Maximising H-Colourings of Graphs

Hannah Guggiari[*], Alex Scott[*†]

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Abstract

For graphs G and H, an H-colouring of G is a map $\psi : V(G) \to V(H)$ such that $ij \in E(G) \Rightarrow \psi(i)\psi(j) \in E(H)$. The number of H-colourings of G is denoted by $\hom(G, H)$.

We prove the following: for all graphs H and $\delta \geq 3$, there is a constant $\kappa(\delta, H)$ such that, if $n \geq \kappa(\delta, H)$, the graph $K_{\delta,n-\delta}$ maximises the number of H-colourings among all connected graphs with n vertices and minimum degree δ. This answers a question of Engbers.

We also disprove a conjecture of Engbers on the graph G that maximises the number of H-colourings when the assumption of the connectivity of G is dropped.

Finally, let H be a graph with maximum degree k. We show that, if H does not contain the complete looped graph on k vertices or $K_{k,k}$ as a component and $\delta \geq \delta_0(H)$, then the following holds: for n sufficiently large, the graph $K_{\delta,n-\delta}$ maximises the number of H-colourings among all graphs on n vertices with minimum degree δ. This partially answers another question of Engbers.

1 Introduction

Let G be a simple, loopless graph and let H be a simple graph, possibly with loops. A *graph homomorphism* from G to H is a map $\psi : V(G) \to V(H)$ such that $ij \in E(G) \Rightarrow \psi(i)\psi(j) \in E(H)$. An *$H$-colouring* of G is a graph homomorphism from G to H. We denote by $\hom(G, H)$ the number of H-colourings of G.

Given a class of graphs $\mathcal{G}$ and a fixed graph H, it is natural to ask which $G \in \mathcal{G}$ maximises $\hom(G, H)$. Various classes of graphs have been considered

[*]Mathematical Institute, University of Oxford, Oxford, OX2 6GG, United Kingdom.
[†]Supported by a Leverhulme Trust Research Fellowship.

(see Cutler [1] for a survey). For instance, a number of authors, such as Galvin [6], have studied the class of all δ-regular graphs for fixed δ; others, including Loh, Pikhurko and Sudakov [7], have investigated the class of all graphs with n vertices and m edges. In this paper, we consider the class of all graphs with minimum degree at least δ. This class was studied by Engbers [4, 5] who raised a number of questions and conjectures. We will answer two of these and provide a partial answer to a third.

In Section 2, we consider the case when $\mathcal{G}$ is the set of all *connected* graphs on n vertices with minimum degree at least δ. For this $\mathcal{G}$ and any non-regular graph H, Engbers [5] showed that, for any fixed $\delta \geq 2$ and n sufficiently large, $\hom(G, H)$ is maximised uniquely by $G = K_{\delta,n-\delta}$. In this paper, we will extend this result by showing that it holds for all $\delta \geq 3$ and for all graphs H. This answers a question posed by Engbers [5]. In the case where $\delta = 2$ and H is any graph, Engbers [4] showed that the number of H-colourings is maximised by one of $K_{2,n-2}$, $\frac{n}{3}K_3$ or $\frac{n}{4}K_{2,2}$ (depending on the structure of H).

An H-colouring of G requires that each component of G is mapped to a component of H. As we are only considering connected graphs G, each H-colouring of G maps G to a single component of H. We therefore begin with the case when H is connected.

Theorem 1.1. *For every $\delta \geq 3$ and every connected graph H, there exists a constant $\kappa(\delta, H)$ such that the following holds: if $n \geq \kappa(\delta, H)$ and G is a connected graph on n vertices with minimum degree at least δ, then we have* $\hom(G, H) \leq \hom(K_{\delta,n-\delta}, H)$. *Further, if H is not a complete looped graph or a complete balanced bipartite graph, we have equality if and only if $G = K_{\delta,n-\delta}$.*

Extending this result to all graphs H follows as an easy corollary. If H has h components $H_1, \dots H_h$, then $\hom(G, H) = \hom(G, H_1) + \cdots + \hom(G, H_h)$ because G is a connected graph. For n sufficiently large, $G = K_{\delta,n-\delta}$ maximises $\hom(G, H_i)$ for each component H_i and so $G = K_{\delta,n-\delta}$ also maximises $\hom(G, H)$.

Corollary 1.2. *For every $\delta \geq 3$ and every graph H, there exists a constant $\kappa(\delta, H)$ such that the following holds: if $n \geq \kappa(\delta, H)$ and G is a connected graph on n vertices with minimum degree at least δ, then we have* $\hom(G, H) \leq \hom(K_{\delta,n-\delta}, H)$. *Further, if H has a component which is neither a complete looped graph nor a complete balanced bipartite graph, we have equality if and only if $G = K_{\delta,n-\delta}$.*

We may identify a proper q-colouring of a graph G with a graph homomorphism from G into K_q. Therefore, counting the number of proper q-colourings of G corresponds to counting the number of proper graph homomorphisms from G into K_q. As K_q is a connected graph, the following corollary also follows immediately from Theorem 1.1. This answers another question posed by Engbers [5].

Corollary 1.3. *Fix $\delta \geq 3$ and $q > 2$. Then, for n sufficiently large, $K_{\delta,n-\delta}$ uniquely maximizes the number of proper q-colourings amongst all connected graphs on n vertices with minimum degree at least δ.*

A natural extension to Corollary 1.2 is to allow G to have more than one component. Here the picture is less complete.

If H is the graph consisting of a single edge with one of the vertices looped, then counting the number of H-colourings of a graph G is equivalent to counting the number of independent sets in G. Extending previous work on this topic, Cutler and Radcliffe [2] gave complete results for all values of n and δ. In particular, if $n \geq 2\delta$, then $K_{\delta,n-\delta}$ is the unique graph which maximises $\hom(G, H)$.

