

Maximising H -Colourings of Graphs

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Abstract

For graphs G and H , an H -colouring of G is a map $\psi : V(G) \rightarrow V(H)$ such that $ij \in E(G) \Rightarrow \psi(i)\psi(j) \in E(H)$. The number of H -colourings of G is denoted by $\text{hom}(G, H)$.

We prove the following: for all graphs H and $\delta \geq 3$, there is a constant $\kappa(\delta, H)$ such that, if $n \geq \kappa(\delta, H)$, the graph $K_{\delta, n-\delta}$ maximises the number of H -colourings among all connected graphs with n vertices and minimum degree δ . This answers a question of Engbers.

We also disprove a conjecture of Engbers on the graph G that maximises the number of H -colourings when the assumption of the connectivity of G is dropped.

Finally, let H be a graph with maximum degree k . We show that, if H does not contain the complete looped graph on k vertices or $K_{k,k}$ as a component and $\delta \geq \delta_0(H)$, then the following holds: for n sufficiently large, the graph $K_{\delta, n-\delta}$ maximises the number of H -colourings among all graphs on n vertices with minimum degree δ . This partially answers another question of Engbers.

1 Introduction

Let G be a simple, loopless graph and let H be a simple graph, possibly with loops. A *graph homomorphism* from G to H is a map $\psi : V(G) \rightarrow V(H)$ such that $ij \in E(G) \Rightarrow \psi(i)\psi(j) \in E(H)$. An *H -colouring* of G is a graph homomorphism from G to H . We denote by $\text{hom}(G, H)$ the number of H -colourings of G .

Given a class of graphs \mathcal{G} and a fixed graph H , it is natural to ask which $G \in \mathcal{G}$ maximises $\text{hom}(G, H)$. Various classes of graphs have been considered

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(see Cutler [1] for a survey). For instance, a number of authors, such as Galvin [6], have studied the class of all δ -regular graphs for fixed δ ; others, including Loh, Pikhurko and Sudakov [7], have investigated the class of all graphs with n vertices and m edges. In this paper, we consider the class of all graphs with minimum degree at least δ . This class was studied by Engbers [4, 5] who raised a number of questions and conjectures. We will answer two of these and provide a partial answer to a third.

In Section 2, we consider the case when \mathcal{G} is the set of all *connected* graphs on n vertices with minimum degree at least δ . For this \mathcal{G} and any non-regular graph H , Engbers [5] showed that, for any fixed $\delta \geq 2$ and n sufficiently large, $\text{hom}(G, H)$ is maximised uniquely by $G = K_{\delta, n-\delta}$. In this paper, we will extend this result by showing that it holds for all graphs H . This answers a question posed by Engbers [5].

An H -colouring of G requires that each component of G is mapped to a component of H . As we are only considering connected graphs G , each H -colouring of G maps G to a single component of H . We therefore begin with the case when H is connected.

Theorem 1.1. *For every $\delta \geq 3$ and every connected graph H , there exists a constant $\kappa(\delta, H)$ such that the following holds: if $n \geq \kappa(\delta, H)$ and G is a connected graph on n vertices with minimum degree at least δ , then we have $\text{hom}(G, H) \leq \text{hom}(K_{\delta, n-\delta}, H)$. Further, if H is not a complete looped graph or a complete balanced bipartite graph, we have equality if and only if $G = K_{\delta, n-\delta}$.*

Extending this result to all graphs H follows as an easy corollary. If H has h components H_1, \dots, H_h , then $\text{hom}(G, H) = \text{hom}(G, H_1) + \dots + \text{hom}(G, H_h)$ because G is a connected graph. For n sufficiently large, $G = K_{\delta, n-\delta}$ maximises $\text{hom}(G, H_i)$ for each component H_i and so $G = K_{\delta, n-\delta}$ also maximises $\text{hom}(G, H)$.

Corollary 1.2. *For every $\delta \geq 3$ and every graph H , there exists a constant $\kappa(\delta, H)$ such that the following holds: if $n \geq \kappa(\delta, H)$ and G is a connected graph on n vertices with minimum degree at least δ , then we have $\text{hom}(G, H) \leq \text{hom}(K_{\delta, n-\delta}, H)$. Further, if H has a component which is neither a complete looped graph nor a complete balanced bipartite graph, we have equality if and only if $G = K_{\delta, n-\delta}$.*

We may identify a proper q -colouring of a graph G with a graph homomorphism from G into K_q . Therefore, counting the number of proper q -colourings of G corresponds to counting the number of proper graph homomorphisms

from G into K_q . As K_q is a connected graph, the following corollary also follows immediately from Theorem 1.1. This answers another question posed by Engbers [5].

Corollary 1.3. *Fix $\delta \geq 3$ and $q > 2$. Then, for n sufficiently large, $K_{\delta, n-\delta}$ uniquely maximizes the number of proper q -colourings amongst all connected graphs on n vertices with minimum degree at least δ .*

A natural extension to Corollary 1.2 is to allow G to have more than one component. Here the picture is less complete.

If H is the graph consisting of a single edge with one of the vertices looped, then counting the number of H -colourings of a graph G is equivalent to counting the number of independent sets in G . Extending previous work on this topic, Cutler and Radcliffe [2] gave complete results for all values of n and δ . In particular, if $n \geq 2\delta$, then $K_{\delta, n-\delta}$ is the unique graph which maximises $\text{hom}(G, H)$.

