Maximum directed cuts in acyclic digraphs

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Abstract

It is easily shown that every digraph with m edges has a directed cut of size at least $m/4$, and that $1/4$ cannot be replaced by any larger constant. We investigate the size of a largest directed cut in *acyclic* digraphs, and prove a number of related results concerning cuts in digraphs and acyclic digraphs.

1 Introduction

Results on maximum cuts in an undirected graph have a huge literature (see Poljak and Tuza [21] and Laurent [19]), and the extremal Max Cut problem is now quite well understood. Given a graph and a partition of its vertex set into sets X, Y, a *cut* (X, Y) means the edge set $E(X, Y)$ with one endpoint in X and the other endpoint in Y. The *size* of the cut (X, Y) is $e(X, Y) = |E(X, Y)|$. Similarly, we write $e(X)$ for the number of edges with both ends in X. For a graph G, $f(G)$ denotes the maximum of $e(X, Y)$ over all cuts (X, Y) of G and $f(m)$ is the minimum value of $f(G)$ taken over all graphs with m edges. It is easily seen that every graph with m edges has a cut of size at least $m/2$.

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Edwards [12] improved on this, showing that

$$f(m) \geq \frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8}, \tag{1}$$

with equality for complete graphs of odd order; in fact, the same bound holds for multigraphs. From the other side, Alon [2] proved that there is a constant $c > 0$ such that

$$f(m) \leq \frac{m}{2} + \sqrt{\frac{m}{8}} + cm^{1/4} \tag{2}$$

for every m, and showed that the $O(m^{1/4})$ term is necessary: there is a constant $c' > 0$ such that $f(m) \geq m/2 + \sqrt{m/8} + c'm^{1/4}$ for infinitely many m. (For further work, see [4] and [8].) A lower bound in terms of size and order was proved by Edwards [12], who showed that if G is a connected graph with m edges then

$$f(G) \geq \frac{m}{2} + \frac{|G| - 1}{4}. \tag{3}$$

In this paper we concentrate on cuts in directed graphs. Let X, Y be a partition of the vertex set of a directed graph D. The *directed cut* (X, Y) is the set of edges $E(X, Y)$ with starting point in X and with endpoint in Y. The *size* of the cut (X, Y) is $e(X, Y) = |E(X, Y)|$. As in the undirected case, we can define $g(D)$ as the maximum of $e(X, Y)$ over all directed cuts and $g(m)$ as the minimum of $g(D)$ over all directed graphs D with m edges. Random bipartitions show instantly that $g(m) \geq m/4$. The stronger bound

$$g(m) \geq \frac{m}{4} + \sqrt{\frac{m}{32} + \frac{1}{256}} - \frac{1}{16}$$

follows from (1); regular orientations of complete graphs of odd order show that this bound can be achieved. In particular, it follows that $g(m) \sim m/4$ (see [8] for further discussion).

A more delicate extremal problem emerges when we restrict our attention to *acyclic* directed graphs (or, equivalently, ordered graphs). Let $h(m)$ be the minimum of $g(G)$ over all acyclic directed graphs with m edges. Clearly, $h(m) \geq g(m)$. It is easily seen that $h(m) \geq f(m)/2$, and so $h(m) \geq m/4$, but it is not so obvious whether $1/4$ can be replaced by a larger constant. For instance, it is not hard to show that acyclic orientations of random graphs tend to have rather large directed cuts: for fixed $p \in (0, 1)$, almost every $G \in \mathcal{G}(n, p)$ has $g(\tilde{G}) = (1 + o(1))e(G)/2$ for every acyclic orientation $\tilde{G}$ of G (while $g(\tilde{G}) = (1 + o(1))e(G)/4$ for almost every other orientation). Furthermore, while there are natural families of extremal graphs for $f(m)$ and $g(m)$, there do not appear to be natural candidates for extremal graphs for $h(m)$.

