JUDICIOUS PARTITIONS OF 3-UNIFORM HYPERGRAPHS

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Abstract. A conjecture of Bollobás and Thomason asserts that, for $r \geq 1$, every r-uniform hypergraph with m edges can be partitioned into r classes such that every class meets at least $rm/(2r-1)$ edges. Bollobás, Reed and Thomason [3] proved that there is a partition in which every edge meets at least $(1-1/e)m/3 \approx 0.21m$ edges. Our main aim is to improve this result for $r = 3$. We prove that every 3-uniform hypergraph with m edges can be partitioned into 3 classes, each of which meets at least $(5m-1)/9$ edges. We also prove that for $r > 3$ we may demand $0.27m$ edges.

Note: For the final version of this paper, see the journal publication.

1. Introduction

Many classical partitioning problems ask for the maximum or minimum of a given quantity over all partitions of a combinatorial structure. For instance, the Max Cut problem asks for the maximum size of a bipartite subgraph of a graph G; this is equivalent to solving the problem of finding the minimum over partitions $V(G) = V_1 \cup V_2$ of $e(G[V_1]) + e(G[V_2])$. More generally, the Max k-Cut problem asks for the maximum size of a k-partite subgraph of G, or equivalently for the minimum over partitions $V(G) = V_1 \cup \cdots \cup V_k$ of $\sum_{i=1}^k e(G[V_i])$. Max Cut is NP-Hard [10], and has been the subject of much research both in computer science and combinatorics (see Edwards [6], [7]; Erdős, Gyárfás and Kohayakawa [9]; Alon [1]; Andersen, Grant and Linial [2]; Erdős, Faudree, Pach and Spencer [8]).

Partitioning problems such as Max Cut involve maximizing or minimizing a single quantity. However, in applications it is often the case that many quantities must be maximized or minimized simultaneously (one can think of many practical examples, such as sharing out sweets among a group of children): we shall refer to such problems as *judicious partitioning problems*. For instance, given a graph G and an integer k, we ask for the minimum over all partitions $V(G) = V_1 \cup \cdots \cup V_k$ of

$$\max\{e(G[V_1]), \dots, e(G[V_k])\}.$$

In [4] it was proved that every graph G with m edges has a vertex-partition into k classes, each of which contains at most $m/\binom{k+1}{2}$ edges; there is also a vertex-partition into k classes in which each class contains at most $(1 + o(1))m/k^2$ edges. Thus the asymptotic bound is just over half the extremal bound: this seems to be a common feature of judicious partitioning problems. In [5], the analogous problem for hypergraphs was considered. It was shown that, for every integer k, every 3-uniform hypergraph with m edges has a partition into k sets, each of which contains at most $(1 + o(1))m/k^3$ edges, and a similar result was conjectured for r-uniform hypergraphs. (For $r = 1$ we get the trivial problem of partitioning a set; however, the weighted version of the problem is not trivial. Results for the weighted problem are given by van Lint [11].)

In this paper we consider partitions in which every vertex class *meets* many edges. More specifically, given an r-uniform hypergraph H with m edges and an integer $k \geq 2$, what is the maximum over all partitions $V(H) = V_1 \cup \cdots \cup V_k$ of

$$\min\{d(V_1), \ldots, d(V_k)\},$$

where $d(S)$ denotes the number of edges incident with S? Bollobás and Thomason have conjectured that every r-uniform hypergraph with m edges has a partition into r classes in which each class meets at least

$$\frac{rm}{2r-1}$$

edges. For $r = 2$, this follows immediately from the first result cited from [4] above. For $r \geq 3$, Bollobás, Reed and Thomason [3] have proved that there is a partition in which each class meets at least $(1-\frac{1}{e})m/3 \approx 0.21m$ edges. Our main aim in this paper is to address the case $r = 3$. We prove that every 3-uniform hypergraph with m edges has a partition into three sets, each of which meets at least $(5m-1)/9$ edges (note that the conjectured bound is $3m/5$). For $r \geq 3$, we give an improvement on the bound of [3], showing that there is a partition into r sets, each of which meets at least $0.27m$ edges. We conclude with some open problems.