Galvin [6] conjectured that, for any H, if G was a δ-regular graph on n vertices, then $\hom(G, H) \leq \max\{\hom(K_{\delta,\delta}, H)^{n/2\delta}, \hom(K_{\delta+1}, H)^{n/(\delta+1)}\}$. If this were true, it would mean that, whenever $2\delta(\delta+1)|n$, the δ-regular graph on n vertices which maximises the number of H-colourings is either $\frac{n}{2\delta}K_{\delta,\delta}$ or $\frac{n}{\delta+1}K_{\delta+1}$. Galvin's conjecture was shown to be false by Sernau [8]. He produced an infinite family of counterexamples as follows: fix δ and any simple loopless graph H with no $(\delta+1)$-clique. Take any connected δ-regular graph G on $n < 2\delta$ vertices with $\hom(G, H) > 0$. He proved that there existed $k \in \mathbb{N}$ such that $\hom(G, kH) > \max\{\hom(K_{\delta+1}, kH)^{n/(\delta+1)}, \hom(K_{\delta,\delta}, kH)^{n/2\delta}\}$ and hence that Galvin's conjecture was false.

Engbers [4] considered a similar question to Galvin but only when the order of G was sufficiently large. He asked which graph on n vertices with minimum degree δ maximises the number of H-colourings as the value of n increases.

For general H and $\delta = 1$ or $\delta = 2$, Engbers showed that $\hom(G, H)$ is maximised by one of $\frac{n}{\delta+1}K_{\delta+1}$, $\frac{n}{2\delta}K_{\delta,\delta}$ or $K_{\delta,n-\delta}$ (where the graph that maximises $\hom(G, H)$ depends on the structure of H). These results led him to make the following conjecture.

Conjecture 1.4 [4]**.** *Fix $\delta \geq 1$ and any graph H. Let G be a graph on n vertices with minimum degree at least δ. There exists a constant $c(\delta, H)$ such that, for $n \geq c(\delta, H)$, we have*

$$\hom(G, H) \leq \max\left\{\hom(K_{\delta+1}, H)^{\frac{n}{\delta+1}}, \hom(K_{\delta,\delta}, H)^{\frac{n}{2\delta}}, \hom(K_{\delta,n-\delta}, H)\right\}.$$

In Section 3, we will use similar ideas to Sernau to construct counterexamples to Conjecture 1.4 whenever $\delta \geq 3$.

On the other hand, we can show that Conjecture 1.4 does hold in certain circumstances. In Section 4, we will consider the case when the graph H is fixed and δ and n are sufficiently large. In particular, for each $k \in \mathbb{N}$, we consider the family $\mathcal{H}_k$ of all graphs with maximum degree k that do not contain the complete looped graph on k vertices or $K_{k,k}$ as a component. We will prove the following theorem.

Theorem 1.5. *Fix any $k \in \mathbb{N}$. For every graph $H \in \mathcal{H}_k$ and every $\delta \geq \delta_0(H)$, the following holds: there exists a constant $n_0(\delta, H)$ such that, if $n \geq n_0(\delta, H)$ and G is a graph on n vertices with minimum degree δ, then* $\hom(G, H) \leq \hom(K_{\delta,n-\delta}, H)$. *Equality holds if and only if $G = K_{\delta,n-\delta}$.*

The graph $K_{\delta,n-\delta}$ need not maximise the number of H-colourings if H has maximum degree k and contains either the complete looped graph on k vertices or $K_{k,k}$ as a component (i.e. $H \notin \mathcal{H}_k$). This is discussed in more detail in Section 5.

Convention. Throughout this paper, G will be a simple graph without loops. We will adopt the same convention for vertex degrees as Engbers [5]: for any vertex $v \in V(H)$, we define $d(v) = |\{w \in V(H) : vw \in E(H)\}|$. In particular, adding a loop to a vertex in H increases the degree by one.

2 Proof of Theorem 1.1

The following definition was introduced by Engbers [4]. We will use it in the proof of Theorem 1.1 as well as in Section 4.

Definition. For any graph H with maximum degree k and $\delta \geq 1$, we define $S(\delta, H)$ to be the set of vectors in $V(H)^\delta$ such that the elements of the vector have k neighbours in common. We define $s(\delta, H) = |S(\delta, H)|$. As H has at least one vertex of degree k, we have $s(\delta, H) \geq 1$.

We will need the following theorem of Erdős and Pósa.

Theorem 2.1 [3]**.** *There is a function $f : \mathbb{N} \to \mathbb{R}$ such that, given any $d \in \mathbb{N}$, every graph contains either d disjoint cycles or a set of at most $f(d)$ vertices meeting all its cycles.*

We will frequently use the following lemma of Engbers.

Lemma 2.2 [4]**.** *Suppose H is not the complete looped graph on k vertices or $K_{k,k}$. Then, for any two vertices i, j of H and for $r \geq 4$, there are at most $(k^2-1)k^{r-4}$ H-colourings of P_r that map the initial vertex of that path to i and the terminal vertex to j.*

We will also need the following simple observation.

Proposition 2.3. *Let G and H be graphs with G connected and $X \subseteq V(G)$. Suppose the vertices of X have already been mapped to vertices of H. The remaining vertices of G can be mapped into $V(H)$ in such a way that there are at most $\Delta(H)$ choices for each vertex of $V(G)\backslash X$.*

Proof. Because G is connected, there is a path from each vertex of $V(G)\backslash X$ to X. We order the vertices of $V(G)\backslash X$ by increasing distance from X. Each vertex $v \in V(G)\backslash X$ either has a neighbour in X or a neighbour before it in the ordering. Therefore, when we come to colour v, one of its neighbours has already been coloured so there are at most $\Delta(H)$ choices for v. □

Proof of Theorem 1.1. Let $\delta \geq 3$ be fixed and let H be a connected graph with maximum degree $k \in \mathbb{N}$. We have $|V(H)| \geq k$. There are two special cases to look at before we consider a general H.