One motivation for studying $h(m)$ comes from the Hall ratio of graphs (see [9], and Johnson [11]), $\rho(G) = \max\{|V(H)|/\alpha(H)\}$ where H runs over all induced subgraphs of G. In [9] $\rho(G)$ was studied for Kneser graphs and for Mycielski graphs, and the study of $\rho(G)$ for shift graphs was proposed in [10]. The shift graph SH_n (defined by Erdős and Hajnal in [13]) is the underlying graph of the directed line graph of the transitive tournament on n vertices. The main open problem ([10]) was whether there is $\epsilon > 0$ such that $\rho(SH_n) \le (4-\epsilon)$ for every n. It is easy to see that the problem is equivalent to deciding whether $h(m) \ge cm$ for some $c > 1/4$.

In Section 2, we give a negative answer to this problem, showing in fact that $h(m) = m/4 + O(m^{4/5})$ by giving an explicit construction. Since $g(m) = m/4 + \Theta(\sqrt{m})$, the question arises of whether $h(m) - m/4$ grows more quickly than $g(m) - m/4$. We show that this is indeed the case, proving that $h(m) \ge m/4 + \Omega(m^{2/3})$.

In Section 3, we examine maximum cuts in digraphs (acyclic or general) with degree restrictions. We consider both the class of digraphs with maximum outdegree at most k and the class of digraphs $D(k,l)$ in which every vertex has either indegree at most k or outdegree at most l. (For instance, $D(0,0)$ is the class of bipartite graphs with all edges directed from one vertex class to the other.) We also note a characterization of graphs that have an orientation lying in one of these classes.

Finally, in Section 4, we consider the problem of covering the edges of a digraph with as few cuts as possible.

2 Max Directed Cut for acyclic digraphs

The first aim of this section is to construct an acyclic digraph with m edges and no directed cut of size greater than $(1+o(1))m/4$.

Theorem 1. *For $m \ge 1$,*

$$h(m) = \frac{m}{4} + O(m^{4/5}).$$

Proof. Fix $n \ge 1$ and let $r = \lfloor n^{1/3} \rfloor$. We construct an acyclic digraph D' with $m = m(n) = (1+o(1))n^{5/3}$ edges, and no directed cut of size more than $m/4 + O(m^{4/5})$.

We first define a digraph D as follows. By a well known theorem of Singer [22] there is a set A of r natural numbers such that all differences $a - b$, with $a, b \in A$ and $a \ne b$, are distinct and $\max A \le (1+o(1))r^2$. We define the digraph D to have vertex set $\mathbb{Z}_n$, and directed edges from $i+a$ to $i+b$, for all $i \in \mathbb{Z}_n$, and all $a, b \in A$ with $a < b$ (here sums are written modulo n). By the construction of A, there are no multiple edges, and so $e(D) = n\binom{r}{2}$.

Let G be the underlying graph of D. Since G is a union of n copies of K_r (where the ith copy has vertex set $\{i + a : a \in A\}$), we have

$$f(G) \le nf(K_r) \le \frac{nr^2}{4}.$$

Since D is Eulerian, it follows that

$$g(D) = \frac{1}{2}f(G) \le \frac{nr^2}{8}.$$

We now modify D to obtain an acyclic digraph D' by deleting all edges ij with $i > j$ (where we identify the vertices of $\mathbb{Z}_n$ with the integers $0, \ldots, n-1$). Since $\max A \leq (1+o(1))r^2$, we have deleted at most

$$(1+o(1))r^2\binom{r}{2} \leq (1+o(1))\frac{r^4}{4}$$

edges. Thus D' is acyclic, has

$$m \geq n\binom{r}{2} - (1+o(1))\frac{r^4}{4} = \frac{nr^2}{2} - \frac{nr}{2} - (1+o(1))\frac{r^4}{4}$$

edges, and

$$\begin{aligned} g(D') &\leq \frac{nr^2}{8} \\ &\leq \frac{m}{4} + \frac{nr}{8} + (1+o(1))\frac{r^4}{16} \\ &\leq \frac{m}{4} + (1+o(1))\frac{n^{4/3}}{4}. \end{aligned}$$

A simple calculation shows that $m = (1+o(1))n^{5/3}/2$, and so

$$g(D') \leq \frac{m}{4} + (1+o(1))\frac{m^{4/5}}{2}.$$

The construction is easily extended to other values of m by deleting edges. $\square$

We note that the constants in the proof above could be improved at several points. For instance, it follows from eigenvalue techniques that no bipartition will split all the copies of K_r in a balanced way, and our estimates elsewhere are crude. However, it is probably not worth pursuing this, as the exponent $4/5$ may well not be optimal.

