Alon–Seymour–Thomas Revisited

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Abstract

Alon, Seymour, and Thomas [*J. Amer. Math. Soc.* 1990] proved that every K_t-minor-free graph on n vertices has treewidth less than $t^{3/2}n^{1/2}$. We prove the following product structure strengthening of this result: every K_t-minor-free graph on n vertices is a subgraph of $H \boxtimes K_p$ for some graph H with $\operatorname{tw}(H) \leqslant t-1$, where $p \leqslant \sqrt{(t-3)n}$. We also prove the following qualitative strengthening: every K_t-minor-free graph is a subgraph of $H \boxtimes K_p$, where $\operatorname{tw}(H) \leqslant t-2$ and $p = \operatorname{tw}(G)+1$. Similarly, we prove that every $K_{s,t}$-minor-free graph on n vertices is a subgraph of $H \boxtimes K_p$, where $\operatorname{tw}(H) \leqslant s$ and $p \leqslant \min\{2\sqrt{(s-1)(t-1)n}, (t-1)(\operatorname{tw}(G)+1)\}$. In all these results, the values of p are tight up to a multiplicative constant (for fixed s and t), while the bounds on $\operatorname{tw}(H)$ are best possible when p is solely a function of $\operatorname{tw}(G)$.

1 Introduction

In one of the cornerstone results of Graph Minor Theory, Alon, Seymour, and Thomas [1] proved that every K_t-minor-free graph has a balanced separator of size at most $t^{3/2}n^{1/2}$. In fact, they proved the following stronger result.[1]

Theorem 1 ([1])**.** *Every K_t-minor-free graph on n vertices has treewidth less than $t^{3/2}n^{1/2}$.*

We prove a stronger structural result that implies Theorem 1. This paper is part of a body of research called 'Graph Product Structure Theory' which describes complicated graphs as subgraphs of products of simpler graphs; see [2, 4–7, 9, 10]. In this paper we only consider products of the form $H \boxtimes K_p$, which is the 'blow-up' of the graph H, obtained by replacing

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[1]The balanced separator result follows by the separator lemma of Robertson and Seymour [17, (2.6)].

each vertex of H by a copy of the complete graph K_p and each edge of H by a copy of the complete bipartite graph $K_{p,p}$.

Our first result is the following product structure strengthening of Theorem 1.

Theorem 2. *Let $t \geqslant 4$ be an integer and G be an n-vertex K_t-minor-free graph. Then*

(a) *G is isomorphic to a subgraph of $H \boxtimes K_{\lfloor p \rfloor}$, where $\operatorname{tw}(H) \leqslant t-1$ and $p := \sqrt{(t-3)n}$;*

(b) *G is isomorphic to a subgraph of $H \boxtimes K_{\lfloor p \rfloor}$, where $\operatorname{tw}(H) \leqslant t-2$ and $p := 2\sqrt{(t-3)n}$.*

Theorem 2(a) immediately implies Theorem 1, since

$$\operatorname{tw}(G) \leqslant \operatorname{tw}(H \boxtimes K_p) \leqslant (\operatorname{tw}(H)+1)p - 1 \leqslant t\sqrt{(t-3)n}.$$

Theorem 2 describes K_t-minor-free graphs (which can be very complicated, in the sense that they can have arbitrarily large treewidth for fixed $t \geqslant 5$) as blow-ups of much simpler graphs, namely graphs with bounded treewidth. It is not possible to improve the dependence on n in the blow-up factor, p. Indeed, the $n^{1/2} \times n^{1/2}$ planar grid graph G is K_5-minor-free and has treewidth $n^{1/2}$. If G is isomorphic to a subgraph of $H \boxtimes K_p$ where H has bounded treewidth, then $n^{1/2} \leqslant \operatorname{tw}(G) \leqslant (\operatorname{tw}(H)+1)p - 1$ and so $p = \Omega(n^{1/2})$.

While our proof of Theorem 2 uses several ideas from [1], it is in fact significantly simpler, avoiding the use of havens or any form of treewidth duality. Instead, the proof directly constructs an isomorphism from G to $H \boxtimes K_p$ where H is a graph obtained by repeated clique-sums (which implies the desired treewidth bound).

We also prove the following analogous theorem for complete bipartite minors. Let $K^*_{s,t}$ be the graph whose vertex-set can be partitioned A, B, where $|A| = s$ and $|B| = t$, A is a clique, and every vertex in A is adjacent to every vertex in B, that is, $K^*_{s,t}$ is obtained from $K_{s,t}$ by adding all the edges inside the part of size s.

Theorem 3. *For all integers $s, t \geqslant 2$, every n-vertex $K^*_{s,t}$-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_{\lfloor p \rfloor}$, where $\operatorname{tw}(H) \leqslant s$ and $p := 2\sqrt{(s-1)(t-1)n}$.*

Again the $n^{1/2} \times n^{1/2}$ planar grid (which is $K_{3,3}$-minor-free) shows the dependence on n in the blow-up factor is best possible—we must have $p = \Omega(n^{1/2})$.