1. *H is the complete looped graph on k vertices.*
   If G is any graph on n vertices, we find that $\hom(G,H) = k^n$ because any vertex of G can be mapped to any vertex of H. Hence, as any graph on n vertices with minimum degree δ maximises the number of H-colourings, we have $\hom(G,H) \leq \hom(K_{\delta,n-\delta},H)$ as required.

2. *$H = K_{k,k}$.*
   H is bipartite so $\hom(G,H) \neq 0$ if and only if G is bipartite. For any connected bipartite graph G on n vertices, $\hom(G,H) = 2k^n$. This means that any connected bipartite graph on n vertices with minimum degree δ maximises the number of H-colourings and hence $\hom(G,H) \leq \hom(K_{\delta,n-\delta},H)$ as required.

As the theorem is true in these two cases, we may assume that H is not the complete looped graph on k vertices or $K_{k,k}$. We may also assume that $k \geq 2$ as we have already dealt with the cases when H is a single looped vertex and when $H = K_{1,1}$. Hence we may apply Lemma 2.2 when required.

Let G be a graph on n vertices with minimum degree δ that has the maximum number of H-colourings. We know that H has at least one vertex v of degree k. When considering H-colourings of $K_{\delta,n-\delta}$, we can map the vertex class of size δ to v and the other vertex class to the neighbours of v. Hence, $\hom(K_{\delta,n-\delta},H) \geq k^{n-\delta}$.

We will proceed to determine the structure of G. The assumption that G has most H-colourings tells us that $\hom(G, H) \geq \hom(K_{\delta,n-\delta}, H)$. We will show that, for n sufficiently large, we must have $G = K_{\delta,n-\delta}$.

Claim 1: G has a bounded number of disjoint cycles.
Suppose that G has d disjoint cycles. We colour G in the following way. Pick any vertex of G and map it to any vertex of H. Take a shortest path from the starting vertex to a vertex on one of the disjoint cycles. There are at most k ways to map each vertex on this path to vertices of H. We then consider the other vertices on the cycle (as the end vertex of the path has already been mapped to a vertex of H). Lemma 2.2 gives at most $(k^2 - 1)k^{t-3}$ ways to map these vertices to H, where t is the number of vertices in the cycle. We then repeat this process of finding a shortest path from the already mapped vertices to one of the disjoint cycles and mapping the vertices in the path and cycle to H. Once all of the vertices in disjoint cycles have been considered, any remaining vertices can be mapped greedily with at most k choices for each by Proposition 2.3. Therefore

$$\hom(G, H) \leq |V(H)|(k^2 - 1)^d k^{n-2d-1} < |V(H)|k^{n-1}e^{-\frac{d}{k^2}}.$$

This is strictly smaller than $k^{n-\delta}$ whenever $d > k^2 \log |V(H)| + k^2(\delta - 1) \log k$. As $\hom(G, H)$ is maximal, it follows that G has bounded number of disjoint cycles. This bound only depends on H and δ. Hence we have proved the claim.

Applying Theorem 2.1 to G, we find that there exists a constant $\alpha = \alpha(\delta, H)$ such that G can be made acyclic by removing at most α vertices. We can therefore partition the vertices of G into a set A of size at most α and a set F such that $G[F]$ is a forest.

We will show that we can make F into an independent set by moving at most a constant number of vertices from F to A. This constant depends only on δ and H and not on the number of vertices in G.

We say that a component of a graph is *non-trivial* if it contains at least one edge.

Claim 2: The forest F has a bounded number of non-trivial components.
Suppose F has a non-trivial components, $G_1, \dots G_a$. Each G_i is a tree and so contains a maximal path P_i. As every vertex in G has degree at least $\delta \geq 3$, each end-vertex of P_i must have a neighbour in A. We colour G in the following way. First map A into H. There are at most $|V(H)|^{|A|}$ ways to do this. We then consider each G_i in turn. By Lemma 2.2, there are at most

$(k^2-1)k^{|P_i|-2}$ ways to colour P_i and at most k ways to colour each of the other vertices of G_i. Finally, we consider the remaining vertices of G, each of which has at most k possible choices by Proposition 2.3. Hence

$$\hom(G,H) \leq |V(H)|^{|A|}(k^2-1)^a k^{n-|A|-2a} < |V(H)|^{\alpha} k^{n-\alpha} e^{-\frac{a}{k^2}}.$$

This is strictly less than $k^{n-\delta}$ whenever $a > k^2\alpha \log|V(H)| + k^2(\delta-\alpha)\log k$. The maximality of $\hom(G,H)$ means that there exists a constant depending only on δ and H that bounds the number of non-trivial components of F and hence proves the claim.

Let T be any non-trivial component of F. Define T' to be the subtree obtained from T by deleting all of the leaves. We will show that the size of T' is bounded by a constant that only depends on δ and H. This is done in two steps: first we show that the maximal length of a path in T is bounded and then we show that T' can only have a bounded number of leaves. Together, these two claims bound the size of T'.

Claim 3: The length of the longest path in T is bounded.
Suppose the longest path P in T is $u_1v_1u_2v_2\ldots$ and has length b. We may write $b = 2b' + r$ where $r \in \{0,1\}$. The minimum degree of G is at least $\delta \geq 3$ and T is acyclic. Therefore, each vertex of P has a neighbour which is not on P. Further, every leaf of T must have a neighbour in A.

We colour the vertices of G as follows. First, colour A. Next, we colour the vertices of P using the following algorithm. Initially, $i=1$. The algorithm colours vertices u_i and v_i at step i (and possibly some other vertices of T that do not lie on P).