We have now shown that $h(m) = (1+o(1))m/4$, but the question remains of how $h(m)$ compares with $g(m)$. The interest here is in the $o(m)$ term: in other words, how much better do we do than the trivial $m/4$? We know that $g(m) - m/4 = \Theta(\sqrt{m})$. Here we show that $h(m) - m/4$ is rather larger.

Theorem 2. *For $m \geq 1$,*

$$h(m) \geq \frac{m}{4} + \Omega(m^{2/3}).$$

Proof. Let D be an acyclic digraph with m edges. Identifying one vertex from each component, we may assume that the underlying graph of D is connected. Let G be the underlying multigraph of D, and let $n = |D| = |G|$. It is known that (3) holds for connected multigraphs (see, for instance, [20]), and so

$$f(G) \geq \frac{m}{2} + \frac{n-1}{4}.$$

Therefore

$$g(D) \geq \frac{m}{4} + \frac{n-1}{8}. \tag{4}$$

Now let

$$\gamma : V(D) \to [-n/2, n/2) \cap \mathbb{Z}$$

be an injective order-preserving map (i.e. if $uv \in E(D)$ then $\gamma(u) > \gamma(v)$). We define the *length* of an edge $uv \in E(D)$ to be the positive integer $\gamma(u) - \gamma(v)$.

Consider a random directed cut (V^+, V^-) of D, where each $u \in V(D)$ independently belongs to V^+ with probability

$$\frac{1}{2} + \frac{\gamma(u)m}{10n^3}.$$

(Note that, as $|\gamma(u)| \leq n/2$ and $m \leq 2\binom{n}{2}$, this is a real number in $(0,1)$.) Now consider an edge uv with length l. The probability that u lies in the directed cut is

$$\begin{aligned}
\mathbb{P}(u \in V^+, v \in V^-) &= \left(\frac{1}{2} + \frac{\gamma(u)m}{10n^3}\right)\left(\frac{1}{2} - \frac{\gamma(v)m}{10n^3}\right) \\
&= \frac{1}{4} + \frac{lm}{20n^3} - \frac{\gamma(u)\gamma(v)m^2}{100n^6} \\
&\geq \frac{1}{4} + \frac{lm}{20n^3} - \frac{m^2}{400n^4}.
\end{aligned}$$

It follows that the expected number of edges in the cut is at least

$$\frac{m}{4} + \frac{m}{20n^3}L - \frac{m^3}{400n^4},$$

where L is the sum of lengths of all edges of D. Since D contains at most $n-1$ edges of any given length, we have

$$L \geq (n-1)(1 + 2 + \cdots + \lfloor m/(n-1) \rfloor).$$

Let $\alpha = \lfloor m/(n-1) \rfloor \geq m/2(n-1)$ (note that $m \geq n-1$, as the underlying graph of D is connected). Then

$$L \geq (n-1) \cdot \frac{1}{2}\alpha(\alpha+1) \geq \frac{m^2}{8(n-1)},$$

so

$$\begin{aligned}
g(D) &\geq \frac{m}{4} + \frac{m}{20n^3} \cdot \frac{m^2}{8(n-1)} - \frac{m^3}{400n^4} \\
&\geq \frac{m}{4} + \frac{m^3}{320n^3(n-1)}. \qquad (5)
\end{aligned}$$

It now follows from (4) and (5) that

$$\begin{aligned} g(D) &\geq \frac{m}{4} + \frac{1}{2}\left(\frac{m^3}{320n^3(n-1)} + \frac{n-1}{8}\right) \\ &\geq \frac{m}{4} + \Omega(m^{3/5}), \end{aligned}$$

where the final inequality follows by minimizing over n. $\square$

It would be interesting to know the optimal power of m in the theorems above.

Problem 1. *What is the infimum α_0 of $\alpha > 0$ such that, for $m \geq 1$,*

$$h(m) = \frac{m}{4} + O(m^\alpha)?$$

Clearly $\alpha_0 \in [2/3, 4/5]$, from the two theorems above.