Given Theorem 1, it is natural to try to bound the blow-up factor in Theorems 2 and 3 by a function of the treewidth of G, and ideally by a linear function of $\operatorname{tw}(G)$ since if $G \subseteq H \boxtimes K_p$ and $\operatorname{tw}(H) = \mathcal{O}(1)$, then $p = \Omega(\operatorname{tw}(G))$. In this direction, Campbell et al. [4, Thm. 18] proved that every K_t-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_p$ where $\operatorname{tw}(H) \leqslant t-2$ and $p = \mathcal{O}_t(\operatorname{tw}(G)^2 \log \operatorname{tw}(G))$. Similarly, they proved [4, Thm. 19] that every $K_{s,t}$-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_p$ where $\operatorname{tw}(H) \leqslant s$ and $p = \mathcal{O}_{s,t}(\operatorname{tw}(G)^2 \log \operatorname{tw}(G))$. Here $\mathcal{O}_{s,t}(\cdot)$ and $\Omega_{s,t}(\cdot)$ hide dependence on s and t.

We achieve a blow-up factor that is linear in $\operatorname{tw}(G)$. For K_t-minor-free graphs, it is also independent of t.

Theorem 4. *For any integer $t \geqslant 2$, every K_t-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_p$, where $\operatorname{tw}(H) \leqslant t-2$ and $p \coloneqq \operatorname{tw}(G)+1$.*

The value of p in Theorem 4 is within a factor $t-1$ of best possible, since

$$\operatorname{tw}(G) \leqslant \operatorname{tw}(H \boxtimes K_p) \leqslant (\operatorname{tw}(H)+1)p-1 \leqslant (t-1)p-1.$$

Furthermore, the $t-2$ bound on the treewidth of H is best possible, since Campbell et al. [4, Thm. 18] proved that, for any function f and for all t, there is a K_t-minor-free graph G that is not a subgraph of $H \boxtimes K_{f(\operatorname{tw}(G))}$ if $\operatorname{tw}(H) \leqslant t-3$.

For $K^*_{s,t}$-minor-free graphs we also obtain a blow-up factor that is linear in $\operatorname{tw}(G)$.

Theorem 5. *For all integers $s, t \geqslant 2$, every $K^*_{s,t}$-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_{\lfloor p \rfloor}$, where $\operatorname{tw}(H) \leqslant s$ and $p \coloneqq (\operatorname{tw}(G)+1)(t-1)$.*

Here the value of p is within a factor $(s+1)(t-1)$ of best possible and the $\operatorname{tw}(H) \leqslant s$ bound is best possible [4, Thm. 19].

Since graphs of Euler genus[2] g contain no $K_{3,2g+3}$-minor, the next result follows from Theorems 3 and 5.

Corollary 6. *For any integer $g \geqslant 0$, every n-vertex graph G of Euler genus g is isomorphic to a subgraph of $H \boxtimes K_{\lfloor p \rfloor}$, where $\operatorname{tw}(H) \leqslant 3$ and*

$$p \coloneqq \min\{4\sqrt{(g+1)n},\, 2(g+1)(\operatorname{tw}(G)+1)\}.$$

When $s = 2$ we can do much better than Theorem 3 and avoid all dependence upon n (since $K^*_{2,t}$ is planar, and thus $K^*_{2,t}$-minor-free graphs have bounded treewidth). Campbell et al. [4] showed that every $K^*_{2,t}$-minor-free graph is isomorphic to a subgraph of $H \boxtimes K_{\mathcal{O}(t^3)}$ where $\operatorname{tw}(H) \leqslant 2$. They state as an open problem whether this $\mathcal{O}(t^3)$ bound can be improved to $\mathcal{O}(t)$. The following result solves this problem. Note that Theorem 7 implies $K^*_{2,t}$-minor-free graphs have treewidth $\mathcal{O}(t)$, a result that has also been proved by Leaf and Seymour [12, (4.4)].

Theorem 7. *For every integer $t \geqslant 2$, every $K^*_{2,t}$-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_{t-1}$, where $\operatorname{tw}(H) \leqslant 2$.*

2 Preliminaries

We consider simple finite undirected graphs G with vertex-set $V(G)$ and edge-set $E(G)$. For each vertex $v \in V(G)$, let $N_G(v) = \{w \in V(G) \colon vw \in E(G)\}$. For $S \subseteq V(G)$, let $N_G(S) = \bigcup\{N_G(v) \setminus S \colon v \in S\}$.

[2]The *Euler genus* of a surface with h handles and c cross-caps is $2h + c$. The *Euler genus* of a graph G is the minimum integer $g \geqslant 0$ such that G embeds in a surface of Euler genus g; see [14] for more about graph embeddings in surfaces.

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. Say G is *H-minor-free* if H is not a minor of G. A *K_h-model* in a graph G consists of pairwise-disjoint vertex-sets $(U_1, \dots, U_h)$ such that, for each i, $G[U_i]$ is connected and, for all distinct i, j, there is an edge between U_i and U_j. Clearly K_h is a minor of a graph G if and only if G contains a K_h-model.