Galvin [6] conjectured that, for any H , if G was a δ -regular graph on n vertices, then $\text{hom}(G, H) \leq \max\{\text{hom}(K_{\delta, \delta}, H)^{n/2\delta}, \text{hom}(K_{\delta+1}, H)^{n/(\delta+1)}\}$. If this were true, it would mean that, whenever $2\delta(\delta+1) \mid n$, the δ -regular graph on n vertices which maximises the number of H -colourings is either $\frac{n}{2\delta}K_{\delta, \delta}$ or $\frac{n}{\delta+1}K_{\delta+1}$. Galvin's conjecture was shown to be false by Sernau [8]. He produced an infinite family of counterexamples as follows: fix δ and any simple loopless graph H with no $(\delta+1)$ -clique. Take any connected δ -regular graph G on $n < 2\delta$ vertices with $\text{hom}(G, H) > 0$. He proved that there existed $k \in \mathbb{N}$ such that $\text{hom}(G, kH) > \max\{\text{hom}(K_{\delta+1}, kH)^{n/(\delta+1)}, \text{hom}(K_{\delta, \delta}, kH)^{n/2\delta}\}$ and hence that Galvin's conjecture was false.

Engbers [4] considered a similar question to Galvin but only when the order of G was sufficiently large. He asked which graph on n vertices with minimum degree δ maximises the number of H -colourings as the value of n increases.

For general H and $\delta = 1$ or $\delta = 2$, Engbers showed that $\text{hom}(G, H)$ is maximised by one of $\frac{n}{\delta+1}K_{\delta+1}$, $\frac{n}{2\delta}K_{\delta, \delta}$ or $K_{\delta, n-\delta}$ (where the graph that maximises $\text{hom}(G, H)$ depends on the structure of H). These results led him to make the following conjecture.

Conjecture 1.4 [4]. *Fix $\delta \geq 1$ and any graph H . Let G be a graph on n vertices with minimum degree at least δ . There exists a constant $c(\delta, H)$ such that, for $n \geq c(\delta, H)$, we have*

$$\text{hom}(G, H) \leq \max \left\{ \text{hom}(K_{\delta+1}, H)^{\frac{n}{\delta+1}}, \text{hom}(K_{\delta, \delta}, H)^{\frac{n}{2\delta}}, \text{hom}(K_{\delta, n-\delta}, H) \right\}.$$

In Section 3, we will use similar ideas to Sernau to construct counterexamples to Conjecture 1.4 whenever $\delta \geq 3$.

On the other hand, we can show that Conjecture 1.4 does hold in certain circumstances. In Section 4, we will consider the case when the graph H is fixed and δ and n are sufficiently large. In particular, for each $k \in \mathbb{N}$, we consider the family \mathcal{H}_k of all graphs with maximum degree k that do not contain the complete looped graph on k vertices or $K_{k,k}$ as a component. We will prove the following theorem.

Theorem 1.5. *Fix any $k \in \mathbb{N}$. For every graph $H \in \mathcal{H}_k$ and every $\delta \geq \delta_0(H)$, the following holds: there exists a constant $n_0(\delta, H)$ such that, if $n \geq n_0(\delta, H)$ and G is a graph on n vertices with minimum degree δ , then $\text{hom}(G, H) \leq \text{hom}(K_{\delta, n-\delta}, H)$. Equality holds if and only if $G = K_{\delta, n-\delta}$.*

The graph $K_{\delta, n-\delta}$ need not maximise the number of H -colourings if H has maximum degree k and contains either the complete looped graph on k vertices or $K_{k,k}$ as a component (i.e. $H \notin \mathcal{H}_k$). This is discussed in more detail in Section 5.

Convention. Throughout this paper, G will be a simple graph without loops. We will adopt the same convention for vertex degrees as Engbers [5]: for any vertex $v \in V(H)$, we define $d(v) = |\{w \in V(H) : vw \in E(H)\}|$. In particular, adding a loop to a vertex in H increases the degree by one.

2 Proof of Theorem 1.1

The following definition was introduced by Engbers [4]. We will use it in the proof of Theorem 1.1 as well as in Section 4.

Definition. For any graph H with maximum degree k and $\delta \geq 1$, we define $S(\delta, H)$ to be the set of vectors in $V(H)^\delta$ such that the elements of the vector have k neighbours in common. We define $s(\delta, H) = |S(\delta, H)|$. As H has at least one vertex of degree k , we have $s(\delta, H) \geq 1$.

We will need the following theorem of Erdős and Pósa.

Theorem 2.1 [3]. *There is a function $f : \mathbb{N} \rightarrow \mathbb{R}$ such that, given any $d \in \mathbb{N}$, every graph contains either d disjoint cycles or a set of at most $f(d)$ vertices meeting all its cycles.*

We will frequently use the following lemma of Engbers.

Lemma 2.2 [4]. *Suppose H is not the complete looped graph on k vertices or $K_{k,k}$. Then, for any two vertices i, j of H and for $r \geq 4$, there are at most $(k^2 - 1)k^{r-4}$ H -colourings of P_r that map the initial vertex of that path to i and the terminal vertex to j .*

We will also need the following simple observation.

Proposition 2.3. *Let G and H be graphs with G connected and $X \subseteq V(G)$. Suppose the vertices of X have already been mapped to vertices of H . The remaining vertices of G can be mapped into $V(H)$ in such a way that there are at most $\Delta(H)$ choices for each vertex of $V(G) \setminus X$.*

Proof. Because G is connected, there is a path from each vertex of $V(G) \setminus X$ to X . We order the vertices of $V(G) \setminus X$ by increasing distance from X . Each vertex $v \in V(G) \setminus X$ either has a neighbour in X or a neighbour before it in the ordering. Therefore, when we come to colour v , one of its neighbours has already been coloured so there are at most $\Delta(H)$ choices for v . \square

Proof of Theorem 1.1. Let $\delta \geq 3$ be fixed and let H be a connected graph with maximum degree $k \in \mathbb{N}$. We have $|V(H)| \geq k$. There are two special cases to look at before we consider a general H .

1. H is the complete looped graph on k vertices.

If G is any graph on n vertices, we find that $\text{hom}(G, H) = k^n$ because any vertex of G can be mapped to any vertex of H . Hence, as any graph on n vertices with minimum degree δ maximises the number of H -colourings, we have $\text{hom}(G, H) \leq \text{hom}(K_{\delta, n-\delta}, H)$ as required.

2. $H = K_{k,k}$.