For related results and conjectures concerning undirected graphs, see [2, 3, 5].

3 Maximum directed cuts in digraphs with degree restrictions

The following simple result follows easily from results by a number of authors, beginning with Andersen, Grant and Linial [6], Locke [17] and Lehel and Tuza [16] (see [2] for a compact proof): for k an odd integer, every graph with m edges and maximum degree at most k has a cut of size at least

$$\frac{k+1}{2k}m. \tag{6}$$

The bound is sharp for unions of complete graphs on $k+1$ vertices. For even k, the sharp bound is $\frac{k+2}{2(k+1)}m$ (that is (6) with $k+1$ instead of k) and complete graphs again yield the optimum.

We can obtain similar bounds for digraphs by controlling the chromatic number.

Lemma 3. *Let D be a digraph with m edges and maximum outdegree at most k. Then the underlying graph of D has chromatic number at most $2k+1$.*

Proof. Any digraph with maximum outdegree at most k contains a vertex with indegree at most k. It follows that the underlying graph of D is $2k$-degenerate (that is, every induced subgraph has a vertex of degree at most $2k$), and so has chromatic number at most $2k+1$. $\square$

This is clearly attained when D is an orientation of K_{2k+1} in which every vertex has outdegree k. We can now use Lemma 3 to bound the maximum size of a directed cut.

Corollary 4. *Let D be a digraph with m edges and maximum outdegree at most k. Then D has a directed cut of size at least*

$$\frac{k+1}{4k+2}m.$$

Proof. Let G be the underlying multigraph of D (so G may have some double edges). Then $\chi(G) \leq 2k+1$, so we can pick a $(2k+1)$-colouring of G. Identifying the vertices in each colour class, we obtain an edge-weighted copy K of K_{2k+1}. Now take a random partition of $V(K)$ into vertex classes of size $[k]$ and $[k+1]$, chosen uniformly from all such partitions. The expected weight of the resulting cut is $\frac{k+1}{2k+1}m$. Taking a cut with at least this weight yields a corresponding cut (X, Y) in G with the same weight. Finally, either (X, Y) or (Y, X) has weight at least half of this, as required. $\square$

This bound is sharp for k-out-regular orientations of K_{2k+1}, since we then have $e(X, Y) = e(Y, X)$ for every cut (X, Y).

Similar results can be obtained for graphs in which each vertex has a constrained indegree *or* outdegree. We therefore turn to considering $D(k, l)$, the set of digraphs in which every vertex has indegree at most k or outdegree at most l. Let us note first that there is a natural characterization of graphs that have an orientation in $D(k, l)$, using the following theorem ([14], for generalizations see [15],[7]).

Theorem 5. *Let G be a graph with vertex set V and let $(d_v)_{v\in V}$. Then there is an orientation of G with $d^+(v) \leq d_v$ for every vertex v if and only if*

$$e(X) \leq \sum_{v\in X} d_v \tag{7}$$

for all $X \subseteq V$.

The following corollary is immediate.

Corollary 6. *A graph G with vertex set V has an orientation with maximum outdegree at most k if and only if $e(X) \leq k|X|$ for every $X \subseteq V$. If in addition $e(G) = k|V|$ then G has a k-out-regular orientation.*

We can now characterize graphs with an orientation in $D(k, l)$.

Corollary 7. *A graph G with vertex set V has an orientation in $D(k, l)$ if and only if there is a partition $V(G) = V_1 \cup V_2$ such that $e(X) \leq k|X|$ for all $X \subseteq V_1$ and $e(X) \leq l|X|$ for all $X \subseteq V_2$.*

Proof. If D is in $D(k, l)$ then let V_1 be the set of vertices in D with outdegree at most k, and let $V_2 = V(G) \setminus V_1$. This partition clearly satisfies the conditions. Conversely, suppose G is a graph with a partition $V_1 \cup V_2$ satisfying the conditions. By the theorem, we can orient the subgraphs induced by V_1 and V_2 so that they have maximum outdegree at most k and at most l respectively. But then reversing the orientation of the subgraph induced by V_1 and orienting all edges in $E(V_1, V_2)$ from V_1 to V_2 gives an orientation in $D(k, l)$. $\square$

A similar result to Corollary 4 can be proved for graphs in $D(k,l)$.