A *tree-decomposition* of a graph G consists of a collection $(B_x : x \in V(T))$ of subsets of $V(G)$, called *bags*, indexed by the nodes of a tree T, such that:

- for each vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty (connected) subtree of T; and
- for each edge $vw \in E(G)$, there is a node $x \in V(T)$ for which $v, w \in B_x$.

The *width* of such a tree-decomposition is $\max\{|B_x| : x \in V(T)\} - 1$. The *treewidth* of a graph G is the minimum width of a tree-decomposition of G. Treewidth is the standard measure of how similar a graph is to a tree. Indeed, a connected graph has treewidth 1 if and only if it is a tree. Treewidth is of fundamental importance in structural and algorithmic graph theory; see [3, 8, 15] for surveys.

We use the following property to prove treewidth upper bounds. A graph G is a *clique-sum* of graphs G_1 and G_2, if for some clique $\{v_1, \dots, v_k\}$ in G_1 and for some clique $\{w_1, \dots, w_k\}$ in G_2, G is obtained from the disjoint union of G_1 and G_2 by identifying v_i and w_i for each i. In this case, it is well known and easily seen that $\operatorname{tw}(G) = \max\{\operatorname{tw}(G_1), \operatorname{tw}(G_2)\}$.

Our proofs use hitting set results which say that (under certain conditions) a collection of connected subgraphs of a graph either has a small 'hitting set' (a small set of vertices that meets every subgraph in the collection) or contains some suitable graphs. We first give such results in terms of the treewidth of G. The following folklore lemma (see [18, (8.7)]) follows from the perfectness of the intersection graph of any set of subtrees of a tree, and the Helly property of subtrees.

Lemma 8. *For any non-negative integer ℓ and any collection $\mathcal{F}$ of subtrees of a tree T, either*:

(*a*) *there are $\ell + 1$ vertex-disjoint trees in $\mathcal{F}$, or*
(*b*) *there is set S of at most ℓ vertices such that $S \cap V(T) \neq \varnothing$ for all $T \in \mathcal{F}$.*

Lemma 8 immediately implies the following result.

Corollary 9. *For any non-negative integer ℓ and any collection $\mathcal{F}$ of connected subgraphs of a graph G, either*:

(*a*) *there are $\ell + 1$ vertex-disjoint graphs in $\mathcal{F}$, or*
(*b*) *there is set S of at most $\ell(\operatorname{tw}(G) + 1)$ vertices such that $S \cap V(F) \neq \varnothing$ for all $F \in \mathcal{F}$.*

While Corollary 9 is useful, the fact that the size of the hitting set depends on ℓ means that it is insufficient to prove Theorems 4 and 5 (only giving $p = \mathcal{O}(\operatorname{tw}(G)^2)$). The following result does not give vertex-disjoint graphs but gives a good upper bound on the size of the hitting set. Given a graph G and $V_1, V_2 \subseteq V(G)$, a set S *separates* V_1 and V_2 if no connected component of $G - S$ contains a vertex of both V_1 and V_2. In particular, if $V_1 \subseteq S$, then S separates V_1 and V_2.

Lemma 10. *For any collection $\mathcal{F}$ of connected subgraphs of a graph G, either*:

(*a*) *there exists $F_1, F_2 \in \mathcal{F}$ and a set S of at most $\operatorname{tw}(G)$ vertices that separates $V(F_1)$ and $V(F_2)$, or*

(*b*) *there is set S of at most $\operatorname{tw}(G)+1$ vertices such that $S \cap V(F) \neq \varnothing$ for all $F \in \mathcal{F}$.*

Proof. Let $(B_x : x \in V(T))$ be a tree-decomposition of G with bags of size at most $\operatorname{tw}(G)+1$. We may assume that $B_x \neq B_y$ for each edge $xy \in E(T)$, otherwise contract xy. For each $F \in \mathcal{F}$, let $T_F \coloneqq T[\{x \in V(T) : B_x \cap V(F) \neq \varnothing\}]$. Since F is connected, T_F is a subtree of T. First suppose there exist $F_1, F_2 \in \mathcal{F}$ such that T_{F_1} and T_{F_2} are vertex-disjoint. Let xy be any edge on the shortest path from T_{F_1} to T_{F_2}. Then $S \coloneqq B_x \cap B_y$ separates F_1 and F_2. Since $B_x \neq B_y$, we have $|S| \leqslant |B_x| - 1 \leqslant \operatorname{tw}(G)$. Otherwise, the T_F pairwise intersect. By the Helly property of subtrees, there is a node x in every T_F. Then $S \coloneqq B_x$ has size at most $\operatorname{tw}(G)+1$ and meets every $F \in \mathcal{F}$. □

Lemma 10 has the following immediate corollary: if G contains pairwise disjoint trees $T_1, \dots, T_k$ such that, for all distinct i, j, there are ℓ pairwise disjoint paths between T_i and T_j, then $\operatorname{tw}(G) \geqslant \min\{k-1, \ell\}$. Reed and Wood [16, Lem. 3.2] proved the following converse to this. For any positive integers k and ℓ, if a graph G has treewidth at least $k\ell - 1$, then G contains k pairwise disjoint paths $P_1, \dots, P_k$ such that for all distinct i, j there are ℓ pairwise disjoint paths between P_i and P_j. These two results provide a rough characterisation of treewidth that is of independent interest.