At the i^{th} step, consider vertices u_i and v_i on P. If $i=1$, u_i is an end-vertex of P and so has a neighbour in A; if $i \neq 1$, u_i has v_{i-1} as a neighbour. Hence, we know u_i is adjacent to a vertex which has already been coloured. Consider the vertex v_i. If v_i has a neighbour in A, we have a path of length 4 starting and ending at vertices which have already been coloured. Lemma 2.2 tells us there are at most k^2-1 choices for u_i and v_i (see Figure 1). If v_i does not have a neighbour in A, it must have another neighbour in T which does not lie on P. Take a maximal path Q_i in T, which starts at v_i and avoids P. The end-vertex of Q_i that is not v_i must be a leaf in T and hence has a neighbour in A (see Figure 1). We therefore have a path of length $|Q_i|+3$ which starts and ends with vertices that have already been coloured and has $u_i \cup Q_i$ as the internal vertices. Lemma 2.2 gives at most $(k^2-1)k^{|Q_i|-1}$ ways to colour the path $u_i \cup Q_i$. We then proceed to the $(i+1)^{\text{th}}$ step of the algorithm.

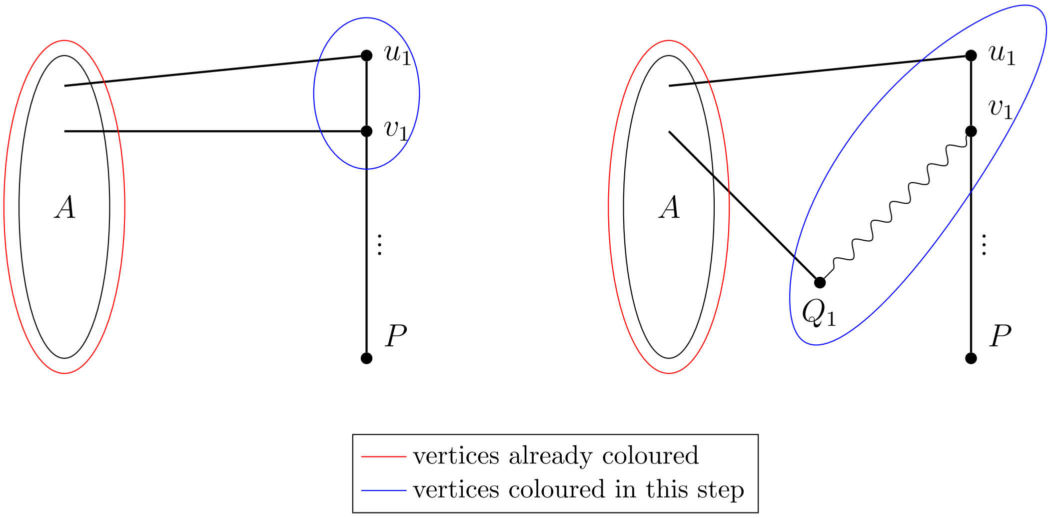

Figure 1: On the left, v_1 has a neighbour in A; on the right, v_1 does not.

After b' steps, we have coloured $2b'$ vertices of P (and possibly some other vertices of T). We finish by colouring all of the remaining vertices of G, each of which has at most k choices by Proposition 2.3. Therefore

$$\hom(G, H) \leq |V(H)|^{|A|}(k^2 - 1)^{b'} k^{n-|A|-2b'} < |V(H)|^{\alpha} k^{n-\alpha} e^{-\frac{b'}{k^2}}.$$

This is strictly less than $k^{n-\delta}$ whenever $b' > k^2 \alpha \log |V(H)| + k^2(\delta - \alpha) \log k$. Because $\hom(G, H)$ is maximal, there exists a constant depending only on δ and H which bounds the length of a maximal path in any non-trivial component of F as required.

Claim 4: T' has a bounded number of leaves.
Suppose T' has l leaves. Each leaf of T' has at least two neighbours which are not in T' because the minimum degree of G is at least $\delta \geq 3$. At least one of these neighbours is a leaf of T. Similarly, every leaf of T has a neighbour in A.

We colour G by first colouring the vertices of A. For each leaf v of T', there are two possibilities. If v has two neighbours u and w which are leaves of T, there is a path of length 5 with end vertices in A and internal vertices u, v and w. By Lemma 2.2 there are at most $(k^2 - 1)k$ ways to colour the path uvw. If v only has one neighbour u which is a leaf of T, then v must also have a neighbour in A because it has at least δ neighbours and only one of these can be in T' (see Figure 2). Apply Lemma 2.2 to the path with end vertices in A and internal vertices u and v. There are at most $k^2 - 1$ choices for the colours of u and v.

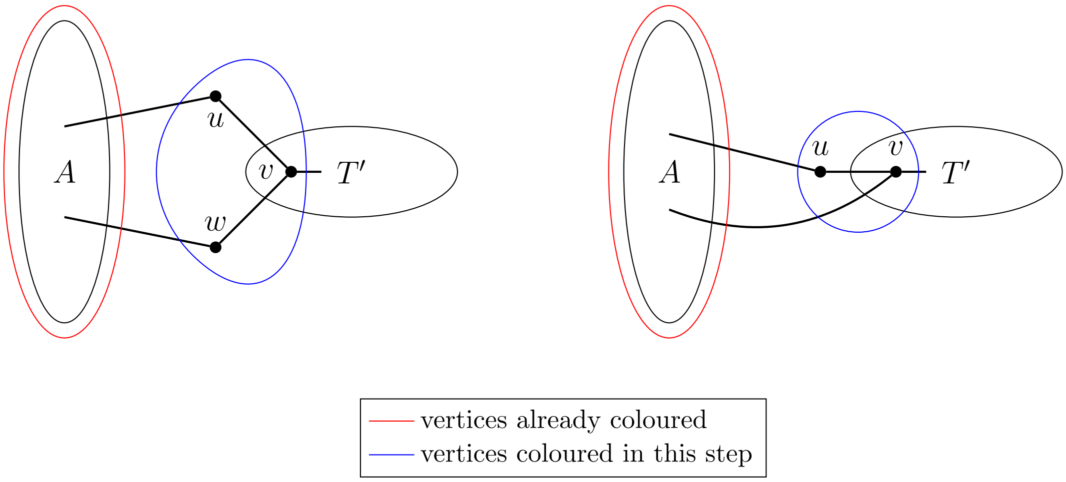

Figure 2: On the left, v has two leaves as neighbours; on the right, v has one.