H is bipartite so $\text{hom}(G, H) \neq 0$ if and only if G is bipartite. For any connected bipartite graph G on n vertices, $\text{hom}(G, H) = 2k^n$. This means that any connected bipartite graph on n vertices with minimum degree δ maximises the number of H -colourings and hence $\text{hom}(G, H) \leq \text{hom}(K_{\delta, n-\delta}, H)$ as required.

As the theorem is true in these two cases, we may assume that H is not the complete looped graph on k vertices or $K_{k,k}$. We may also assume that $k \geq 2$ as we have already dealt with the cases when H is a single looped vertex and when $H = K_{1,1}$. Hence we may apply Lemma 2.2 when required.

Let G be a graph on n vertices with minimum degree δ that has the maximum number of H -colourings. We know that H has at least one vertex v of degree k . When considering H -colourings of $K_{\delta, n-\delta}$, we can map the vertex class of size δ to v and the other vertex class to the neighbours of v . Hence, $\text{hom}(K_{\delta, n-\delta}, H) \geq k^{n-\delta}$.

We will proceed to determine the structure of G . The assumption that G has most H -colourings tells us that $\text{hom}(G, H) \geq \text{hom}(K_{\delta, n-\delta}, H)$. We will show that, for n sufficiently large, we must have $G = K_{\delta, n-\delta}$.

Claim 1: G has a bounded number of disjoint cycles.

Suppose that G has d disjoint cycles. We colour G in the following way. Pick any vertex of G and map it to any vertex of H . Take a shortest path from the starting vertex to a vertex on one of the disjoint cycles. There are at most k ways to map each vertex on this path to vertices of H . We then consider the other vertices on the cycle (as the end vertex of the path has already been mapped to a vertex of H). Lemma 2.2 gives at most $(k^2 - 1)k^{t-3}$ ways to map these vertices to H , where t is the number of vertices in the cycle. We then repeat this process of finding a shortest path from the already mapped vertices to one of the disjoint cycles and mapping the vertices in the path and cycle to H . Once all of the vertices in disjoint cycles have been considered, any remaining vertices can be mapped greedily with at most k choices for each by Proposition 2.3. Therefore

$$\text{hom}(G, H) \leq |V(H)|(k^2 - 1)^d k^{n-2d-1} < |V(H)|k^{n-1}e^{-\frac{d}{k^2}}.$$

This is strictly smaller than $k^{n-\delta}$ whenever $d > k^2 \log |V(H)| + k^2(\delta - 1) \log k$. As $\text{hom}(G, H)$ is maximal, it follows that G has bounded number of disjoint cycles. This bound only depends on H and δ . Hence we have proved the claim.

Applying Theorem 2.1 to G , we find that there exists a constant $\alpha = \alpha(H, \delta)$ such that G can be made acyclic by removing at most α vertices. We can therefore partition the vertices of G into a set A of size at most α and a set F such that $G[F]$ is a forest.

We will show that we can make F into an independent set by moving at most a constant number of vertices from F to A . This constant depends only on δ and H and not on the number of vertices in G .

We say that a component of a graph is *non-trivial* if it contains at least one edge.

Claim 2: The forest F has a bounded number of non-trivial components.

Suppose F has a non-trivial components, G_1, \dots, G_a . Each G_i is a tree and so contains a maximal path P_i . As every vertex in G has degree at least $\delta \geq 3$, each end-vertex of P_i must have a neighbour in A . We colour G in the following way. First map A into H . There are at most $|V(H)|^{|A|}$ ways to do this. We then consider each G_i in turn. By Lemma 2.2, there are at most $(k^2 - 1)k^{|P_i|-2}$ ways to colour P_i and at most k ways to colour each of the other vertices of G_i . Finally, we consider the remaining vertices of G , each of which has at most k possible choices by Proposition 2.3. Hence

$$\text{hom}(G, H) \leq |V(H)|^{|A|}(k^2 - 1)^a k^{n-|A|-2a} < |V(H)|^a k^{n-\alpha} e^{-\frac{\alpha}{k^2}}.$$

This is strictly less than $k^{n-\delta}$ whenever $a > k^2\alpha \log |V(H)| + k^2(\delta - \alpha) \log k$. The maximality of $\text{hom}(G, H)$ means that there exists a constant depending only on δ and H that bounds the number of non-trivial components of F and hence proves the claim.

Let T be any non-trivial component of F . Define T' to be the subtree obtained from T by deleting all of the leaves. We will show that the size of T' is bounded by a constant that only depends on δ and H . This is done in two steps: first we show that the maximal length of a path in T is bounded and then we show that T' can only have a bounded number of leaves. Together, these two claims bound the size of T' .

Claim 3: The length of the longest path in T is bounded.

Suppose the longest path P in T is $u_1v_1u_2v_2\dots$ and has length b . We may write $b = 2b' + r$ where $r \in \{0, 1\}$. The minimum degree of G is at least $\delta \geq 3$ and T is acyclic. Therefore, each vertex of P has a neighbour which is not on P . Further, every leaf of T must have a neighbour in A .

We colour the vertices of G as follows. First, colour A . Next, we colour the vertices of P using the following algorithm. Initially, $i = 1$. The algorithm colours vertices u_i and v_i at step i (and possibly some other vertices of T that do not lie on P).