In the setting $p = \mathcal{O}(\sqrt{n})$ we need a different hitting set result. The following lemma is due to Alon et al. [1]. Let $\mathcal{F}$ be the collection of connected subgraphs of G that intersect all of $A_1, \dots, A_k$. Lemma 11 says that $\mathcal{F}$ either contains a small graph or has a small hitting set.

Lemma 11 ([1, (1.2)])**.** *Let G be a graph, $A_1, \dots, A_k$ be non-empty subsets of $V(G)$, and $r \geqslant 1$ be a real. Then either*:

(*a*) *there is a tree X in G with $|V(X)| \leqslant r$ such that $V(X) \cap A_i \neq \varnothing$ for each i, or*

(*b*) *there is a set Y of at most $(k-1)|V(G)|/r$ vertices such that no component of $G - Y$ intersects all of $A_1, \dots, A_k$.*

The next result is a straightforward extension of Lemma 11.

Lemma 12. *Let G be a graph, $A_1, \dots, A_k$ be non-empty subsets of $V(G)$, $r \geqslant 1$ be a real, and $\ell \geqslant 1$ be an integer. Then either*:

(*a*) *there are pairwise disjoint trees* $X_1, \dots, X_\ell$ *in* G *with* $|V(X_j)| \leqslant r$ *and such that* $V(X_j) \cap A_i \neq \varnothing$ *for each* i *and* j, *or*

(*b*) *there is a set* Y *of at most* $(\ell-1)r + (k-1)|V(G)|/r$ *vertices such that no component of* $G - Y$ *intersects all of* $A_1, \dots, A_k$.

Proof. We proceed by induction on ℓ. Lemma 11 proves the result if $\ell = 1$. Now assume that $\ell \geqslant 2$ and the result holds for $\ell - 1$. If outcome (b) holds for $\ell - 1$, then the same set Y satisfies outcome (b) for ℓ. So assume that (a) holds for $\ell - 1$. That is, there are pairwise disjoint trees $X_1, \dots, X_{\ell-1}$ in G with $|V(X_j)| \leqslant r$ and such that $V(X_j) \cap A_i \neq \varnothing$ for each i and j. Apply Lemma 11 to $G' \coloneqq G - V(X_1 \cup \dots \cup X_{\ell-1})$. If there is a tree X_ℓ in G' with $|V(X_\ell)| \leqslant r$ such that $V(X_\ell) \cap A_i \neq \varnothing$ for each i, then $X_1, \dots, X_\ell$ are the desired set of trees, and outcome (a) holds. Otherwise there exists $Y' \subseteq V(G')$ with $|Y'| \leqslant (k-1)|V(G)|/r$ such that no component of $G' - Y'$ intersects all of $A_1, \dots, A_k$. Let $Y \coloneqq V(X_1 \cup \dots \cup X_{\ell-1}) \cup Y'$. Thus $|Y| \leqslant (\ell-1)r + (k-1)|V(G)|/r$ and no component of $G - Y$ intersects all of $A_1, \dots, A_k$ (since $G' - Y' = G - Y$). That is, Y satisfies (b). $\square$

The *strong product* $A \boxtimes B$ of graphs A and B has vertex-set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y)$ are adjacent if $v = w$ and $xy \in E(B)$, or $x = y$ and $vw \in E(A)$, or $vw \in E(A)$ and $xy \in E(B)$.

Instead of working with products, it is convenient to present our proofs using the following definition. A *partition* of a graph G is a graph H such that:

- each vertex of H is a set of vertices of G,
- each vertex of G is in exactly one vertex of H, and
- for each edge vw of G, if $v \in X \in V(H)$ and $w \in Y \in V(H)$ then $XY \in E(H)$ or $X = Y$.

A *(k, p)-partition* of a graph G is a partition H such that $\operatorname{tw}(H) \leqslant k$ and $|X| \leqslant p$ for every $X \in V(H)$. The next observation follows from the definitions.

Observation 13. *A graph* G *has a* (k, p)*-partition if and only if* G *is isomorphic to a subgraph of* $H \boxtimes K_{\lfloor p \rfloor}$ *for some graph* H *with treewidth at most* k.

So to prove all our results it suffices to find a suitable partition. The following definition enables inductive proofs. A partition H of a graph G is *rooted* at a K_h-model $(U_1, \dots, U_h)$ in G if $U_1, \dots, U_h$ are vertices of H. Then $U_1, \dots, U_h$ are the vertices of an h-clique in H.

3 Blow-up $\mathcal{O}(\sqrt{n})$

The following definition allows the proofs of Theorems 2, 3 and 7 to be combined. Let $\mathcal{J}_{s,t}$ be the class of graphs G whose vertex-set has a partition $A \cup B$, where $|A| = s$ and $|B| = t$,

A is a clique, every vertex in A is adjacent to every vertex in B, and $G[B]$ is connected. A graph is *$\mathcal{J}_{s,t}$-minor-free* if it contains no graph in $\mathcal{J}_{s,t}$ as a minor.