Lemma 8. *Every digraph in $D(k,l)$ has an underlying graph with chromatic number at most $2k+2l+2$.*

Proof. Let U be the set of vertices of indegree at most k and let W be the remaining vertices. Then the digraph induced by U has indegree at most k, and its underlying graph is therefore $2k$-degenerate and so has chromatic number at most $2k+1$; similarly, the digraph induced by W has outdegrees at most l, and it follows that its underlying graph has chromatic number at most $2l+1$. □

Note that this bound is attained by taking a k-out-regular orientation of K_{2k+1} and an l-out-regular orientation of K_{2l+1}, adding all edges between them and orienting from the first graph to the second.

Corollary 9. *Every digraph in $D(k,l)$ with m edges has a directed cut of size at least*

$$\frac{k+l+2}{4k+4l+6}m. \tag{8}$$

Proof. Suppose $D \in D(k,l)$ and let G be its underlying multigraph. Then $\chi(G) < 2k+2l+3$, and proceeding as in the proof of Corollary 4, it follows from Lemma 8 that there is a cut of the desired size. □

For an upper bound on the constant in Corollary 9, let $t = \max\{k,l\}$. A regular orientation of K_{2t+1} belongs to $D(k,l)$ and has no directed cut of size more than

$$\frac{t(t+1)}{2} = \frac{t+1}{4t+2}e(K_{2t+1}). \tag{9}$$

Thus, by (8) and (9), we see that the correct bound lies between

$$\frac{1}{4} + \frac{1}{8k+8l+12}$$

and

$$\frac{1}{4} + \frac{1}{8\max\{k,l\}+4}.$$

It would be interesting to know the correct value.

The problem is also interesting when we restrict our attention to acyclic digraphs. For instance, when $k=l=1$ we have the following result.

Theorem 10. *Every acyclic digraph $D \in D(1,1)$ has a cut of size at least $2|E(D)|/5$.*

Proof. The proof is by induction on $m = |E(D)|$. If D is the union of disjoint directed paths, then D clearly has a directed cut of size $\lceil m/2 \rceil > 2m/5$.

Assume that this is not the case, w.l.o.g. the set L of all vertices with outdegree at least two is nonempty. Because D is acyclic, the subgraph induced by L has a vertex x_0 with outdegree zero in L, let $f_1 = x_0x_1$ and $f_2 = x_0x_2$ be edges of D for some $x_1, x_2 \notin L$. Note that there is at most one edge entering

x_0 and at most one edge leaving x_i, for $i = 1, 2$; remove those edges from G together with f_1 and f_2. By induction, the remaining graph has a cut of size at least $2(m-5)/5$ that can be extended by f_1 and f_2 to get a cut of size at least $2m/5$ in D. □

An extremal digraph for Theorem 10 is given by taking the directed path $(1, 2, 3, 4, 5)$ with the chord 24.

4 Covering by directed cuts

In previous sections, we concentrated on finding the largest cut that can be guaranteed in a digraph satisfying various conditions. We now turn to the problem of covering the edges of a digraph with directed cuts, and in particular ask for an upper bound on the number of directed cuts required to cover the edges of graphs in $D(k, l)$.

We start with the *complete directed graph on n vertices*, in which every pair of vertices is joined by a pair of edges, one in each direction. Surprisingly, covering with directed cuts turns out to be closely related to Sperner systems.