Once each leaf of T' has been assigned to a vertex of H, there are at most k choices for each of the remaining vertices of G by Proposition 2.3. Therefore

$$\hom(G,H) \leq |V(H)|^{|A|}(k^2-1)^l k^{n-|A|-2l} < |V(H)|^{\alpha} k^{n-\alpha} e^{-\frac{l}{k^2}}.$$

This is strictly less than $k^{n-\delta}$ whenever $l > k^2\alpha \log|V(H)| + k^2(\delta-\alpha)\log k$. The maximality of $\hom(G,H)$ means that the maximum number of leaves T' can have is bounded above by a constant depending only on δ and H as required.

Claims 3 and 4 show that, for each non-trivial component T of F, the subtree T' consisting of T without its leaves has maximal size bounded by a constant $t(\delta,H)$. Claim 2 shows that there are at most $a(\delta,H)$ non-trivial components of F for some constant $a(\delta,H)$.

We can make F into an independent set by moving some (possibly all) of the vertices of each T' from F to A. If any non-trivial component has $T'=\emptyset$, then T is a single edge and in this case we just move one of the end vertices from F to A. Hence, by moving at most $a(\delta,H)t(\delta,H)$ vertices from F to A, we can turn the forest into an independent set.

We have now partitioned the vertices of G into sets of vertices L and R where $|L| \leq \alpha(\delta,H) + a(\delta,H)t(\delta,H)$ and R is an independent set. The size of L is bounded above by a constant that only depends on δ and H; it does not depend on the size of G.

Each vertex in R has at least δ neighbours in L because of the minimum degree of the vertices in G. By the pigeonhole principle, there exists a set $Y \subseteq L$ of size δ such that Y is contained in the neighbourhood of at least $(n - |L|)/\binom{|L|}{\delta} \geq cn$ vertices of R for some constant $c = c(\delta, H)$. Hence, G contains the subgraph $K_{\delta,cn}$.

If G does not contain $K_{\delta,n-\delta}$ as a subgraph, then Y is not a dominating set for G. Therefore, the subgraph induced by $G\backslash Y$ has a non-trivial component. If $G\backslash Y$ contains a non-trivial tree, take a maximal path X in this tree. Otherwise, choose X to be a cycle together with a shortest path from the cycle to Y.

We may colour the vertices of G in such a way that Y is always coloured first. Recall the definition of $S(\delta, H)$ given at the beginning of Section 2.

If Y is coloured using a vector from $S(\delta, H)$, we then colour the vertices of X. There are at most $(k^2 - 1)k^{|X|-2}$ ways do this. Finally, we colour the remaining vertices, each of which has at most k choices by Proposition 2.3. This gives at most $s(\delta, H)(k^2 - 1)k^{n-\delta-2}$ such colourings.

Alternatively, if Y is not coloured using a vector from $S(\delta, H)$, then there are at most $k - 1$ ways to map each of the other cn vertices of the $K_{\delta,cn}$ subgraph into H. There are then at most k choices for each of the remaining vertices of G by Proposition 2.3. There are at most $|V(H)|^\delta(k-1)^{cn}k^{n-\delta-cn}$ such colourings.

Combining the above gives

$$\begin{aligned}
\hom(G, H) &\leq s(\delta, H)(k^2 - 1)k^{n-\delta-2} + |V(H)|^\delta(k-1)^{cn}k^{n-cn-\delta} \\
&= s(\delta, H)k^{n-\delta} - s(\delta, H)k^{n-\delta-2} + |V(H)|^\delta(k-1)^{cn}k^{n-cn-\delta} \\
&< s(\delta, H)k^{n-\delta}
\end{aligned}$$

for sufficiently large values of n.

If G contains $K_{\delta,n-\delta}$ as a subgraph and $G \neq K_{\delta,n-\delta}$, then we know that G contains at least one extra edge between two vertices in the same partition class. Clearly, every mapping of G into H is also a mapping of $K_{\delta,n-\delta}$ into H. We will show below that the converse is not true.

If ij is an edge in H, then mapping the size δ partition class of $K_{\delta,n-\delta}$ to i and the other partition class to j is a proper mapping of $K_{\delta,n-\delta}$ into H. However, it is only a proper mapping of G to H if the partition class containing the extra edge is mapped to a looped vertex. Therefore, if H has a non-looped vertex, $\hom(G, H) < \hom(K_{\delta,n-\delta})$.

Suppose every vertex of H is looped. We assumed that H was connected and not the complete looped graph so there will be non-adjacent vertices j and k which have a common neighbour i. We may map the partition class with the extra edge to vertices j and k and the other partition class to i.

If the extra edge has one endpoint in j and the other in k, we do not get a proper H-colouring of G but it is a valid H-colouring of $K_{\delta,n-\delta}$. Hence $\hom(G,H) < \hom(K_{\delta,n-\delta})$.

Therefore, if $\hom(G,H)$ is maximal and n is sufficiently large, then we must have $G = K_{\delta,n-\delta}$. $\square$

3 Counterexample to Conjecture 1.4

We write $T_t(x)$ for the *t-partite Turán graph on x vertices* (i.e. the complete t-partite graph on x vertices with the vertex classes as equal as possible).

For every $\delta \geq 3$, we will construct a graph H such that, for infinitely many values of n, the number of H-colourings is uniquely maximised by a disjoint union of complete multipartite graphs. This shows that Conjecture 1.4 does not hold. For simplicity, we first assume that $(t-1)|\delta$ for some $3 \leq t \leq \delta$.

