromatic number, having high degeneracy implies that there are two vertices x and y such that the out-neighbourhood of $\{x, y\}$ has high degeneracy.

Proof of Theorem 6. Let H be a bipartite subgraph of G with $\delta(H) \geqslant 6k$ and let $A \cup B$ be a bipartition of H with $|A| \geqslant |B|$. Define $T_1 = T[A]$ and $T_2 = T[B]$. Pick $x \in A$ satisfying $|N_{T_1}^+[x]| \geqslant |A|/2$ and define $A' = N_{T_1}^+[x]$. Now let $H_1 = H[A', B]$. It can be shown using linear programming duality that every tournament has a probability distribution on its vertex set which assigns weight at least $1/2$ to every closed in-neighbourhood (see [FR95, Sec. 1.2]). Let w be such a probability distribution for T_2. Take a random vertex $\boldsymbol{y} \in B$ according to w and note that $\mathbb{P}(u \in N_{T_2}^+[\boldsymbol{y}]) \geqslant 1/2$ for every $u \in B$. Let $\boldsymbol{H_2} = H_1[A', N_{T_2}^+[\boldsymbol{y}]]$ so that for every $e \in E(H_1)$, $\mathbb{P}[e \in E(\boldsymbol{H_2})] \geqslant 1/2$. We have $\mathbb{E}[e(\boldsymbol{H_2})] \geqslant e(H_1)/2 \geqslant 3k|A'| \geqslant k(|A'| + |B|)$, from which it follows, since $|N_{T_2}^+[y]| \leqslant |B|$, that there exists $y \in B$ such that $e(H_2) \geqslant k|V(H_2)|$. Removing x and y from H_2, we obtain a subgraph G' of $G[N_T^+(\{x, y\})]$ with $e(G') \geqslant (k-2)|V(G')|$. Thus G', and therefore also $G[N^+(\{x, y\})]$, has degeneracy greater than $k - 2$. □

4. Fractional chromatic number

We remind the reader that a graph G has fractional chromatic number $\chi_f(G) \leqslant r$ if and only if there is a probability distribution on the independent sets of G such that the random independent set $\boldsymbol{I}$ obtained and every vertex v satisfy $\mathbb{P}(v \in \boldsymbol{I}) \geqslant 1/r$. In this section we demonstrate that the modified version of Conjecture 1 in which chromatic number is replaced

[4]The *closed in-neighbourhood* of a vertex v in tournament T is $N_T^-[v] := \{v\} \cup N_T^-(v)$. The closed out-neighbourhood is defined analogously.

by fractional chromatic number is true, as observed by Scott and Seymour [SS16] without proof.

Theorem 9. *For $c \geqslant 1$, let G be a graph and T be a tournament on the same vertex set such that $\chi_f(G[N_T^+(v)]) \leqslant c$ for every vertex v. Then $\chi_f(G) \leqslant 2(c+1)$.*

Proof. Let w be a probability distribution on the vertex set of T that assigns weight at least $1/2$ to every closed in-neighbourhood. For each vertex v, since $\chi_f(G[N_T^+(v)]) \leqslant c$, there is a random independent set $\boldsymbol{I}_v$ of $G[N_T^+(v)]$ such that $\mathbb{P}(u \in \boldsymbol{I}_v) \geqslant 1/c$ for every $u \in N_T^+(v)$.

We sample a random independent set $\boldsymbol{I}$ of G as follows. First pick a vertex $\boldsymbol{v}$ according to w. Then with probability $1/(c+1)$ take $\boldsymbol{I} = \{\boldsymbol{v}\}$ and with probability $c/(c+1)$ take $\boldsymbol{I} = \boldsymbol{I}_{\boldsymbol{v}}$. Note that, for any vertex u, if $\boldsymbol{v} \in N^-[u]$, then $u \in \boldsymbol{I}$ with probability at least $1/(c+1)$. Hence, by the defining property of w, $\mathbb{P}(u \in \boldsymbol{I}) \geqslant 1/(2c+2)$ and so $\chi_f(G) \leqslant 2(c+1)$. □

5. Closing remarks

We have been unable to determine whether high chromatic number forces an out-neighbourhood with high degeneracy, and we would be interested to know if this is the case.

Question 10. *Does there exist, for each integer d, an integer χ such that for every graph G with $\chi(G) \geqslant \chi$ and every tournament T on the same vertex set, there is a vertex v for which $G[N_T^+(v)]$ has degeneracy at least d?*

We do, however, suspect that this is true for $d = 2$, that is, it should be possible to force some out-neighbourhood to contain a cycle.

Conjecture 11. *For every graph G with sufficiently large chromatic number, and every tournament T on the same vertex set, there exists a vertex v such that $G[N_T^+(v)]$ contains a cycle.*

We have shown that for certain very structured tournaments T there are graphs on the same vertex set with large chromatic number, in which every out-neighbourhood of T induces a bipartite subgraph. We conjecture that (with high probability) we cannot replace T with a random tournament.