Theorem 14. *Let s,t,n be positive integers and define*

$$p \coloneqq \begin{cases} \max\{t-1,1\} & \text{if } s=1 \text{ or } 2,\\ \sqrt{(s-2)n} & \text{if } s \geqslant 3 \text{ and } t=1,\\ 2\sqrt{(s-1)(t-1)n} & \text{otherwise.} \end{cases}$$

Then every $\mathcal{J}_{s,t}$-minor-free graph G on n vertices is isomorphic to a subgraph of $H \boxtimes K_{\lfloor p \rfloor}$ for some graph H of treewidth at most s.

Observe that $\mathcal{J}_{t-1,1} = \mathcal{J}_{t-2,2} = \{K_t\}$. So every K_t-minor-free graph is $\mathcal{J}_{t-1,1}$ and $\mathcal{J}_{t-2,2}$-minor-free. Hence Theorem 14 implies Theorem 2. Clearly, $K^*_{s,t}$ is a subgraph of every graph in $\mathcal{J}_{s,t}$ and so every $K^*_{s,t}$-minor-free graph is $\mathcal{J}_{s,t}$-minor-free. Hence, Theorem 14 implies Theorems 3 and 7.

Theorem 14 is implied by Observation 13 and the following lemma.

Lemma 15. *Let s,t,n be positive integers and define p as in Theorem 14. Suppose G is a $\mathcal{J}_{s,t}$-minor-free graph on n vertices and $(U_1,\dots,U_h)$ is a K_h-model in G where $h \leqslant s$ and $|U_i| \leqslant p$ for all i. Then there is an (s,p)-partition of G rooted at $(U_1,\dots,U_h)$.*

Proof. Let $U \coloneqq U_1 \cup \dots \cup U_h$. We proceed by induction on n. If $n \leqslant h+p$, then $(U_1,\dots,U_h,V(G)-U)$ is the desired H-partition with $H = K_{h+1}$, which has treewidth $h \leqslant s$. Now assume that $n > h+p$. Let $A_i \coloneqq N_G(U_i) \setminus U$ for each i.

First suppose that some A_i is empty, say $A_1 = \varnothing$. By induction on n, $G-U_1$ has an (s,p)-partition H_1 rooted at $(U_2,\dots,U_h)$. Add a new part U_1 adjacent to each of $U_2,\dots,U_h$ to get the desired H-partition of G. The neighbourhood of U_1 is a clique on $h-1$ vertices, so $\operatorname{tw}(H) = \max\{\operatorname{tw}(H_1), h-1\} \leqslant s$. Thus we may assume that A_i is non-empty for all i.

Next suppose that $G-U$ is disconnected. Then there is a partition $U \cup V_1 \cup V_2$ of $V(G)$ into three non-empty vertex-sets such that there is no edge from V_1 to V_2. Let $G_1 = G[U \cup V_1]$ and $G_2 = G[U \cup V_2]$. By induction, each G_i has an (s,p)-partition H_i rooted at $(U_1,\dots,U_h)$. Let H be the partition of G obtained from H_1 and H_2 by identifying the vertex U_j in H_1 with the vertex U_j in H_2 (for each $j \in \{1,\dots,h\}$). The graph H is a clique-sum of H_1 and H_2, so $\operatorname{tw}(H) = \max\{\operatorname{tw}(H_1), \operatorname{tw}(H_2)\} \leqslant s$. Thus we may assume that $G-U$ is connected.

If $h \leqslant s-1$ and there is some U_{h+1} of size at most p such that $(U_1,\dots,U_{h+1})$ is a K_{h+1}-model in G, then Lemma 15 for $U_1,\dots,U_{h+1}$ would imply it is also true for $U_1,\dots,U_h$ (with the same partition). In particular, if $h \leqslant s-1$, then we may assume there is no U_{h+1} of size at most p such that $(U_1,\dots,U_{h+1})$ is a K_{h+1}-model in G. Call this property the 'maximality of h'.

We now show that there is some vertex-set $Y \subseteq V(G) - U$ satisfying

$$|Y| \leqslant p \text{ and no component of } G - U - Y \text{ meets every } A_i. \tag{$\dagger$}$$

First suppose that $s = 1$ and so $U = U_1$. Suppose that $|A_1| \geqslant t$. Let $v_1, \ldots, v_t$ be distinct vertices in A_1. As $G - U$ is connected, it is possible to partition $V(G) - U$ into vertex-sets $Q_1, \ldots, Q_t$ such that for all i, $v_i \in Q_i$ and $G[Q_i]$ is connected. Now contract each Q_i into a single vertex q_i and U_1 into a single vertex u_1 to get a graph G' on vertex-set $\{u_1, q_1, \ldots, q_t\}$. As $G - U$ is connected, $G'[\{q_1, \ldots, q_t\}]$ is connected and so $G' \in \mathcal{J}_{1,t}$, a contradiction. Hence $|A_1| \leqslant t - 1 \leqslant p$. Then $Y = A_1$ satisfies ($\dagger$).