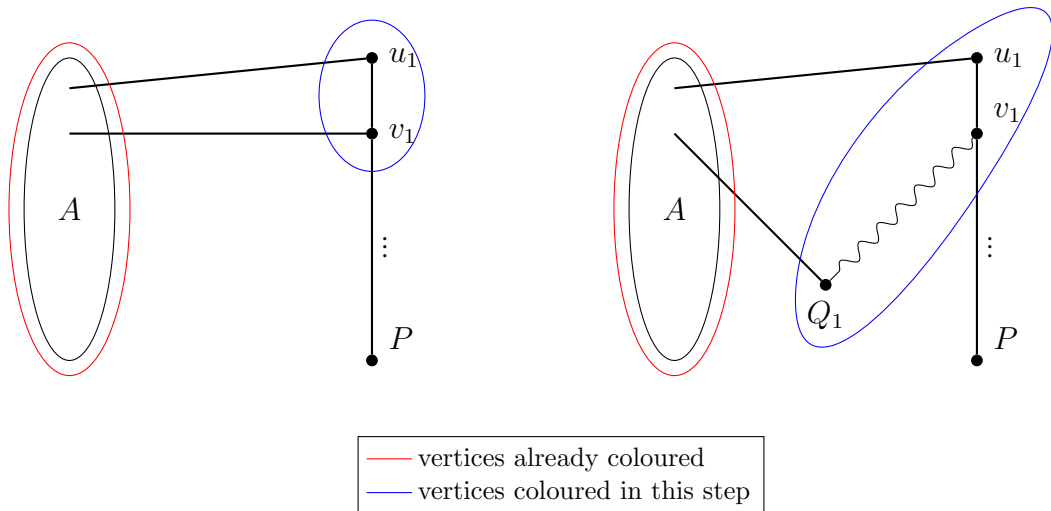


Figure 1: On the left, v_1 has a neighbour in A ; on the right, v_1 does not.

At the i^{th} step, consider vertices u_i and v_i on P . If $i = 1$, u_i is an end-vertex of P and so has a neighbour in A ; if $i \neq 1$, u_i has v_{i-1} as a neighbour.

Hence, we know u_i is adjacent to a vertex which has already been coloured. Consider the vertex v_i . If v_i has a neighbour in A , we have a path of length 4 starting and ending at vertices which have already been coloured. Lemma 2.2 tells us there are at most $k^2 - 1$ choices for u_i and v_i (see Figure 1). If v_i does not have a neighbour in A , it must have another neighbour in T which does not lie on P . Take a maximal path Q_i in T , which starts at v_i and avoids P . The end-vertex of Q_i that is not v_i must be a leaf in T and hence has a neighbour in A (see Figure 1). We therefore have a path of length $|Q_i| + 3$ which starts and ends with vertices that have already been coloured and has $u_i \cup Q_i$ as the internal vertices. Lemma 2.2 gives at most $(k^2 - 1)k^{|Q_i|-1}$ ways to colour the path $u_i \cup Q_i$. We then proceed to the $(i + 1)^{\text{th}}$ step of the algorithm.

After b' steps, we have coloured $2b'$ vertices of P (and possibly some other vertices of T). We finish by colouring all of the remaining vertices of G , each of which has at most k choices by Proposition 2.3. Therefore

$$\text{hom}(G, H) \leq |V(H)|^{|A|} (k^2 - 1)^{b'} k^{n-|A|-2b'} < |V(H)|^\alpha k^{n-\alpha} e^{-\frac{b'}{k^2}}.$$

This is strictly less than $k^{n-\delta}$ whenever $b' > k^2 \alpha \log |V(H)| + k^2 (\delta - \alpha) \log k$. Because $\text{hom}(G, H)$ is maximal, there exists a constant depending only on δ and H which bounds the length of a maximal path in any non-trivial component of F as required.

Claim 4: T' has a bounded number of leaves.

Suppose T' has l leaves. Each leaf of T' has at least two neighbours which are not in T' because the minimum degree of G is at least $\delta \geq 3$. At least one of these neighbours is a leaf of T . Similarly, every leaf of T has a neighbour in A .

We colour G by first colouring the vertices of A . For each leaf v of T' , there are two possibilities. If v has two neighbours u and w which are leaves of T , there is a path of length 5 with end vertices in A and internal vertices u , v and w . By Lemma 2.2 there are at most $(k^2 - 1)k$ ways to colour the path uvw . If v only has one neighbour u which is a leaf of T , then v must also have a neighbour in A because it has at least δ neighbours and only one of these can be in T' (see Figure 2). Apply Lemma 2.2 to the path with end vertices in A and internal vertices u and v . There are at most $k^2 - 1$ choices for the colours of u and v .

Once each leaf of T' has been assigned to a vertex of H , there are at most k choices for each of the remaining vertices of G by Proposition 2.3. Therefore

$$\text{hom}(G, H) \leq |V(H)|^{|A|} (k^2 - 1)^l k^{n-|A|-2l} < |V(H)|^\alpha k^{n-\alpha} e^{-\frac{l}{k^2}}.$$

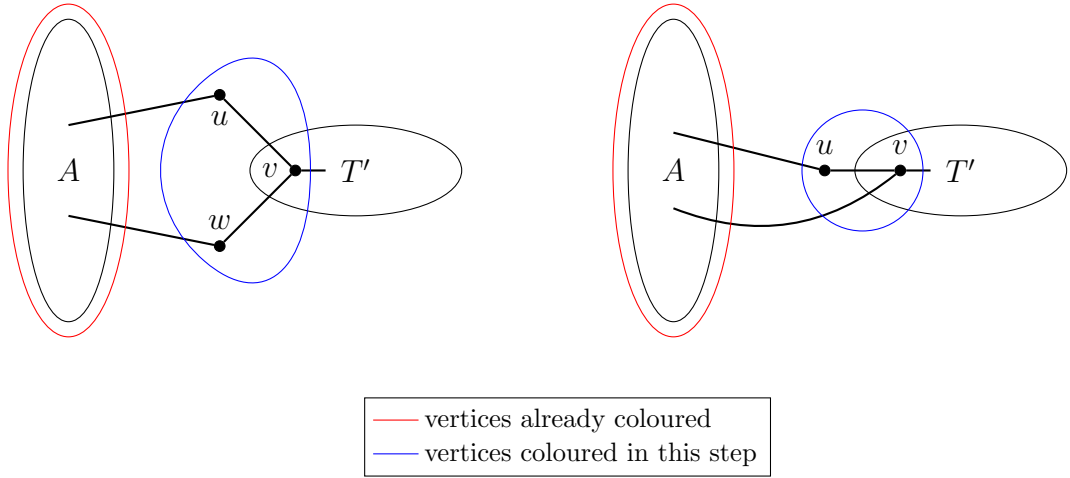


Figure 2: On the left, v has two leaves as neighbours; on the right, v has one.