Theorem 11. *The minimal number of directed cuts required to cover the edges of the complete directed graph on n vertices is equal to*

$$c(n) := \min\left\{k : \binom{k}{\lfloor k/2 \rfloor} \geq n\right\} = \log_2 n + \frac{1}{2}\log_2 \log_2 n + O(1).$$

Proof. Given a sequence S of directed cuts $(X_1, Y_1), \ldots, (X_m, Y_m)$ of the complete digraph with vertex set V, let us associate with each vertex v the set $S(v) = \{i : v \in X_i\}$. Then S covers the edges of the complete digraph if and only if $\{S_v\}_{v \in V}$ forms an antichain. The bound in the theorem then follows from Sperner's Lemma. □

We note here that Theorem 11 and its subsequent applications can be derived from Exercise 9.26 (b) in [18] (attributed to A. Hajnal), or from the results in [1]. It follows immediately from the theorem that any digraph whose underlying graph has chromatic number at most n can be covered by $c(n) \sim \log_2 n$ directed cuts, as we can use a colouring of the underlying graph to group the vertices of our digraph. On the other hand, any graph with chromatic number n requires at least $\log_2 n$ cuts to be covered, since we can colour its vertices with sets S as in the proof above.

Lemma 12. *Let D be a digraph in which every vertex has outdegree at most k. Then the edges of D can be covered with at most $c(2k+1)$ cuts.*

Proof. Since the underlying graph of D is at most $(2k+1)$-chromatic, we can identify vertices of D to obtain an edge-weighted copy of the complete digraph with $2k+1$ vertices, which can be covered with $c(2k+1)$ directed cuts. □

We now concentrate on graphs in $D(k,l)$. When $k = l = 1$ we have the following result.

Theorem 13. *The edge set of any $G \in D(1,1)$ can be partitioned into at most three cuts.*

Proof. Let V_0 be the set of all vertices of outdegree at most 1, and set $V_1 = V(D) \setminus V_0$. Every component of the subgraph induced by V_0 is either an *in-branching*, defined as a rooted tree with all edges directed towards the root, or a *function graph* defined as an in-branching extended with one *root edge* leaving the root. Similarly, every connected component of the subgraph induced by V_1 is an out-branching or a function graph with all edges reversed.

The set of edges of a branching considered as a bipartite graph has a natural bipartition into two directed cuts. Use colours 1 and 2 to colour these cuts in every component induced by V_0 and by V_1; the root edges will get a new colour 3. Every edge from V_0 to V_1 connects roots of two branching components; they will be coloured 3 as well. For any edge going from $x \in V_1$ to $y \in V_0$ we choose a colour different from the colour of the edge entering x and that of leaving y. Each colour class is obviously a directed cut, finishing the proof. $\square$

The directed path $(1,2,3,4,5)$ with the chord 24 (or odd directed cycles) cannot be decomposed into two directed cuts, showing that Theorem 13 is sharp. For larger k and l, we have the following result.

Theorem 14. *Every $D \in D(k,l)$ can be covered by at most $c(2k+2l+2)$ directed cuts.*

Proof. As the underlying graph of every digraph in $D(k,l)$ is at most $(2k+2l+2)$-chromatic, we can identify vertices of D to obtain an edge-weighted copy of the complete digraph with $2k+2l+2$ vertices, which can be covered with $c(2k+2l+2)$ directed cuts. $\square$

For acyclic digraphs we can do (very slightly) better.

Theorem 15. *Every acyclic digraph in $D(k,l)$ can be covered with at most $c(k+l+2)$ directed cuts.*

Proof. We first note that the underlying graph of any acyclic graph D in $D(k,l)$ is at most $k+l+2$-chromatic. Indeed, let U be the set of vertices with indegree at most k and let W be the remaining vertices. Consider the digraph D' induced by U. Since G is acyclic, every subgraph of D' has a vertex with outdegree 0, and so the underlying graph of D' is k-degenerate and has chromatic number at most $k+1$. Similarly, the digraph induced by W has chromatic number at most $l+1$, and so the underlying graph is $(k+l+2)$-colourable. The proof is completed as before. $\square$

The bounds in Lemma 12 and Theorem 14 are within $O(1)$ of being optimal, as can be seen (from Theorem 11) by looking at complete directed graphs, and it would be interesting to know the correct values. The bound in Theorem 15 may be less good: a lower bound of $\log_2(k+l+1)$ comes from looking at the transitive tournament T_{k+l+1}, which still leaves a gap of $O(\log\log(k+l+1))$. A good starting point is the following problem, which may not be too difficult.

Problem 2. *What is the smallest $d(k)$ such that every acyclic digraph with maximum outdegree at most k can be covered by $d(k)$ directed cuts?*

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