Theorem 3.1. *Fix $\delta \geq 3$ and $3 \leq t \leq \delta$ such that $\delta = (t-1)\alpha$ for some $\alpha \in \mathbb{N}$. Then there exists a constant $k_0(\delta)$ such that the following holds for all values of $m \in \mathbb{N}$: if $k \geq k_0(\delta)$ and G is any graph on $n = mt\alpha$ vertices with minimum degree at least δ, then we have $\hom(G, kK_t) \leq \hom(mT_t(t\alpha), kK_t)$ with equality if and only if $G = mT_t(t\alpha)$.*

Proof. Fix $\delta \geq 3$ and $3 \leq t \leq \delta$ as above where $\delta = (t-1)\alpha$. Take k sufficiently large that $(t!k)^{1/(t\alpha)} > tk^{1/(t\alpha+1)}$.

Clearly, $\hom(K_{t+1}, kK_t) = 0$ and so we only need to consider graphs which are K_{t+1}-free.

Any K_{t+1}-free graph with minimum degree at least δ has at least $t\alpha$ vertices. Turán's theorem tells us that $T_t(t\alpha)$ is the only such graph with exactly $t\alpha$ vertices. It is easy to see that $\hom(T_t(t\alpha), kK_t) = t!k$.

Let $m \in \mathbb{N}$ and take G to be any graph on $n = mt\alpha$ vertices with minimum degree at least δ. We may assume that G has a components $G_1, \ldots G_a$ with $|G_1| \geq \cdots \geq |G_a| \geq t\alpha$. Then $\hom(G, kK_t) = \prod_{i=1}^{a} \hom(G_i, kK_t)$. If $|G_1| = t\alpha$, then $|G_i| = t\alpha$ for all i and hence $G = mT_t(t\alpha)$.

Suppose that $|G_1| > t\alpha$. We know that, if $|G_i| = t\alpha$, then $G_i = T_t(t\alpha)$ and $\hom(G_i, kKt) = t!k$. If $|G_i| > t\alpha$, then we may colour the vertices of G_i greedily to get $\hom(G_i, kK_t) \leq tk(t-1)^{|G_i|-1} < kt^{|G_i|}$. We chose k such that $(t!k)^{1/(t\alpha)} > tk^{1/(t\alpha+1)}$. Using this and the fact that $|G_i| \geq t\alpha + 1$, we have $\hom(G_i, kK_t) < (t!k)^{|G_i|/(t\alpha)}$. Combining these two observations, we get

$$\hom(G, kK_t) = \prod_{i=1}^{a} \hom(G_i, kK_t) < (t!k)^{n/(t\alpha)} = (t!k)^m = \hom(mT_t(t\alpha), kK_t).$$

Therefore, if G is any graph on $n = mt\alpha$ vertices with minimum degree at least δ, we have $\hom(G, kK_t) \leq \hom(mT_t(t\alpha), kK_t)$. We have equality if and only if $G = mT_t(t\alpha)$. $\square$

We may use the techniques above to show that, if $(t-1)|(\delta+1)$, then a similar result holds – there is a graph H such that the number of H-colourings is uniquely maximised by a union of complete t-partite graphs. Therefore, for every $\delta \geq 3$, by taking $t = 3$, we can produce a counterexample to Conjecture 1.4.

In all of the examples we have seen so far, the number of H-colourings has been maximised by the union of complete multipartite graphs. We will now give an example where this is not the case.

Take $\delta = 7$ and $t = 4$ and choose k as in Theorem 3.1. Let $H = kK_4$, $m \in \mathbb{N}$ and take G to be any graph on $n = 10m$ vertices with minimum degree at least 7. As before, we may assume that G is 4-colourable. If G has a component with at least 11 vertices, then we can show, in a similar way to Theorem 3.1, that $\hom(G, kK_4) < \hom(mT_4(10), kK_4)$. Any union of complete multipartite graphs except $mT_4(10)$ is either not 4-colourable or contains a component with at least 11 vertices. Therefore, $mT_4(10)$ maximises the number of H-colourings among unions of complete multipartite graphs. However, the number of H-colourings is not maximised overall by $mT_4(10)$. Let T' be the graph formed from $T_4(10)$ by removing a perfect matching between the two vertex classes of size 2. Then $\hom(mT', kK_4) = 2\hom(mT_4(10), kK_4)$.

4 Proof of Theorem 1.5

We will need the following simple observation.

Proposition 4.1. *Fix $d \in \mathbb{N}$. Let G be any graph with minimum degree at least $3d$. Then G has at least d disjoint cycles.*

Proof. If $d = 1$, the minimum degree of G is at least 3 and so G contains a cycle. If $d > 1$, take C to be a shortest cycle in G. Each vertex in G has at most 3 neighbours on C or else we would be able to find a shorter cycle. Removing the vertices in C reduces the minimum degree by at most 3. Therefore, by induction, we can find at least $d - 1$ disjoint cycles in $G \backslash V(C)$. $\square$

Before proving Theorem 1.5, we will prove a couple of useful lemmas. Recall the definitions of $S(\delta, H)$ and $s(\delta, H)$ given at the start of Section 2.

Lemma 4.2. *Fix $\delta \geq 1$ and $k \geq 2$. Fix H to be any graph with maximum degree k. Then there exists a constant $\beta(\delta, H)$ such that, for $n \geq \beta(\delta, H)$, we have* $\hom(K_{\delta,n-\delta}, H) \leq s(\delta, H)k^{n+1-\delta}$.

Proof. The graph $K_{\delta,n-\delta}$ has two vertex classes. Denote the class of size δ by Z. When we are counting the number of H-colourings of $K_{\delta,n-\delta}$, we will colour vertices in Z first and then the remaining vertices may be coloured greedily. There are two possibilities: either Z is coloured so that all of the vertices used in H have k common neighbours (i.e. we use a vector from $S(\delta, H)$) or the vertices in H used to colour Z have strictly fewer than k neighbours in common.

First, we consider the case where Z is coloured using a vector from $S(\delta, H)$. When we come to colour the vertices of $G\setminus Z$, there are exactly k choices for each one. Therefore, there are exactly $s(\delta, H)k^{n-\delta}$ such colourings.