This is strictly less than $k^{n-\delta}$ whenever $l > k^2\alpha \log |V(H)| + k^2(\delta - \alpha) \log k$. The maximality of $\text{hom}(G, H)$ means that the maximum number of leaves T' can have is bounded above by a constant depending only on δ and H as required.

Claims 3 and 4 show that, for each non-trivial component T of F , the subtree T' consisting of T without its leaves has maximal size bounded by a constant $t(\delta, H)$. Claim 2 shows that there are at most $a(\delta, H)$ non-trivial components of F for some constant $a(\delta, H)$.

We can make F into an independent set by moving some (possibly all) of the vertices of each T' from F to A . If any non-trivial component has $T' = \emptyset$, then T is a single edge and in this case we just move one of the end vertices from F to A . Hence, by moving at most $a(\delta, H)t(\delta, H)$ vertices from F to A , we can turn the forest into an independent set.

We have now partitioned the vertices of G into sets of vertices L and R where $|L| \leq \alpha(H, \delta) + a(\delta, H)t(\delta, H)$ and R is an independent set. The size of L is bounded above by a constant that only depends on δ and H ; it does not depend on the size of G .

Each vertex in R has at least δ neighbours in L because of the minimum degree of the vertices in G . By the pigeonhole principle, there exists a set $Y \subseteq L$ of size δ such that Y is contained in the neighbourhood of at least $(n - |L|) / \binom{|L|}{\delta} \geq cn$ vertices of R for some constant $c = c(\delta, H)$. Hence, G contains the subgraph $K_{\delta, cn}$.

If G does not contain $K_{\delta, n-\delta}$ as a subgraph, then Y is not a dominating set for G . Therefore, the subgraph induced by $G \setminus Y$ has a non-trivial component.

If $G \setminus Y$ contains a non-trivial tree, take a maximal path X in this tree. Otherwise, choose X to be a cycle together with a shortest path from the cycle to Y .

We may colour the vertices of G in such a way that Y is always coloured first. Recall the definition of $S(\delta, H)$ given at the beginning of Section 2.

If Y is coloured using a vector from $S(\delta, H)$, we then colour the vertices of X . There are at most $(k^2 - 1)k^{|X|-2}$ ways to do this. Finally, we colour the remaining vertices, each of which has at most k choices by Proposition 2.3. This gives at most $s(\delta, H)(k^2 - 1)k^{n-\delta-2}$ such colourings.

Alternatively, if Y is not coloured using a vector from $S(\delta, H)$, then there are at most $k - 1$ ways to map each of the other cn vertices of the $K_{\delta, cn}$ subgraph into H . There are then at most k choices for each of the remaining vertices of G by Proposition 2.3. There are at most $|V(H)|^\delta (k - 1)^{cn} k^{n-\delta-cn}$ such colourings.

Combining the above gives

$$\begin{aligned} \text{hom}(G, H) &\leq s(\delta, H)(k^2 - 1)k^{n-\delta-2} + |V(H)|^\delta (k - 1)^{cn} k^{n-cn-\delta} \\ &= s(\delta, H)k^{n-\delta} - s(\delta, H)k^{n-\delta-2} + |V(H)|^\delta (k - 1)^{cn} k^{n-cn-\delta} \\ &< s(\delta, H)k^{n-\delta} \end{aligned}$$

for sufficiently large values of n .

If G contains $K_{\delta, n-\delta}$ as a subgraph and $G \neq K_{\delta, n-\delta}$, then we know that G contains at least one extra edge between two vertices in the same partition class. Clearly, every mapping of G into H is also a mapping of $K_{\delta, n-\delta}$ into H . We will show below that the converse is not true.

If ij is an edge in H , then mapping the size δ partition class of $K_{\delta, n-\delta}$ to i and the other partition class to j is a proper mapping of $K_{\delta, n-\delta}$ into H . However, it is only a proper mapping of G to H if the partition class containing the extra edge is mapped to a looped vertex. Therefore, if H has a non-looped vertex, $\text{hom}(G, H) < \text{hom}(K_{\delta, n-\delta})$.

Suppose every vertex of H is looped. We assumed that H was connected and not the complete looped graph so there will be non-adjacent vertices j and k which have a common neighbour i . We may map the partition class with the extra edge to vertices j and k and the other partition class to i . If the extra edge has one endpoint in j and the other in k , we do not get a proper H -colouring of G but it is a valid H -colouring of $K_{\delta, n-\delta}$. Hence $\text{hom}(G, H) < \text{hom}(K_{\delta, n-\delta})$.

Therefore, if $\text{hom}(G, H)$ is maximal and n is sufficiently large, then we must have $G = K_{\delta, n-\delta}$. \square

3 Counterexample to Conjecture 1.4

We write $T_t(x)$ for the t -partite Turán graph on x vertices (i.e. the complete t -partite graph on x vertices with the vertex classes as equal as possible).

For every $\delta \geq 3$, we will construct a graph H such that, for infinitely many values of n , the number of H -colourings is uniquely maximised by a disjoint union of complete multipartite graphs. This shows that Conjecture 1.4 does not hold. For simplicity, we first assume that $(t-1)|\delta$ for some $3 \leq t \leq \delta$.

Theorem 3.1. *Fix $\delta \geq 3$ and $3 \leq t \leq \delta$ such that $\delta = (t-1)\alpha$ for some $\alpha \in \mathbb{N}$. Then there exists a constant $k_0(\delta)$ such that the following holds for all values of $m \in \mathbb{N}$: if $k \geq k_0(\delta)$ and G is any graph on $n = mt\alpha$ vertices with minimum degree at least δ , then we have $\text{hom}(G, kK_t) \leq \text{hom}(mT_t(t\alpha), kK_t)$ with equality if and only if $G = mT_t(t\alpha)$.*

Proof. Fix $\delta \geq 3$ and $3 \leq t \leq \delta$ as above where $\delta = (t-1)\alpha$. Take k sufficiently large that $(t!k)^{1/(t\alpha)} > tk^{1/(t\alpha+1)}$.