Next suppose that $s = 2$. If $h = 1$, then for any $x \in A_1$, the pair $(U_1, \{x\})$ is a K_2-model in G, which contradicts the maximality of h. Hence $h = 2$ and $U = U_1 \cup U_2$. Suppose $G - U$ contains t pairwise vertex-disjoint paths $P_1, \ldots, P_t$ from A_1 to A_2. As $G - U$ is connected, there is a partition of $V(G) - U$ into vertex-sets $Q_1, \ldots, Q_t$ such that, for all i, $V(P_i) \subseteq Q_i$ and $G[Q_i]$ is connected. Now contract each Q_i to a single vertex q_i and each U_i to a single vertex u_i to get a graph G' on vertex-set $\{u_1, u_2, q_1, \ldots, q_t\}$. As $G - U$ is connected, $G'[\{q_1, \ldots, q_t\}]$ is connected and so $G' \in \mathcal{J}_{2,t}$, a contradiction. Thus, by Menger's theorem, there is a set $Y \subseteq V(G) - U$ of at most $t - 1 \leqslant p$ vertices such that there is no path from A_1 to A_2 in $G - U - Y$. In particular, no component of $G - U - Y$ meets both A_1 and A_2 and so Y satisfies ($\dagger$). Thus we may assume that $s \geqslant 3$.

Suppose that $h \leqslant s - 1$. Apply Lemma 11 to $G - U$ with $r = \sqrt{(s-2)n} \geqslant 1$ and $k = h$. If (a) occurs, then there is a tree T on at most $r \leqslant p$ vertices intersecting each A_i. Then $(U_1, \ldots, U_h, T)$ is a K_{h+1}-model in G with all parts of size at most p, which contradicts the maximality of h. Hence, (b) occurs. That is, there is a vertex-set Y of size at most $(h-1)n/r \leqslant (s-2)n/r = r \leqslant p$ such that no component of $G - U - Y$ meets every A_i. This Y satisfies ($\dagger$).

Now assume that $h = s$. For $t = 1$ we are done: since $G - U$ is connected, contracting each of $U_1, \ldots, U_s, G - U$ to a single vertex gives a K_{s+1}-minor in G, which is a contradiction as $K_{s+1} \in \mathcal{J}_{s,1}$. Thus $t \geqslant 2$. Apply Lemma 12 to $G - U$ with $\ell = t$, $k = h = s$ and $r = \sqrt{\frac{s-1}{t-1}n} \geqslant 1$. Suppose (a) occurs. Then there are pairwise disjoint trees $T_1, \ldots, T_t$ in $G - U$ such that each T_j meets each A_i. As $G - U$ is connected, it is possible to partition $V(G) - U$ into vertex-sets $Q_1, \ldots, Q_t$ such that, for all i, $V(T_i) \subseteq Q_i$ and $G[Q_i]$ is connected. Now contract each Q_i to a single vertex q_i and each U_i to a single vertex u_i to get a graph G' on vertex-set $\{u_1, \ldots, u_s, q_1, \ldots, q_t\}$. As $G - U$ is connected, $G'[\{q_1, \ldots, q_t\}]$ is connected and so $G' \in \mathcal{J}_{s,t}$, a contradiction. Hence, (b) occurs: there is a vertex-set Y of size at most $(t-1)r + (s-1)n/r = p$ such that no component of $G - U - Y$ meets every A_i. This Y satisfies ($\dagger$).

We have shown in all cases that there is some vertex-set $Y \subseteq V(G) - U$ satisfying ($\dagger$). Take a minimal such Y and let $G_1, \ldots, G_q$ be the components of $G - U - Y$. Fix a G_j and let Y_j be the set of vertices $w \in Y$ that have a neighbour in G_j. By ($\dagger$), there is an i' such that $A_{i'} \cap V(G_j) = \varnothing$. Since $G - U$ is connected and $A_{i'}$ is non-empty, both Y and Y_j are non-empty. We claim that for each $w \in Y_j$ there is a path P_w from w to $A_{i'}$ that avoids $U \cup V(G_j)$. By the minimality of Y, some component Q of $G - U - (Y \setminus \{w\})$ meets every A_i. Since Y satisfies (†), w is a cut-vertex of Q. Also w has a neighbour in G_j, so G_j is a subgraph of Q and, furthermore, G_j is a component of $Q - w$. Since Q meets every A_l, there is a path P_w from w to $A_{i'}$ inside Q. But $V(G_j)$ does not meet $A_{i'}$ and G_j is a component of $Q - w$, so P_w avoids G_j. Also P_w is in Q, so P_w avoids U. Hence, P_w has the required properties. Let Z_j be the subgraph induced by the union of $U_{i'}$ and all P_w (where $w \in Y_j$). By construction, Z_j is connected and disjoint from $V(G_j) \cup (U \setminus U_{i'})$.