Next, we consider the case where Z is coloured so that the vertices used do not have k common neighbours in H. This leaves at most $k-1$ ways to map the vertices of $G\setminus Z$ into H. Hence, there are at most $|V(H)|^{\delta}(k-1)^{n-\delta}$ such colourings.

Combining the above gives

$$\hom(K_{\delta,n-\delta}, H) \leq s(\delta, H)k^{n-\delta} + |V(H)|^{\delta}(k-1)^{n-\delta}.$$

Hence, for n sufficiently large, we have

$$\begin{aligned}\hom(K_{\delta,n-\delta}, H) &\leq s(\delta, H)k^{n-\delta} + k^{n-\delta} \\ &\leq s(\delta, H)k^{n+1-\delta}.\end{aligned}$$

This proves the required result. $\square$

Lemma 4.3. *Fix H to be any graph with maximum degree $k \in \mathbb{N}$ that does not have the complete looped graph on k vertices or $K_{k,k}$ as a component. There exists a constant $\delta_0(H)$ such that, if $\delta \geq \delta_0(h)$ and G is a connected graph on n vertices with minimum degree δ, then* $\hom(G, H) < k^{n-1}$.

Proof. The minimum degree condition on G ensures that $n \geq \delta + 1$. The restrictions on H mean that $k \geq 2$.

Let H have h components $H_1, \dots H_h$. As G is connected, any H-colouring of G maps G to a single component H_i and so $\hom(G, H) = \sum_{i=1}^{h} \hom(G, H_i)$. We therefore first count the number of H_i-colourings of G for each $i \in [h]$. There are three cases to consider.

Case 1. Let H_i be a complete looped graph on l vertices where $l < k$. Then $\hom(G, H_i) = l^n \leq (k-1)^n$. This is strictly less than k^{n-h-1} whenever $n > \frac{(h+1)\log k}{\log k - \log(k-1)}$.

Case 2. Let $H_i = K_{l,l}$ where $l < k$. Then $\hom(G, H_i) = 2l^n \leq 2(k-1)^n$. This is strictly less than k^{n-h-1} whenever $n > \frac{\log 2 + (h+1)\log k}{\log k - \log(k-1)}$.

Case 3. Let H_i be any connected graph which is not the complete looped graph on l vertices or $K_{l,l}$ for some $l \leq k$. Suppose G has d vertex disjoint cycles $C_1, \ldots C_d$. We colour G in the following way:

1. Pick any vertex of G and map it to any vertex of H_i.
2. Find a shortest path P from the already coloured vertices of G to an uncoloured vertex on one of the cycles C_j. There are at most k ways to map each vertex on this path to vertices of H_i.
3. The end vertex of P has already been mapped to a vertex of H_i so we consider the other vertices on the cycle C_j. Lemma 2.2 gives at most $(k^2-1)k^{|C_j|-3}$ ways to map these vertices to H_i.
4. If, for some $j' \in \{1, \ldots d\}$, the cycle $C_{j'}$ has not yet been coloured, go back to step 2.
5. Colour any remaining uncoloured vertices in a greedy fashion. By Proposition 2.3, there are at most k choices for each vertex.

By colouring G in this way, we find that

$$\hom(G, H_i) \leq |V(H_i)|(k^2-1)^d k^{n-2d-1} < |V(H_i)| k^{n-1} e^{-\frac{d}{k^2}}.$$

This is strictly less than k^{n-h-1} whenever $d > k^2 \log |V(H_i)| + k^2 h \log k$.

Choose $\delta \geq \max\left\{3k^2 \log |V(H)| + 3k^2 h \log k, \frac{(h+1)\log k}{\log k - \log(k-1)}\right\}$ and note that $n \geq \delta + 1$. If H_i is in either Case 1 or Case 2, then n is large enough that $\hom(G, H_i) < k^{n-h-1}$. If H_i is in Case 3, then, by Proposition 4.1, we have that the number of disjoint cycles in G is at least $k^2 \log |V(H)| + k^2 h \log k$ and hence $\hom(G, H_i) < k^{n-h-1}$. Then

$$\hom(G, H) = \sum_{i=1}^{h} \hom(G, H_i) < hk^{n-h-1} < k^{n-1}.$$

Hence, if H does not contain the complete looped graph on k vertices or $K_{k,k}$ as a component, we have $\hom(G, H) < k^{n-1}$ for δ sufficiently large as required. $\square$

We are now ready to prove the main result.

Proof of Theorem 1.5. Let H be any graph with maximum degree k that does not have the complete looped graph on k vertices or $K_{k,k}$ as a component. This allows us to apply Lemma 4.3 as required.

Choose $\delta \geq \delta_0(H)$ where $\delta_0(H)$ is the constant found in Lemma 4.3. Set $\lambda(\delta, H) = \max\{\kappa(\delta, H), \beta(\delta, H)\}$ where $\kappa(\delta, H)$ is the constant found in Theorem 1.1 and $\beta(\delta, H)$ is the constant found in Lemma 4.2. Now, choose $n > (\delta - 1)(\lambda(\delta, H) - 1)$.

Let G be a graph on n vertices with minimum degree δ that has the maximum number of H-colourings. Clearly, $\hom(G, H) \geq \hom(K_{\delta,n-\delta}, H) \geq s(\delta, H)k^{n-\delta} \geq k^{n-\delta}$.

Let G have t components $G_1, \ldots G_t$. An H-colouring of G comprises of separate H-colourings of each component G_i and therefore $\hom(G, H) = \prod_{i=1}^{t} \hom(G_i, H)$. As G has the most H-colourings among all graphs on n vertices with minimum degree δ, we must also have that G_i has the most H-colourings among all graphs on $|G_i|$ vertices with minimum degree δ for each $i \in \{1, \ldots t\}$.