For a hypergraph H and $W \subset V(H)$ we write $d(W)$ for the number of edges meeting W and $e(W)$ for the number of edges contained in W. We shall also write $d_i(W)$ for the number of edges of size i meeting W and $e_i(W)$ for the number of edges of size i contained in W. Similarly, $d(V_j, V_k)$ denotes the number of edges meeting both V_j and V_k and $d_i(V_j, V_k)$ for the number of edges of size i meeting both V_j and V_k.

2. The main result

Our main aim in this paper is to prove a result for 3-uniform hypergraphs. The constant we obtain in Theorem 1 is 5/9, while the conjectured bound has constant 3/5.

Theorem 1. *Let G be a 3-uniform hypergraph with m edges. Then there is a partition of $V(G)$ into three sets, each of which meets at least*

$$\frac{5m-1}{9} \tag{1}$$

edges.

We shall use two lemmas in the proof of Theorem 1. The first lemma asserts that we can find a 'good' random partition of a 3-uniform hypergraph, and the second is a general partitioning result for hypergraphs. Much of the detail in Lemma 2 and the proof of Theorem 1 (for instance, the $2s/9$ term in (2)) is needed only for the constant term in (1) and could be omitted if we were happy with a bound of form $(5m-C)/9$.

Note that, by considering random partitions, it follows immediately that for every 3-uniform hypergraph G there is some partition $V(G) = V_1 \cup V_2 \cup V_3$ with

$$d(V_1) + d(V_2) + d(V_3) \geq \frac{19}{9}e(G).$$

The constant 19/9 is clearly best possible, as can be seen by considering large complete triple systems. However, we can improve on this in two ways. First of all, if there are two vertices that share many edges then we can consider random partitions in which those vertices are in different classes: we obtain a slight improvement on $19e(G)/9$. Secondly, by partitioning a little more carefully, we may ensure that the sums of degrees in each class do not differ by too much.

Lemma 1. *Let G be a 3-uniform hypergraph with vertices $v_1, v_2, \ldots, v_n$, where $d(v_1) \geq d(v_2) \geq \cdots d(v_n)$, and suppose that there are s edges that contain at least two of v_1, v_2 and v_3. Then there is a partition $V(G) = V_1 \cup V_2 \cup V_3$ with v_1, v_2 and v_3 in different vertex classes, such that*

$$d(V_1) + d(V_2) + d(V_3) \geq \frac{19}{9}e(G) + \frac{2}{9}s \tag{2}$$

and, for $i \neq j$,

$$\sum_{v \in V_i} d(v) - \sum_{v \in V_j} d(v) \leq \max_{v \in V_i}\{d(v)\}. \tag{3}$$

Proof. Adding one or two isolated vertices if required, we may assume that $n = 3k$ for some integer k, so $V(G) = \{v_1, \dots, v_{3k}\}$, where $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_{3k})$. We pick independently, for $j = 0, \dots, k-1$, a random permutation $\sigma_j \in \Sigma_3$ and, for $i = 1, 2, 3$, let

$$V_i = \{v_{3j+\sigma_j(i)} : j = 0, \dots, k-1\}.$$

Thus we have partitioned $V(G)$ into three sets of size k, each of which contains one vertex from $\{v_{3j+1}, v_{3j+2}, v_{3j+3}\}$, for $j = 0, \dots, k-1$. It is easily seen that each edge meets each vertex class with probability at least $19/27$. Since v_1, v_2 and v_3 belong to different vertex classes, every edge containing at least two vertices from v_1, v_2 and v_3 meets each vertex class with probability at least $7/9$ (there are two cases to check: when the edge is $\{v_1, v_2, v_3\}$, and when the third vertex is v_i for some $i > 3$). Thus

$$\begin{aligned} \mathbb{E}\left(\sum_{i=1}^{s} d(V_i)\right) &\geq \frac{19}{9}(e(G) - s) + \frac{7}{3}s \\ &= 199e(G) + 29s. \end{aligned}$$

Hence there is a partition of this form that satisfies (2).