Take the subgraph of G induced by $V(G_j) \cup Z_j \cup U$ and contract Z_j to a new vertex z_j. Call the graph obtained G'_j: this has vertex-set $V(G_j) \cup (U \setminus U_{i'}) \cup \{z_j\}$. Now $\{z_j\}$ and the U_i (for $i \neq i'$) are a K_h-model in G'_j. By induction, G'_j has an (s,p)-partition H_j rooted at $(\{z_j\}, U_i \colon i \neq i')$. Add to H_j the vertex $U_{i'}$ adjacent to all other U_i and to $\{z_j\}$. Since the neighbourhood of this added vertex is a clique of order $h \leqslant s$, H_j still has treewidth at most s. Let H be obtained from the disjoint union of $H_1, \dots, H_q$, where corresponding U_i are identified and the vertices $z_1, \dots, z_q$ from $H_1, \dots, H_q$ are identified and replaced by Y. The graph H is a clique-sum of $H_1, \dots, H_q$, so $\operatorname{tw}(H) \leqslant \max_j \operatorname{tw}(H_j) \leqslant s$. Each vertex of each H_j has size at most p and $|Y| \leqslant p$, so each vertex of H has size at most p.

We finally check that H is a partition of G. The vertices $U_1, \dots, U_h, Y$ form a clique in H so all edges of G inside $Y \cup U$ appear in H. Every edge inside G_j appears in $G'_j - z_j$ so appears in H_j and hence H. Any edge between U and G_j is, by definition of i', an edge between G_j and $U \setminus U_{i'}$ so appears in $G'_j - z_j$ and hence in H. Finally consider edges between Y and G_j. Let vw be an edge with $v \in V(G_j)$ and $w \in Y$. By definition, $w \in Y_j$ and so the edge vz_j is present in G'_j and hence in H_j. Since z_j is replaced by Y, the edge vw is in H. □

4 Blow-up $\mathcal{O}(\operatorname{tw}(G))$

This section proves Theorems 4 and 5 from Section 1 via the following combined result.

Theorem 16. *For all integers $s, t \geqslant 2$, every $\mathcal{J}_{s,t}$-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_{\lfloor p \rfloor}$, where $\operatorname{tw}(H) \leqslant s$ and $p \coloneqq (\operatorname{tw}(G) + 1)(t - 1)$.*

Theorem 16 implies Theorems 4 and 5 since $\mathcal{J}_{t-2,2} = \{K_t\}$ and $K^*_{s,t}$ is a subgraph of every graph in $\mathcal{J}_{s,t}$.

Theorem 16 follows from the next lemma.

Lemma 17. *Let $s, t \geqslant 2$ be integers, G be a $\mathcal{J}_{s,t}$-minor-free graph and p be defined as in Theorem 16. Suppose that $(U_1, \dots, U_h)$ is a K_h-model in G where $h \leqslant s$ and $|U_i| \leqslant p$ for each i. Then there is a (s,p)-partition of G rooted at $(U_1, \dots, U_h)$.*

Proof. Let $U \coloneqq U_1 \cup \cdots \cup U_h$. We proceed by induction on $n = |V(G)|$. If $n \leqslant h + p$, then the partition $(U_1, \ldots, U_h, V(G) - U)$ is the desired H-partition with $H = K_{h+1}$, which has treewidth $h \leqslant s$. Now assume that $n > h + p$. Let $A_i \coloneqq N_G(U_i) \setminus U$ for each i.

By the same argument used in the proof of Lemma 15, we may assume that A_i is non-empty for all i, and that $G - U$ is connected.

We now show that there is some vertex-set $Y \subseteq V(G) - U$ satisfying

$$|Y| \leqslant p \text{ and no component of } G - U - Y \text{ meets every } A_i. \tag{‡}$$

Let $\mathcal{F}$ be the collection of all subgraphs F of $G - U$ such that F is connected and $V(F) \cap A_i \neq \varnothing$ for all i.

First consider the case $h \leqslant s - 1$. If there is some $F \in \mathcal{F}$ with $|V(F)| \leqslant p$, then $(U_1, \ldots, U_h, V(F))$ is a K_{h+1}-model in G with all parts having size at most p. Then Lemma 17 for $U_1, \ldots, U_h, V(F)$ would imply it is also true for $U_1, \ldots, U_h$ (with the same partition). In particular, if $h \leqslant s-1$, then we may assume that every $F \in \mathcal{F}$ has $|V(F)| > p$.

Apply Lemma 10 to the collection $\mathcal{F}$ and graph $G - U$.