Conjecture 12. *For every positive integer k, there exists a χ such that if T is the uniformly random tournament on vertex set $[N]$, then with high probability (as $N \to \infty$), for every graph G on $[N]$ with $\chi(G) \geqslant \chi$ there is a vertex $v \in [N]$ for which $G[N_T^+(v)] \geqslant k$.*

Finally, as remarked after the statement of Theorem 2, if $\chi(G) \geqslant \chi$, then there is a collection of at most $\lceil \log_2(N)/\lfloor \chi/2-1 \rfloor \rceil$ out-neighbourhoods whose union induces a subgraph of chromatic number at least 3. It would be interesting to know if $o(\log(N)/\chi)$ (as $\chi \to \infty$) out-neighbourhoods suffice here. In particular, we conjecture the following.

Conjecture 13. *There exists $f(N)$ satisfying $f(N) = o(\log N)$ such that for every N-vertex graph G with $\chi(G) \geqslant f(N)$, and every tournament T on the same vertex set, there is a vertex v for which $\chi(G[N_T^+(v)]) \geqslant 3$.*

Acknowledgements. We would like to thank Sang-il Oum, Alex Scott, David Wood, and Liana Yepremyan for organising the April 2023 MATRIX-IBS Structural Graph Theory Downunder III workshop where we began this work. Our thanks to Paul Seymour for helpful comments on the paper.

References

[Bár78] Imre Bárány (Nov. 1978). A short proof of Kneser's conjecture. *Journal of Combinatorial Theory, Series A* **25**(3), 325–326.

[BCC+13] Eli Berger, Krzysztof Choromanski, Maria Chudnovsky, Jacob Fox, Martin Loebl, Alex Scott, Paul Seymour, and Stéphan Thomassé (Jan. 2013). Tournaments and colouring. *Journal of Combinatorial Theory, Series B* **103**(1), 1–20.

[Des54] Blanche Descartes (May 1954). k-chromatic graphs without triangles. *American Mathematical Monthly* **61**(5), 352–353.

[Erd63] Paul Erdős (Oct. 1963). On a problem in graph theory. *The Mathematical Gazette* **47**, 220–223.

[FR95] David C. Fisher and Jennifer Ryan (Mar. 1995). Tournament games and positive tournaments. *Journal of Graph Theory* **19**(2), 217–236.

[Gre02] Joshua E. Greene (Dec. 2002). A new short proof of Kneser's conjecture. *The American Mathematical Monthly* **109**(10), 918–920.

[GS71] Ronald L. Graham and Joel H. Spencer (Jan. 1971). A constructive solution to a tournament problem. *Canadian Mathematical Bulletin* **14**, 45–48.

[HLTW19] Ararat Harutyunyan, Tien-Nam Le, Stéphan Thomassé, and Hehui Wu (Sep. 2019). Coloring tournaments: from local to global. *Journal of Combinatorial Theory, Series B* **138**, 166–171.

[Kne55] Martin Kneser (1955). Aufgabe 360. *Jahresbericht der Deutschen Mathematiker-Vereinigung* **58**(2).

[Lov78] László Lovász (Nov. 1978). Kneser's conjecture, chromatic number, and homotopy. *Journal of Combinatorial Theory, Series A* **25**(3), 319–324.

[NSS23] Tung Nguyen, Alex Scott, and Paul Seymour (2023). Some results and problems on tournament structure. url: `https://web.math.princeton.edu/~pds/papers/tourprobs/paper.pdf`. In preparation. Accessed on 18 May 2023.

[Sch78] Alexander Schrijver (1978). Vertex-critical subgraphs of Kneser graphs. *Nieuw Archief voor Wiskunde* **26**(3), 454–461.

[SS16] Alex Scott and Paul Seymour (Mar. 2016). Open problems for the Barbados Graph Theory Workshop 2016, problem 1. url: `https://web.math.princeton.edu/~pds/barbados2016/openproblems.html`. Accessed on 18 May 2023.

[Zyk49] A. Zykov (1949). On some properties of linear complexes (in Russian). *Matematicheskiĭ Sbornik* **24**, 163–188. For an English translation see *American Mathematical Society Translations* **79**, 1952.

(Girão, Illingworth, Michel, Savery) Mathematical Institute, University of Oxford, UK
Email address: `{girao,illingworth,michel,savery}@maths.ox.ac.uk`

(Hendrey) Institute for Basic Science, Daejeon, South Korea
Email address: `kevinhendrey@ibs.re.kr`

(Lehner) Department of Mathematics, University of Auckland, New Zealand
Email address: `florian.lehner@auckland.ac.nz`

(Savery) Heilbronn Institute for Mathematical Research, Bristol, UK

(Steiner) Institute of Theoretical Computer Science, ETH Zürich, Switzerland
Email address: `raphaelmario.steiner@inf.ethz.ch`