Furthermore, for $1 \leq i, j \leq 3$,

$$\begin{aligned} \sum_{v \in V_i} d(v) - \sum_{v \in V_j} d(v) &= \sum_{l=0}^{k-1} d(v_{3l+\sigma_l(i)}) - \sum_{l=0}^{k-1} d(v_{3l+\sigma_l(j)}) \\ &= \sum_{l=0}^{k-1} \big(d(v_{3l+\sigma_l(i)}) - d(v_{3l+\sigma_l(j)})\big) \\ &\leq d(v_{\sigma_1(i)}) \\ &= \max_{v \in V_i} d(v), \end{aligned}$$

since $d(v_{3l+\sigma_l(j)}) \geq d(v_{3(l+1)+\sigma_l(i)})$, for $l < k-1$. □

In an earlier paper [4], we found partitions of graphs such that each vertex class contains few edges. A simple case of this is the assertion that every multigraph G has a vertex partition $V(G) = V_1 \cup V_2$ such that each vertex class contains at most $e(G)/3$ edges; equivalently, each vertex class meets at least $2e(G)/3$ edges. We shall need the following extension of this fact. Although we only need the result for $k = 2$, we give a more general result since it is no harder to prove.

Lemma 2. *Let k be an integer and let G be a hypergraph with m_i edges of size i, for $i = 1, \dots, k$. Then there is a partition of $V(G)$ into two* *sets, each of which meets at least*

$$\frac{m_1 - 1}{3} + \frac{2m_2}{3} + \frac{3m_3}{4} + \ldots + \frac{km_k}{k+1} \tag{4}$$

edges.

Proof. If G contains at least two edges of size one, we choose two such edges, say $\{x\}$ and $\{y\}$, and replace them with a single edge $\{x, y\}$. Clearly, a partition that satisfies (4) for the new hypergraph also satisfies (4) for the original hypergraph. We may therefore assume that G has at most one edge of size 1, so $m_1 \le 1$. It is therefore enough to prove that we can find a partition $V(G) = V_1 \cup V_2$ such that each V_i meets at least

$$\frac{2m_2}{3} + \cdots + \frac{km_k}{k+1}$$

edges.

Let $\lambda_2, \ldots, \lambda_k$ be positive reals and let $V(G) = V_1 \cup V_2$ be a vertex partition minimizing

$$\sum_{i=2}^{k} \lambda_i (e_i(V_1) + e_i(V_2)). \tag{5}$$

For $v \in V_i$, we shall write $f_j(v)$ for the number of edges of size j that are contained in V_i and contain v, and $g_j(v)$ for the number of edges of size j that meet V_i only in the vertex v. Now, for $v \in V_1$, since moving v from V_1 to V_2 does not decrease (5), we have

$$\sum_{j=2}^{k} \lambda_j (f_j(v) - g_j(v)) \le 0.$$

Summing over v,

$$\sum_{j=2}^{k} \lambda_j \sum_{v \in V_1} f_j(v) \le \sum_{j=2}^{k} \lambda_j \sum_{v \in V_1} g_j(v)$$

and so

$$\sum_{j=2}^{k} j \lambda_j e_j(V_1) \le \sum_{j=2}^{k} \lambda_j d_j(V_1, V_2).$$

Therefore

$$\begin{aligned}\sum_{j=2}^{k}(j+1)\lambda_j e_j(V_1) &\leq \sum_{j=2}^{k}\lambda_j(d_j(V_1,V_2)+e_j(V_1))\\ &\leq \sum_{j=2}^{k}\lambda_j m_j,\end{aligned}$$

since $m_j = e_j(V_1) + d_j(V_1, V_2) + e_j(V_2)$. Taking $\lambda_j = 1/(j+1)$, for $j = 2, \ldots, k$, we get

$$\sum_{j=2}^{k} e_j(V_1) \leq \sum_{j=2}^{k}\frac{1}{j+1}m_j.$$

Thus V_2 meets at least $\sum_{j=2}^{k} m_j - \sum_{j=2}^{k} e_j(V_1) \geq \sum_{j=2}^{k}\frac{j}{j+1}m_j$ edges. Arguing similarly for V_1, we obtain (4). □

The bound in Lemma 3 can very likely be improved. In particular, we believe that the term $(m_1 - 1)/3$ can be replaced by $(m_1 - 1)/2$.

We can now proceed with the proof of Theorem 1.

Proof of Theorem 1. Let G be a 3-uniform hypergraph that has no partition satisfying (1). Let $m = e(G)$ and let cm be the largest integer less than $(5m-1)/9$, so $cm = \lfloor(5m-2)/9\rfloor$. We must show that there is a partition of $V(G)$ into three sets, each of which meets more than cm edges.