First suppose there exists $F_1, F_2 \in \mathcal{F}$ and a set S of at most $\operatorname{tw}(G - U) \leqslant \operatorname{tw}(G) \leqslant p$ vertices that separates $V(F_1)$ and $V(F_2)$. Take a minimal such S. Then there is a partition $S \cup V_1 \cup V_2$ of $V(G) - U$ such that $V(F_1) \setminus S \subseteq V_1$, $V(F_2) \setminus S \subseteq V_2$, and there is no edge from V_1 to V_2. Since $|V(F_1)|, |V(F_2)| > p \geqslant |S|$, both V_1 and V_2 are non-empty. We now show that $G[S \cup V_1]$ and $G[S \cup V_2]$ are connected. Consider some $s \in S$. Since S is minimal, either $s \in V(F_1)$ or there is a path from s to $V(F_1) \setminus S$ that is internally disjoint from $S \cup V(F_2)$. Since there is no edge between V_1 and V_2, this path must lie entirely inside $S \cup V_1$. Since F_1 is connected, between any two vertices of S there is a path entirely inside $S \cup V_1$. Since $G - U$ is connected, there is a path from any vertex of V_1 to S inside $S \cup V_1$. Hence $G[S \cup V_1]$ is connected. Similarly for $G[S \cup V_2]$. Let G_1 be the graph obtained from G by contracting all of $S \cup V_1$ to a single vertex v_1 and let G_2 be the graph obtained from G by contracting all of $S \cup V_2$ to a single vertex v_2. For each j, G_j is a minor of G and so is $\mathcal{J}_{s,t}$-minor-free. Furthermore, as $V(F_j) \subseteq S \cup V_j$, $(U_1, \ldots, U_h, \{v_j\})$ is a K_{h+1}-model in G_j. Hence, by induction, each G_j has an (s, p)-partition H_j rooted at $(U_1, \ldots, U_h, \{v_j\})$. Let H be obtained from the disjoint union of H_1 and H_2 where the corresponding U_i are identified and the vertices v_1 and v_2 from H_1 and H_2 are identified and replaced by S. The graph H is a clique-sum of H_1 and H_2, so $\operatorname{tw}(H) \leqslant \max\{\operatorname{tw}(H_1), \operatorname{tw}(H_2)\} \leqslant s$. Each vertex of each H_j has size at most p and $|S| \leqslant p$, so each vertex of H has size at most p.

Otherwise (when we apply Lemma 10), there is a set Y of at most $\operatorname{tw}(G - U) + 1 \leqslant p$ vertices that meets every $F \in \mathcal{F}$. Then $G - U - Y$ contains no graph of $\mathcal{F}$ and so every component of $G - U - Y$ avoids some A_i. This Y satisfies (‡).

Finally consider the case $h = s$. Suppose that $\mathcal{F}$ contains t vertex-disjoint graphs $F_1, \ldots, F_t$. Since $G-U$ is connected, it is possible to partition $V(G)-U$ into vertex-sets $Q_1, \ldots, Q_t$ such that, for all i, $V(F_i) \subseteq Q_i$ and $G[Q_i]$ is connected. Now contract each Q_i to single vertex q_i and each U_i to a single vertex u_i to get a graph G' on vertex-set $\{u_1, \dots, u_s, q_1, \dots, q_t\}$. Since $G - U$ is connected, $G'[\{q_1, \dots, q_t\}]$ is connected and so $G' \in \mathcal{J}_{s,t}$, a contradiction. Hence, there are not t vertex-disjoint graphs in $\mathcal{F}$. Thus, by Corollary 9 applied to $\mathcal{F}$ and $G - U$, there is a vertex-set $Y \subseteq V(G) - U$ of size at most $(\text{tw}(G) + 1)(t - 1) = p$ such that $G - U - Y$ contains no graph of $\mathcal{F}$. This Y satisfies (‡).

We have shown in all cases that there is some vertex-set $Y \subseteq V(G) - U$ satisfying (‡). We may now finish exactly as in the proof of Lemma 15. $\square$

5 Concluding Remarks

We conclude the paper by discussing some possible ways in which Theorem 2 might be strengthened. Similar questions can be asked for $K_{s,t}$-minor-free graphs. Consider the following meta-theorem:

> Every K_t-minor-free graph G is isomorphic to a subgraph of $H \boxtimes K_{p(G)}$ for some function p and some graph H of treewidth at most $f(t)$. $(\star)$

Note that Theorem 2 says that $(\star)$ holds for $p(G) = 2\sqrt{(t-3)n}$ where $n := |V(G)|$ and $f(t) = t - 2$ while Theorem 4 says it holds for $p(G) = \text{tw}(G) + 1$ and $f(t) = t - 2$.

Q1. Is it possible to improve $f(t)$ in Theorem 2 (possibly sacrificing some dependence on t in $p(G)$)? In particular, can $(\star)$ be proved with $p(G) = \mathcal{O}_t(n^{1/2})$ and $f(t) = c$ for some constant c independent of t? It follows from a result of Linial, Matoušek, Sheffet, and Tardos [13] that even for planar graphs, $c \geqslant 2$. On the other hand, $(\star)$ holds with H a star ($c = 1$) and $p(G) = \mathcal{O}_t(n^{2/3})$, and for any $\varepsilon > 0$ there exists c such that $(\star)$ holds with $f(t) \leqslant c$ and $p(G) = \mathcal{O}_t(n^{1/2+\varepsilon})$; see [19]. The real interest is when $p(G) = \mathcal{O}_t(n^{1/2})$.

As noted in Section 1, there is no corresponding improvement to Theorem 4 as, when $p(G)$ is a function of $\text{tw}(G)$, $f(t) = t - 2$ is best possible.

Q2. Optimising the dependence on t in Theorem 2 is an interesting question. Note that Kawarabayashi and Reed [11] proved that K_t-minor-free graphs have balanced separators of order $\mathcal{O}(t\sqrt{n})$, which is best possible. Does $(\star)$ hold with $f(t) \cdot p(G) = \mathcal{O}(t\sqrt{n})$?

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