Clearly, $\text{hom}(K_{t+1}, kK_t) = 0$ and so we only need to consider graphs which are K_{t+1} -free.

Any K_{t+1} -free graph with minimum degree at least δ has at least $t\alpha$ vertices. Turán's theorem tells us that $T_t(t\alpha)$ is the only such graph with exactly $t\alpha$ vertices. It is easy to see that $\text{hom}(T_t(t\alpha), kK_t) = t!k$.

Let $m \in \mathbb{N}$ and take G to be any graph on $n = mt\alpha$ vertices with minimum degree at least δ . We may assume that G has a components G_1, \dots, G_a with $|G_1| \geq \dots \geq |G_a| \geq t\alpha$. Then $\text{hom}(G, kK_t) = \prod_{i=1}^a \text{hom}(G_i, kK_t)$. If $|G_1| = t\alpha$, then $|G_i| = t\alpha$ for all i and hence $G = mT_t(t\alpha)$.

Suppose that $|G_1| > t\alpha$. We know that, if $|G_i| = t\alpha$, then $G_i = T_t(t\alpha)$ and $\text{hom}(G_i, kK_t) = t!k$. If $|G_i| > t\alpha$, then we may colour the vertices of G_i greedily to get $\text{hom}(G_i, kK_t) \leq tk(t-1)^{|G_i|-1} < kt^{|G_i|}$. We chose k such that $(t!k)^{1/(t\alpha)} > tk^{1/(t\alpha+1)}$. Using this and the fact that $|G_i| \geq t\alpha + 1$, we have $\text{hom}(G_i, kK_t) < (t!k)^{|G_i|/(t\alpha)}$. Combining these two observations, we get

$$\text{hom}(G, kK_t) = \prod_{i=1}^a \text{hom}(G_i, kK_t) < (t!k)^{n/(t\alpha)} = (t!k)^m = \text{hom}(mT_t(t\alpha), kK_t).$$

Therefore, if G is any graph on $n = mt\alpha$ vertices with minimum degree at least δ , we have $\text{hom}(G, kK_t) \leq \text{hom}(mT_t(t\alpha), kK_t)$. We have equality if and only if $G = mT_t(t\alpha)$. \square

We may use the techniques above to show that, if $(t-1)|(\delta+1)$, then a similar result holds – there is a graph H such that the number of H -colourings is

uniquely maximised by a union of complete t -partite graphs. Therefore, for every $\delta \geq 3$, by taking $t = 3$, we can produce a counterexample to Conjecture 1.4.

In all of the examples we have seen so far, the number of H -colourings has been maximised by the union of complete multipartite graphs. We will now give an example where this is not the case.

Take $\delta = 7$ and $t = 4$ and choose k as in Theorem 3.1. Let $H = kK_4$, $m \in \mathbb{N}$ and take G to be any graph on $n = 10m$ vertices with minimum degree at least 7. As before, we may assume that G is 4-colourable. If G has a component with at least 11 vertices, then we can show, in a similar way to Theorem 3.1, that $\text{hom}(G, kK_4) < \text{hom}(mT_4(10), kK_4)$. However, the number of H -colourings is not maximised by $mT_4(10)$. Let T' be the graph formed from $T_4(10)$ by removing a perfect matching between the two vertex classes of size 2. Then $\text{hom}(mT', kK_4) = 2 \text{hom}(mT_4(10), kK_4)$.

4 Proof of Theorem 1.5

We will need the following simple observation.

Proposition 4.1. *Fix $d \in \mathbb{N}$. Let G be any graph with minimum degree at least $3d$. Then G has at least d disjoint cycles.*

Proof. If $d = 1$, the minimum degree of G is at least 3 and so G contains a cycle. If $d > 1$, take C to be a shortest cycle in G . Each vertex in G has at most 3 neighbours on C or else we would be able to find a shorter cycle. Removing the vertices in C reduces the minimum degree by at most 3. Therefore, by induction, we can find at least $d - 1$ disjoint cycles in $G \setminus V(C)$. \square

Before proving Theorem 1.5, we will prove a couple of useful lemmas. Recall the definitions of $S(\delta, H)$ and $s(\delta, H)$ given at the start of Section 2.

Lemma 4.2. *Fix $\delta \geq 1$ and $k \geq 2$. Fix H to be any graph with maximum degree k . Then there exists a constant $\beta(\delta, H)$ such that, for $n \geq \beta(\delta, H)$, we have $\text{hom}(K_{\delta, n-\delta}, H) \leq s(\delta, H)k^{n+1-\delta}$.*

Proof. The graph $K_{\delta, n-\delta}$ has two vertex classes. Denote the class of size δ by Z . When we are counting the number of H -colourings of $K_{\delta, n-\delta}$, we will colour vertices in Z first and then the remaining vertices may be coloured greedily. There are two possibilities: either Z is coloured so that all of the vertices used in H have k common neighbours (i.e. we use a vector from

$S(\delta, H)$) or the vertices in H used to colour Z have strictly fewer than k neighbours in common.

First, we consider the case where Z is coloured using a vector from $S(\delta, H)$. When we come to colour the vertices of $G \setminus Z$, there are exactly k choices for each one. Therefore, there are exactly $s(\delta, H)k^{n-\delta}$ such colourings.

Next, we consider the case where Z is coloured so that the vertices used do not have k common neighbours in H . This leaves at most $k - 1$ ways to map the vertices of $G \setminus Z$ into H . Hence, there are at most $|V(H)|^\delta (k - 1)^{n-\delta}$ such colourings.