If there is a vertex $v \in V(G)$ with $d(v) > cm$ then we can take $\{v\}$ as one vertex class and, by Lemma 3, partition $V(G)\setminus\{v\}$ into two classes, each meeting more than cm edges. Thus we may assume $\Delta(G) \leq cm$. We may assume $m > 4$, since smaller cases are easily checked. Let $V(G) = V_1 \cup V_2 \cup V_3$ be the partition guaranteed by Lemma 2. For $i = 1, 2, 3$, let $w_i = d(V_i)$, let $d_i = \max_{v\in V_i}\{d(v)\}$ and let $v_i \in V_i$ be a vertex of degree d_i. We may assume that $w_1 \leq w_2 \leq w_3$ and that v_1, v_2 and v_3 are in different vertex classes. Suppose that v_2 and v_3 have t common edges. Thus a total of $d_2 + d_3 - t$ edges meet v_2 or v_3. If $w_1 > cm$ then we are done. Otherwise, we may assume $w_3 \geq w_2 > cm$, since if $w_2 \leq cm$ then $w_1 + w_2 + w_3 < (2c + 1)m < (19/9)m$, which contradicts (2).

For $0 \leq i \leq 3$, let E_i be the set of edges of G meeting V_2 in exactly i vertices, and set $e_i = |E_i|$. The multiset $\{e \setminus V_2 : e \in E_0 \cup E_1 \cup E_2\}$ is the edge set of a multigraph H with vertex set $V(G) \setminus V_2$ and e_i edges with $3 - i$ vertices, for $i = 0, 1, 2$. Thus, from Lemma 3, we must have

$$\frac{e_2 - 1}{3} + \frac{2e_1}{3} + \frac{3e_0}{4} \leq cm, \tag{6}$$

or else we could partition $V(G) \setminus V_2$ into two sets, each meeting more than cm edges, which together with V_2 would give the required partition. Now

$$\sum_{v\in V_2} d(v) = 3e_3 + 2e_2 + e_1,$$

so it follows from (6) that

$$\begin{aligned}\sum_{v\in V_2} d(v) + 3cm &\geq 3e_3 + 3e_2 + 3e_1 + \frac{9e_0}{4} - 1 \\ &= \frac{9}{4}m + \frac{3}{4}e_3 + \frac{3}{4}e_2 + \frac{3}{4}e_1 - 1.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{v\in V_2} d(v) &\geq \frac{3}{4}(e_3 + e_2 + e_1) + (\frac{9}{4} - 3c)m - 1 \\ &= \frac{3}{4}w_2 + (\frac{9}{4} - 3c)m - 1. \qquad (7)\end{aligned}$$

A similar argument gives

$$\sum_{v\in V_3} d(v) \geq \frac{3}{4}w_3 + (\frac{9}{4} - 3c)m - 1. \qquad (8)$$

Now it follows from (3) that

$$\begin{aligned}\sum_{v\in V_1} d(v) &\geq \max_{i=2,3}\left\{\sum_{v\in V_i} d(v) - d_i\right\} \\ &\geq \frac{1}{2}\left(\sum_{v\in V_2} d(v) + \sum_{v\in V_3} d(v) - d_2 - d_3\right).\end{aligned}$$

Therefore

$$\begin{aligned}3m &= \sum_{v\in V_1} d(v) + \sum_{v\in V_2} d(v) + \sum_{v\in V_3} d(v) \\ &\geq \frac{3}{2}\left(\sum_{v\in V_2} d(v) + \sum_{v\in V_3} d(v)\right) - \frac{1}{2}(d_2 + d_3)\end{aligned}$$

and so, by (7) and (8),

$$\begin{aligned}2m &\geq \sum_{v\in V_2} d(v) + \sum_{v\in V_3} d(v) - \frac{1}{3}(d_2 + d_3) \\ &\geq \frac{3}{4}(w_2 + w_3) + (\frac{9}{2} - 6c)m - 2 - \frac{1}{3}(d_2 + d_3).\end{aligned}$$