Claim 1: G has a bounded number of components.
By Lemma 4.3, we have that $\hom(G_i, H) < k^{|G_i|-1}$ for each $i \in \{1, \ldots t\}$ so

$$\hom(G, H) = \prod_{i=1}^{t} \hom(G_i, H) < \prod_{i=1}^{t} k^{|G_i|-1} = k^{n-t}.$$

If $t \geq \delta$, then we have $\hom(G, H) < k^{n-\delta} \leq \hom(K_{\delta,n-\delta}, H)$ and this contradicts our assumption that G has the maximum number of H-colourings.

Hence we know that G has at most $\delta - 1$ components. By the pigeonhole principle, there is a component of G with at least $\lambda(\delta, H)$ vertices. Without loss of generality, we may assume this component is G_1. By Theorem 1.1, we have that $G_1 = K_{\delta,|G_1|-\delta}$ and, applying Lemma 4.2, we find that $\hom(G_1, H) \leq s(\delta, H)k^{|G_1|+1-\delta}$.

Claim 2: G has exactly one component.
Suppose $t > 1$. We know $\hom(G_1, H) \leq s(\delta, H)k^{|G_1|+1-\delta}$. By Lemma 4.3, we have $\hom(G_2, H) < k^{|G_2|-1}$. Hence

$$\begin{aligned} \hom(G_1 \cup G_2, H) &< s(\delta, H)k^{|G_1|+1-\delta}k^{|G_2|-1} \\ &= s(\delta, H)k^{|G_1|+|G_2|-\delta} \\ &\leq \hom(K_{\delta,|G_1|+|G_2|-\delta}, H). \end{aligned}$$

Replacing $G_1 \cup G_2$ by $K_{\delta,|G_1|+|G_2|-\delta}$ increases the number of H-colourings of G, which contradicts our assumption that G has the maximum number of H-colourings.

We have seen that G has exactly one component G_1 and that this component is $K_{\delta,|G_1|-\delta}$. In other words, if G has the maximum number of H-colourings, then $G = K_{\delta,n-\delta}$ as required. $\square$

5 Conclusion

We have shown that, given any graph H and any $\delta \geq 3$, for sufficiently large n, the graph $G = K_{\delta,n-\delta}$ maximises $\hom(G, H)$ among all connected graphs on n vertices with minimum degree δ. If H has a component which is neither a complete looped graph nor a complete balanced bipartite graph, then $K_{\delta,n-\delta}$ is the unique such maximising graph.

We have also considered the more general question which was asked by Engbers [5]: what happens if we consider all graphs on n vertices with minimum degree δ, rather than just those which are connected? We will look at the case where H is fixed and $\delta \geq \delta_0(H)$. By making δ sufficiently large in relation to $|H|$, we are able to identify the maximising graph for certain graphs H.

In what follows, we take G to be any graph on n vertices with minimum degree δ. We assume that G has t components $G_1, \dots G_t$.

If H is fixed with maximum degree k and δ is sufficiently large, then the graph which maximises the number of H-colourings depends on the structure of H. Some of the different possible graphs which maximise $\hom(G, H)$ are given below.

1. *H is h disjoint copies of the complete looped graph on k vertices.*
   It is easy to see that $\hom(G, H) = \prod_{i=1}^{t} |V(H)|k^{|G_i|-1} = h^t k^n$. When $h = 1$, $\hom(G, H) = k^n$ for any graph G on n vertices and so every graph G maximises the number of H-colourings. When $h > 1$, $\hom(G, H)$ is maximised when G has as many components as possible. The minimum number of vertices in a component of G is $\delta + 1$ which occurs when the component is $K_{\delta+1}$. Writing $n = a(\delta + 1) + b$ where $b \in \{0, \dots \delta\}$, we have that $\hom(G, H)$ is maximised by any graph with a components, e.g. $(a - 1)K_{\delta+1} \cup K_{\delta+b+1}$.

2. *H is h disjoint copies of $K_{k,k}$.*
   It is easy to see that, if a graph is not bipartite, it is not possible to

map it into H. Therefore

$$\hom(G,H) = \begin{cases} \prod_{i=1}^{t} \hom(G_i,H) = (2h)^t k^n & \text{if } G_i \text{ is bipartite} \\ 0 & \text{if } G_i \text{ is not bipartite.} \end{cases}$$

Clearly, the number of H-colourings is maximised when G is bipartite and has as many components as possible. The smallest possible bipartite component of G is $K_{\delta,\delta}$ which has 2δ vertices. Writing $n = 2a\delta + b$ where $b \in \{0, \dots 2\delta - 1\}$, we have that $\hom(G,H)$ is maximised by any bipartite graph with a components, e.g. $(a-1)K_{\delta,\delta} \cup K_{\delta,\delta+b}$.

3. *No component of H is the complete looped graph on k vertices or $K_{k,k}$.* In Section 4, we showed that, for any $\delta \geq \delta_0(H)$, there exists a constant $n_0(\delta,H)$ such that, if $n \geq n_0(\delta,H)$, then $K_{\delta,n-\delta}$ uniquely maximises the number of H-colourings.

From the examples given above, it is clear to see that there is not a simple answer to the question of which graph G maximises $\hom(G,H)$ when H is fixed and δ is sufficiently large. We make the following conjecture.

Conjecture 5.1. *For any graph H and any $\delta \geq \delta_0(H)$, there exists a constant $n_0(\delta,H)$ such that the following holds: if G is a graph with minimum degree δ and at least $n_0(\delta,H)$ vertices, then*

$$\hom(G,H) \leq \max\left\{ \hom(K_{\delta+1},H)^{\frac{|G|}{\delta+1}}, \hom(K_{\delta,\delta},H)^{\frac{|G|}{2\delta}}, \hom(K_{\delta,|G|-\delta},H) \right\}.$$

This conjecture implies that, for a fixed graph H and δ sufficiently large, the following holds: for sufficiently large n satisfying suitable divisibility conditions, the number of H-colourings is always maximised by one of $\frac{n}{\delta+1}K_{\delta+1}$, $\frac{n}{2\delta}K_{\delta,\delta}$ or $K_{\delta,n-\delta}$.

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