Combining the above gives

$$\text{hom}(K_{\delta, n-\delta}, H) \leq s(\delta, H)k^{n-\delta} + |V(H)|^\delta (k - 1)^{n-\delta}.$$

Hence, for n sufficiently large, we have

$$\begin{aligned} \text{hom}(K_{\delta, n-\delta}, H) &\leq s(\delta, H)k^{n-\delta} + k^{n-\delta} \\ &\leq s(\delta, H)k^{n+1-\delta}. \end{aligned}$$

This proves the required result. \square

Lemma 4.3. *Fix H to be any graph with maximum degree $k \in \mathbb{N}$ that does not have the complete looped graph on k vertices or $K_{k,k}$ as a component. There exists a constant $\delta_0(H)$ such that, if $\delta \geq \delta_0(H)$ and G is a connected graph on n vertices with minimum degree δ , then $\text{hom}(G, H) < k^{n-1}$.*

Proof. The minimum degree condition on G ensures that $n \geq \delta + 1$. The restrictions on H mean that $k \geq 2$.

Let H have h components H_1, \dots, H_h . As G is connected, any H -colouring of G maps G to a single component H_i and so $\text{hom}(G, H) = \sum_{i=1}^h \text{hom}(G, H_i)$. We therefore first count the number of H_i -colourings of G for each $i \in [h]$. There are three cases to consider.

Case 1. Let H_i be a complete looped graph on l vertices where $l < k$. Then $\text{hom}(G, H_i) = l^n \leq (k - 1)^n$. This is strictly less than k^{n-h-1} whenever $n > \frac{(h+1)\log k}{\log k - \log(k-1)}$.

Case 2. Let $H_i = K_{l,l}$ where $l < k$. Then $\text{hom}(G, H_i) = 2l^n \leq 2(k - 1)^n$. This is strictly less than k^{n-h-1} whenever $n > \frac{\log 2 + (h+1)\log k}{\log k - \log(k-1)}$.

Case 3. Let H_i be any connected graph which is not the complete looped graph on l vertices or $K_{l,l}$ for some $l \leq k$. Suppose G has d vertex disjoint cycles C_1, \dots, C_d . We colour G in the following way:

1. Pick any vertex of G and map it to any vertex of H_i .
2. Find a shortest path P from the already coloured vertices of G to an uncoloured vertex on one of the cycles C_j . There are at most k ways to map each vertex on this path to vertices of H_i .
3. The end vertex of P has already been mapped to a vertex of H_i so we consider the other vertices on the cycle C_j . Lemma 2.2 gives at most $(k^2 - 1)k^{|C_j|-3}$ ways to map these vertices to H_i .
4. If, for some $j' \in \{1, \dots, d\}$, the cycle $C_{j'}$ has not yet been coloured, go back to step 2.
5. Colour any remaining uncoloured vertices in a greedy fashion. By Proposition 2.3, there are at most k choices for each vertex.

By colouring G in this way, we find that

$$\text{hom}(G, H_i) \leq |V(H_i)|(k^2 - 1)^d k^{n-2d-1} < |V(H_i)|k^{n-1}e^{-\frac{d}{k^2}}.$$

This is strictly less than k^{n-h-1} whenever $d > k^2 \log |V(H_i)| + k^2 h \log k$.

Choose $\delta \geq \max \left\{ 3k^2 \log |V(H)| + 3k^2 h \log k, \frac{(h+1) \log k}{\log k - \log(k-1)} \right\}$ and note that $n \geq \delta + 1$. If H_i is in either Case 1 or Case 2, then n is large enough that $\text{hom}(G, H_i) < k^{n-h-1}$. If H_i is in Case 3, then, by Proposition 4.1, we have that the number of disjoint cycles in G is at least $k^2 \log |V(H)| + k^2 h \log k$ and hence $\text{hom}(G, H_i) < k^{n-h-1}$. Then

$$\text{hom}(G, H) = \sum_{i=1}^h \text{hom}(G, H_i) < h k^{n-h-1} < k^{n-1}.$$

Hence, if H does not contain the complete looped graph on k vertices or $K_{k,k}$ as a component, we have $\text{hom}(G, H) < k^{n-1}$ for δ sufficiently large as required. \square

We are now ready to prove the main result.

Proof of Theorem 1.5. Let H be any graph with maximum degree k that does not have the complete looped graph on k vertices or $K_{k,k}$ as a component. This allows us to apply Lemma 4.3 as required.

Choose $\delta \geq \delta_0(H)$ where $\delta_0(H)$ is the constant found in Lemma 4.3. Set $\lambda(\delta, H) = \max\{\kappa(\delta, H), \beta(\delta, H)\}$ where $\kappa(\delta, H)$ is the constant found in Theorem 1.1 and $\beta(\delta, H)$ is the constant found in Lemma 4.2. Now, choose $n > (\delta - 1)(\lambda(\delta, H) - 1)$.

Let G be a graph on n vertices with minimum degree δ that has the maximum number of H -colourings. Clearly, $\text{hom}(G, H) \geq \text{hom}(K_{\delta, n-\delta}, H) \geq s(\delta, H)k^{n-\delta} \geq k^{n-\delta}$.

Let G have t components G_1, \dots, G_t . An H -colouring of G comprises of separate H -colourings of each component G_i and therefore $\text{hom}(G, H) = \prod_{i=1}^t \text{hom}(G_i, H)$. As G has the most H -colourings among all graphs on n vertices with minimum degree δ , we must also have that G_i has the most H -colourings among all graphs on $|G_i|$ vertices with minimum degree δ for each $i \in \{1, \dots, t\}$.

Claim 1: G has a bounded number of components.

By Lemma 4.3, we have that $\text{hom}(G_i, H) < k^{|G_i|-1}$ for each $i \in \{1, \dots, t\}$ so

$$\text{hom}(G, H) = \prod_{i=1}^t \text{hom}(G_i, H) < \prod_{i=1}^t k^{|G_i|-1} = k^{n-t}.$$

If $t \geq \delta$, then we have $\text{hom}(G, H) < k^{n-\delta} \leq \text{hom}(K_{\delta, n-\delta}, H)$ and this contradicts our assumption that G has the maximum number of H -colourings.