Hence

$$\frac{3}{4}(w_2+w_3) \le (6c-\frac{5}{2})m + \frac{1}{3}(d_2+d_3)+2.$$

Now $w_1 \le cm$, so

$$w_1+w_2+w_3 \le cm + (8c-\frac{10}{3})m + \frac{4}{9}(d_2+d_3)+\frac{8}{3}.$$

It follows from (2) that

$$(9c-\frac{10}{3})m + \frac{4}{9}(d_2+d_3)+\frac{8}{3} \ge \frac{19}{9}m + \frac{2}{9}t,$$

so

$$9cm \ge \frac{49}{9}m - \frac{8}{3} - \frac{4}{9}(d_2+d_3) + \frac{2}{9}t. \tag{9}$$

Now if $d_2+d_3-t \le cm$ then, since $\Delta(G) \le cm$,

$$\frac{4}{9}(d_2+d_3) - \frac{2}{9}t \le \frac{2}{9}cm + \frac{2}{9}(d_2+d_3) \le \frac{2}{3}cm,$$

and so it follows from (9) that

$$9cm \ge \frac{49}{9}m - \frac{8}{3} - \frac{2}{3}cm.$$

Thus

$$\frac{87}{9}cm \ge \frac{49}{9}m - \frac{8}{3}$$

and so

$$cm \ge \frac{49}{87}m - \frac{8}{29},$$

which fails for all $m > 4$.

Otherwise $d_2+d_3-t > cm$. Consider the hypergraph H on $V(G) \setminus \{v_2,v_3\}$ with edge set $\{e \setminus \{v_2,v_3\} : e \in E(G)\}$. It follows from Lemma 3 that there is a bipartition $H_1 \cup H_2$ of $V(H)$ such that, for $i=1,2$,

$$\begin{aligned} d(H_i) &\ge \frac{e_1(H)-1}{3} + \frac{2e_2(H)}{3} + \frac{3e_3(H)}{4} \\ &= \frac{t-1}{3} + \frac{2(d_2+d_3-2t)}{3} + \frac{3(m-d_2-d_3+t)}{4} \\ &= \frac{3m}{4} - \frac{d_2+d_3+3t}{12} - \frac{1}{3}. \end{aligned}$$

If $\min\{d(H_1),d(H_2)\} > cm$ then $\{\{v_2,v_3\},H_1,H_2\}$ is a partition of $V(G)$ in which each class meets more than cm edges. Otherwise

$$\frac{3m}{4} - \frac{d_2+d_3+3t}{12} - \frac{1}{3} \le cm,$$

and so

$$d_2+d_3+3t \ge (9-12c)m - 4,$$

Thus, since $\max\{d_2, d_3\} \le cm$, we have

$$t \ge \left(3 - \frac{14}{3}c\right) m - \frac{4}{3} \tag{10}$$

It follows from (10) and (9) that

$$9cm \ge \frac{49}{9}m - \frac{8}{3} - \frac{4}{9}(d_2 + d_3) + \frac{2}{9}\left(3 - \frac{14}{3}c\right) m - \frac{8}{27}.$$

Since $\max\{d_2, d_3\} \le cm$, we obtain

$$cm \ge \frac{33}{59}m - \frac{16}{59},$$

which fails for all $m > 4$ except $m = 13$. The case $m = 13$ follows by considering the possible values for t, d_2 and d_3 in the argument above. □

In fact, taking $cm = \lfloor(5m - 1)/9\rfloor$ in the proof of Theorem 1 shows that for $m \neq 11, 20, 29, 38$ we can replace $(5m - 1)/9$ by $5m/9$ in (1).

The bound given in Theorem 1 shows that in most cases we can get quite close to the conjecture. For hypergraphs with a large number of edges, however, we believe that it should be possible to do much better. We will return to this at the end of the paper.

3. Partitioning r-uniform hypergraphs

For hypergraphs in general, we cannot get as close to the conjectured $rm/(2r - 1)$ as for 3-uniform hypergraphs. However, we can manage about half of the conjectured bound.

Theorem 2. *Let G be an r-uniform hypergraph with m edges. There is a partition of $V(G)$ into r sets such that each set meets at least $0.27m$ edges.*

We will make use of two lemmas in the proof of Theorem 4.

Lemma 3. *Let $0 < c < 1$ and let G be a hypergraph with maximum degree less than cm. If A and B are disjoint sets of vertices with $\min\{d(A), d(B)\} \ge 2cm$ then there is a partition of $A \cup B$ into three sets, such that two meet at least cm edges and the third meets at least $10cm/9$ edges.*

Proof. We may assume that each edge meets each of A and B in at most one vertex (or else replace it with a smaller edge). Let $A = A_1 \cup A_2 \cup A_3$ be a partition of A into three sets, any two of which meet at least cm edges. Such a partition exists, since we can take A_1 to be a maximal subset of A meeting less than cm edges, A_2 to be a maximal subset of

$A \setminus A_1$ meeting less than cm edges and $A_3 = A_1 \setminus (A_1 \cup A_2)$. Similarly, let $B = B_1 \cup B_2 \cup B_3$ be a partition of B into three sets, any two of which meet at least cm edges.

Now we claim that $A_i \cup B_j$ meets at least $10cm/9$ edges for some i and j. Indeed, if this is not the case then

$$\sum_{i,j=1}^{3} d(A_i \cup B_j) < 10cm.$$

Now since every edge meets each of A and B in at most one vertex, $\sum_{i,j} d(A_i, B_j) = d(A, B) \leq \min\{d(A), d(B)\}$ and so

$$\begin{aligned}\sum_{i,j} d(A_i \cup B_j) &= \sum_{i,j} (d(A_i) + d(B_j) - d(A_i, B_j)) \\ &= 3d(A) + 3d(B) - d(A, B) \\ &\geq 10cm,\end{aligned}$$

which is a contradiction.

Thus $d(A_i \cup B_j) \geq 10cm/9$ for some i and j, say $i = j = 1$. Then $A_1 \cup B_1$, $A_2 \cup A_3$, $B_2 \cup B_3$ gives the required partition of $A \cup B$. □

Lemma 4. *Let $0 < c < 1$, let G be a hypergraph with maximum degree less than cm and suppose A and B are disjoint sets of vertices with $d(A) \geq 3cm$ and $d(A) + 4d(B) > 5cm$. Then there is a partition of $A \cup B$ into two sets, of which one meets at least cm vertices and the other meets at least $2cm$ vertices.*

Proof. If $d(B) \geq cm$ then A and B will do for our sets. Otherwise, we may assume that each edge meets each of A and B at most once. Let $A = A_1 \cup \cdots \cup A_i$ be a partition of A obtained as follows: let $A_1 \subset A$ be a maximal set with $d(A_1) < cm$; let $A_2 \subset A \setminus A_1$ be maximal with $d(A_2) < cm$; and so on. We obtain a partition into i sets, for some $i \geq 4$, such that each sets meets less than cm edges and the union of any two sets meets at least cm edges.

If $i \geq 6$ then $A_1 \cup A_2$, $A_3 \cup A_4$, $A_5 \cup A_6$ each meet at least cm edges, so $A_1 \cup A_2$, $(A \cup B) \setminus (A_1 \cup A_2)$ satisfy the assertion of the lemma, since $d((A \cup B) \setminus (A_1 \cup A_2)) \geq d(A_3 \cup A_4 \cup A_5 \cup A_6) \geq d(A_3 \cup A_4) + d(A_5 \cup A_6) \geq 2cm$.

If $i = 5$ then we claim that $d(A_j \cup B) \geq cm$ for some $j \leq 5$. Indeed, if not then we have

$$\begin{aligned} 5cm &> \sum_{j=1}^{5} d(A_j \cup B) \\ &= \sum_{j=1}^{5} (d(A_j) + d(B) - d(A_j, B)) \\ &= d(A) + 5d(B) - d(A, B) \\ &\geq d(A) + 4d(B), \end{aligned}$$

since $d(A, B) \leq d(B)$, which contradicts the assumption that $d(A) + 4d(B) > 5cm$. Thus $d(A_j \cup B) \geq cm$ for some j. The partition of $A \cup B$ into $A_j \cup B$ and $A \setminus A_j$ satisfies the assertion of the lemma, since $d(A \setminus A_j) = d(A) - d(A_j) \geq 2cm$.

Finally, if $i = 4$, we claim $d(A_j \cup B) \geq cm$ for some $j \leq 4$. If not, then

$$\begin{aligned} 4cm &> \sum_{j=1}^{4} d(A_j \cup B) \\ &= \sum_{j=1}^{4} (d(A_j) + d(B) - d(A_j, B)) \\ &= d(A) + 4d(B) - d(A, B) \\ &\geq d(A) + 3d(B). \end{aligned}$$

Now $d(B) < cm$, so this implies $5cm > d(A) + 4d(B)$, which contradicts the assumptions of the lemma. Thus $d(A_j \cup B) \geq cm$ for some j. Since $d(A_j) < cm$, we have $d(A \setminus A_j) > 2cm$. Therefore the partition of $A \cup B$ into $A_j \cup B$ and $A \setminus A_j$ satisfies the assertion of the lemma. □

We now prove our bound for r-uniform hypergraphs.