Hence we know that G has at most $\delta - 1$ components. By the pigeonhole principle, there is a component of G with at least $\lambda(\delta, H)$ vertices. Without loss of generality, we may assume this component is G_1 . By Theorem 1.1, we have that $G_1 = K_{\delta, |G_1|-\delta}$ and, applying Lemma 4.2, we find that $\text{hom}(G_1, H) \leq s(\delta, H)k^{|G_1|+1-\delta}$.

Claim 2: G has exactly one component.

Suppose $t > 1$. We know $\text{hom}(G_1, H) \leq s(\delta, H)k^{|G_1|+1-\delta}$. By Lemma 4.3, we have $\text{hom}(G_2, H) < k^{|G_2|-1}$. Hence

$$\begin{aligned} \text{hom}(G_1 \cup G_2, H) &< s(\delta, H)k^{|G_1|+1-\delta}k^{|G_2|-1} \\ &= s(\delta, H)k^{|G_1|+|G_2|-\delta} \\ &\leq \text{hom}(K_{\delta, |G_1|+|G_2|-\delta}, H). \end{aligned}$$

Replacing $G_1 \cup G_2$ by $K_{\delta, |G_1|+|G_2|-\delta}$ increases the number of H -colourings of G , which contradicts our assumption that G has the maximum number of H -colourings.

We have seen that G has exactly one component G_1 and that this component is $K_{\delta, |G_1|-\delta}$. In other words, if G has the maximum number of H -colourings, then $G = K_{\delta, n-\delta}$ as required. \square

5 Conclusion

We have shown that, given any graph H and any $\delta \geq 3$, for sufficiently large n , the graph $G = K_{\delta, n-\delta}$ maximises $\text{hom}(G, H)$ among all connected graphs on n vertices with minimum degree δ . If H has a component which is neither a complete looped graph nor a complete balanced bipartite graph, then $K_{\delta, n-\delta}$ is the unique such maximising graph.

We have also considered the more general question which was asked by Engbers [5]: what happens if we consider all graphs on n vertices with minimum degree δ , rather than just those which are connected? There are two situations which arise and we will discuss both.

We will first look at the case where H is fixed and $\delta \geq \delta_0(H)$. By making δ sufficiently large in relation to $|H|$, we are able to identify the maximising graph for certain graphs H . We will then consider what happens when δ is fixed and H is allowed to be any graph, which we will refer to as the *general case*.

In what follows, we take G to be any graph on n vertices with minimum degree δ . We assume that G has t components G_1, \dots, G_t .

If H is fixed with maximum degree k and δ is sufficiently large, then the graph which maximises the number of H -colourings depends on the structure of H . Some of the different possible graphs which maximise $\text{hom}(G, H)$ are given below.

1. H is h disjoint copies of the complete looped graph on k vertices.

It is easy to see that $\text{hom}(G, H) = \prod_{i=1}^t |V(H)|k^{|G_i|-1} = h^t k^n$. When $h = 1$, $\text{hom}(G, H) = k^n$ for any graph G on n vertices and so every graph G maximises the number of H -colourings. When $h > 1$, $\text{hom}(G, H)$ is maximised when G has as many components as possible. The minimum number of vertices in a component of G is $\delta + 1$ which occurs when the component is $K_{\delta+1}$. Writing $n = a(\delta + 1) + b$ where $b \in \{0, \dots, \delta\}$, we have that $\text{hom}(G, H)$ is maximised by any graph with a components, e.g. $(a - 1)K_{\delta+1} \cup K_{\delta+b+1}$.

2. H is h disjoint copies of $K_{k,k}$.

It is easy to see that, if a graph is not bipartite, it is not possible to map it into H . Therefore

$$\text{hom}(G, H) = \begin{cases} \prod_{i=1}^t \text{hom}(G_i, H) = (2h)^t k^n & \text{if } G_i \text{ is bipartite} \\ 0 & \text{if } G_i \text{ is not bipartite.} \end{cases}$$

Clearly, the number of H -colourings is maximised when G is bipartite and has as many components as possible. The smallest possible bipartite component of G is $K_{\delta, \delta}$ which has 2δ vertices. Writing $n = 2a\delta + b$

where $b \in \{0, \dots, 2\delta - 1\}$, we have that $\text{hom}(G, H)$ is maximised by any bipartite graph with a components, e.g. $(a - 1)K_{\delta, \delta} \cup K_{\delta, \delta + b}$.

3. *No component of H is the complete looped graph on k vertices or $K_{k, k}$.*
In Section 4, we showed that, for any $\delta \geq \delta_0(H)$, there exists a constant $n_0(\delta, H)$ such that, if $n \geq n_0(\delta, H)$, then $K_{\delta, n - \delta}$ uniquely maximises the number of H -colourings.

From the examples given above, it is clear to see that there is not a simple answer to the question of which graph G maximises $\text{hom}(G, H)$ when H is fixed and δ is sufficiently large. We make the following conjecture.

Conjecture 5.1. *For any graph H and any $\delta \geq \delta_0(H)$, there exists a constant $n_0(\delta, H)$ such that the following holds: if G is a graph with minimum degree δ and at least $n_0(\delta, H)$ vertices, then*

$$\text{hom}(G, H) \leq \max \left\{ \text{hom}(K_{\delta+1}, H)^{\frac{|G|}{\delta+1}}, \text{hom}(K_{\delta, \delta}, H)^{\frac{|G|}{2\delta}}, \text{hom}(K_{\delta, |G| - \delta}, H) \right\}.$$

This conjecture implies that, for a fixed graph H and δ sufficiently large, the following holds: for sufficiently large n satisfying suitable divisibility conditions, the number of H -colourings is always maximised by one of $\frac{n}{\delta+1}K_{\delta+1}$, $\frac{n}{2\delta}K_{\delta, \delta}$ or $K_{\delta, n - \delta}$.

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