Proof of Theorem 4. Let $c = 0.27$ and $c_r = 1 - (1 - \frac{1}{r})^r$, and suppose that G has no partition satisfying the conditions of Theorem 4. We may clearly assume that $\Delta(G) < cm$ and $r \geq 4$. Let $\mathcal{P} = \{V_1, \ldots, V_r\}$ be a random partition of $V(G)$ into r sets. Then

$$\mathbb{E}\left(\sum_{i=1}^{r} d(V_i)\right) = rm(1 - (1 - \frac{1}{r})^r) = rmc_r. \tag{11}$$

We may therefore choose a partition $V_1, \ldots, V_r$ such that $\sum_{i=1}^{r} d(V_i) > rmc_r$.

We begin by picking out pairs of sets that satisfy the conditions of Lemma 6. Let $A_1, B_1, \ldots, A_s, B_s$ be a sequence of maximal length of distinct sets in $\mathcal{P}$ such that $d(A_i) < cm$, $d(B_i) \geq 3cm$ and $d(B_i) + 4d(A_i) \geq 5c$, and let $\mathcal{S} = \{A_1, B_1, \ldots, A_s, B_s\}$. We now partition the remaining sets V_i depending on $d(V_i)$. Define

$$\begin{aligned}
\mathcal{T} &= \{V_i : d(V_i) < cm \text{ and } V_i \notin \mathcal{S}\}\\
\mathcal{U} &= \{V_i : cm \leq d(V_i) < 2cm\}\\
\mathcal{V} &= \{V_i : 2cm \leq d(V_i) < 3cm\}\\
\mathcal{W} &= \{V_i : d(V_i) \geq 3cm \text{ and } V_i \notin \mathcal{S}\}
\end{aligned}$$

We have partitioned $\mathcal{P}$ as $\mathcal{S} \cup \mathcal{T} \cup \mathcal{U} \cup \mathcal{V} \cup \mathcal{W}$. Let $t = |\mathcal{T}|$, etc, so that

$$r = 2s + t + u + v + w. \tag{12}$$

It follows from Lemma 6 that, for $i = 1, \ldots, s$, there is a partition of $A_i \cup B_i$ into one set C_i meeting at least cm edges and one set D_i meeting at least $2cm$ edges. Adding the resulting sets to $\mathcal{U}$ and $\mathcal{V}$, we have disjoint sets $\mathcal{U}' = \mathcal{U} \cup \{C_1, \ldots, C_s\}$ of $u + s$ sets meeting at least cm vertices, $\mathcal{V}' = \mathcal{V} \cup \{D_1, \ldots, D_s\}$ of $v + s$ sets meeting at least $2cm$ vertices and $\mathcal{W}' = \mathcal{W}$ of w sets meeting at least $3cm$ vertices. Dividing $\mathcal{V}'$ into pairs (with at most one set left over), it follows from Lemma 5 that each pair can be split into three sets, each of which meets at least cm edges; also, since $\Delta(G) < cm$, each set in $\mathcal{W}$ can be split into two sets, each meeting at least cm edges. Therefore, we get at least

$$(u + s) + \frac{3}{2}(v + s - 1) + 1 + 2w = u + \frac{5}{2}s + \frac{3}{2}v + 2w - \frac{1}{2} \tag{13}$$

sets meeting at least cm edges. We shall show that this gives at least r sets. Note that, by (11),

$$(1 + c)sm + ctm + 2cum + 3cvm + wm \geq \sum_{i=1}^{r} d(V_i) > rmc_r. \tag{14}$$

Furthermore, if $\mathcal{T}$ is nonempty, then set $c^*m = \max\{d(V_i) : V_i \in \mathcal{T}\}$: any $V_i \in \mathcal{W}$ satisfies $d(V_i) + 4c^*m < 5c$ (since otherwise V_i and some set from $\mathcal{T}$ would be in $\mathcal{S}$), and so $d(V_i) < (5c - 4c^*)m$.

Case 1. $\mathcal{W} = \emptyset$. We have nonnegative s, t, u, v such that

$$2s + t + u + v = r \tag{15}$$

and

$$(1 + c)s + ct + 2cu + 3cv > rc_r \tag{16}$$

and we want to prove

$$u + \frac{5}{2}s + \frac{3}{2}v \geq r + \frac{1}{2}. \tag{17}$$

Suppose this is not the case, so we have

$$u + \frac{5}{2}s + \frac{3}{2}v \leq r. \tag{18}$$

Since $c > \frac{1}{4}$, (15), (16) and (18) are also satisfied by taking $s' = 0$, $t' = t$, $u' = u + s$ and $v' = v + s$. Thus we may assume

$$t + u + v = r \tag{19}$$

$$ct + 2cu + 3cv > rc_r \tag{20}$$

and

$$u + \frac{3}{2}v \leq r. \tag{21}$$

Substituting (19) into (20), gives

$$c(r - u - v) + 2cu + 3cv > rc_r,$$

and so

$$cu + 2cv > r(c_r - c). \tag{22}$$

Subtracting c times (21) from (22) gives

$$\frac{c}{2}v > r(c_r - 2c).$$

But it follows from (21) that $v < 2r/3$, so

$$\frac{c}{2}(\frac{2}{3}r) > r(c_r - 2c),$$

which gives

$$c > \frac{3c_r}{7} > 0.27,$$

which is a contradiction.

Case 2. $\mathcal{W} \neq \emptyset$. Recall that $c^*m = \max\{d(V_i) : V_i \in \mathcal{T}\}$ if $\mathcal{T}$ is nonempty; if $\mathcal{T} = \emptyset$ then set $c^* = 0$. We have nonnegative s, t, u, v, w such that

$$2s + t + u + v + w = r$$

and, since $d(V_i) \leq c^*m$ for $V_i \in \mathcal{T}$ and $d(V_i) \leq (5c - 4c^*)m$ for $V_i \in \mathcal{W}$,

$$(1 + c)s + c^*t + 2cu + 3cv + (5c - 4c^*)w > rc_r,$$

and we want to prove

$$u + \frac{5}{2}s + \frac{3}{2}v + 2w \geq r + \frac{1}{2}.$$

Suppose this is not the case. As before, we may assume $s = 0$, so we have

$$t + u + v + w = r \tag{23}$$

$$u + \frac{3}{2}v + 2w \leq r. \tag{24}$$

and

$$c^*t + 2cu + 3cv + (5c - 4c^*)w > rc_r \tag{25}$$

Now subtracting $2c$ times (24) from (25) gives

$$c^*t + (c - 4c^*)w > r(c_r - 2c). \tag{26}$$

It follows from (24) that $w \leq r/2$. Since $5c - 4c^* \geq 3c$ by definition of $\mathcal{W}$, we have $c^* \leq c/2$; also, from (23) we have $t \leq r - w$, so

$$\begin{aligned} c^*t + (c - 4c^*)w &\leq \frac{c}{2}(r - w) + cw \\ &= \frac{c}{2}r + \frac{c}{2}w \\ &\leq \frac{3c}{4}r. \end{aligned}$$

Substitution into (26) gives

$$\frac{3c}{4} > r(c_r - 2c),$$

so

$$c > \frac{rc_r}{2r + (3/4)} > 0.27,$$

which is a contradiction. □

Note that there is some leeway in Case 2, so the bound 0.27 could be improved by an improvement in Case 1.

4. Open problems.

In this paper we have considered partitions of r-uniform hypergraphs into r classes. It is of interest to ask more generally about partitions into k classes. For graphs we conjecture that for every graph G with m edges and every integer $k \geq 2$ there is a partition of G into k sets, each of which meets at least

$$\frac{2m}{2k - 1}$$

edges. If this is correct then K_{2k-1} shows the constant to be best possible, and may well be the unique extremal graph.

Asymptotically, it seems likely that it should be possible to obtain partitions that are almost as good as partitions of complete graphs.

We conjecture that, for integers $r, k \geq 2$, every r-uniform hypergraph with m edges has a vertex-partition into k sets, each of which meets at least

$$(1 + o(1)) \left(1 - (1 - \frac{1}{r})^k\right)$$

edges.

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