Asymptotic structure. V. The coarse Menger conjecture in bounded path-width

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Abstract

Menger's theorem tells us that if S, T are sets of vertices in a graph G, then (for $k \geq 0$) either there are $k+1$ vertex-disjoint paths between S and T, or there is a set of k vertices separating S and T. But what if we want the paths to be far apart, say at distance at least c? One might hope that we can find either $k+1$ paths pairwise far apart, or k sets of bounded radius that separate S and T, where the bound on the radius is some ℓ that depends only on k, c (the "coarse Menger conjecture"). We showed in an earlier paper that this is false for all $k \geq 2$ and $c \geq 3$. To do so we gave a sequence of finite graphs, counterexamples for larger and larger values of ℓ with $k = 2$, $c = 3$. Our counterexamples contained subdivisions of uniform binary trees with arbitrarily large depth as subgraphs.

Here we show that for any binary tree T, the coarse Menger conjecture is true for all graphs that contain no subdivision of T as a subgraph, that is, it is true for graphs with bounded path-width (and, further, for graphs with bounded coarse path-width). This is perhaps surprising, since it is false for bounded tree-width.

1 Introduction

Let S, T be sets of vertices of a graph G. (In this paper, all graphs are finite and have no loops or multiple edges.) Menger's theorem [7] tells us that either there are $k+1$ pairwise vertex-disjoint paths between S and T, or there is a set X of at most k vertices such that every S-T path in G meets X. But what if we want the paths to be pairwise far apart? In this case, the question is much harder. Bienstock [3] showed that it is NP-hard to decide whether, given four vertices s_1, s_2, t_1, t_2 of a graph G, there are two paths between between $\{s_1, s_2\}$ and $\{t_1, t_2\}$ that have distance ≥ 2, that is, they are vertex-disjoint and there is no edge joining them. This was recently extended by Baligács and MacManus [2], who showed the same thing for distance $\geq c$, for each $c \geq 3$.

Since the problem is NP-complete, one would not expect to find a necessary and sufficient condition for the existence of $k+1$ S-T paths at distance at least c; but still one could hope for some sort of obstruction that is necessary for excluding $k+1$ S-T paths at distance at least c, and sufficient for excluding $k+1$ S-T ps at distance more than some larger number depending on k, c. Two groups of researchers, Albrechtsen, Huynh, Jacobs, Knappe and Wollan [1], and independently Georgakopoulos and Papasoglu [5], proposed such a statement:

1.1 Coarse Menger Conjecture: *For all integers $k \geq 0$ and $c \geq 1$ there exists $\ell \geq 0$ with the following property. Let G be a graph and let $S, T \subseteq V(G)$. Then either*

- *there are $k+1$ paths between S, T, pairwise at distance at least c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex with distance at most ℓ from some member of X.*

Both groups showed that this is true for $k = 1$, and Gartland, Korhonen and Lokshtanov [4] and Hendrey, Norin, Steiner, and Turcotte [6] proved it for bounded degree graphs when $c = 2$. However, we showed in [8] that the coarse Menger conjecture is false for all $k \geq 2$, for any fixed $c \geq 3$. Indeed, it remains false even if we weaken the bound $|X| \leq k$ in the second bullet to $|X| \leq m$, where m is any constant depending on k, c [12].

Thus, we need to lower our sights a little, and one way to do so is to work in restricted classes of graphs. The counterexamples of [8] have unbounded genus, and contain (as subgraphs) uniform binary trees of arbitrary depth, and therefore they have unbounded "path-width" (defined in the next section). It might be true that the coarse Menger conjecture holds for graphs of bounded genus, but this is open; see [13] for some progress in this direction. Here we prove that the coarse Menger conjecture is true for graphs of bounded path-width.

More exactly, we will prove:

1.2 *Let $k, d \geq 0$ and $c \geq 1$ be integers. Then there exists $\ell \geq 0$, such that for every graph G with path-width at most d, and all $S, T \subseteq V(G)$, either:*

- *there are $k+1$ paths between S, T, pairwise at distance at least c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex with distance at most ℓ from some member of X.*

Thus the coarse Menger conjecture is true for graphs of bounded path-width, and we will deduce in the conclusion that it also holds for graphs of "bounded coarse path-width". Curiously, the coarse Menger conjecture is *not* true for graphs of bounded tree-width, since the counterexamples of [8] have tree-width six.

2 Subdivisions and path-width

The uniform binary tree of depth $d \geq 2$ is the tree H such that for some $r \in V(H)$, r has degree two, all other vertices have degree one or three, and every vertex of degree one has distance exactly $d-1$ from r. Thus, H has $2^d - 1$ vertices. We denote this tree by H_d.

If H is a graph, a *subdivision* of H is a graph obtained from H by replacing each of its edges by a path of length at least one joining the same pair of vertices, where these paths are pairwise vertex-disjoint except for their ends. For $n \geq 0$, let us say an *n-subdivision* of H is a subdivision obtained by replacing each edge by a path of length $\leq n$ (and at least one). (This is inconsistent with the standard term "1-subdivision", which means replacing each edge with a path of length two, but convenient for us.)

Let us define path-width. A graph G has *path-width* at most d if and only if there is a sequence $W_1, \ldots, W_n$ of subsets of its vertex set, satisfying:

- $|W_i| \leq d+1$ for $1 \leq i \leq n$;
- $G[W_1] \cup \cdots \cup G[W_n] = G$; and
- $W_i \cap W_k \subseteq W_j$ for $1 \leq i \leq j \leq k \leq n$.

We do not really need this definition. The only thing about bounded path-width that concerns us is a theorem of Robertson and Seymour [14]:

2.1 *For every integer $d \geq 2$, there exists k, such that every graph that contains no subdivision of H_d as a subgraph has path-width at most k; and conversely, every graph that contains a subdivision of H_d as a subgraph has path-width at least $(d-1)/2$.*

Thus, knowing that there is a bound on path-width is the same as knowing that for some d, no subgraph is a subdivision of H_d. Indeed, in this paper it is more natural to work with the "excluded tree subdivision" version directly, rather than working with path-width. And in that form we can prove a strengthening: instead of excluding all subdivisions of H_d, it is enough that there are no ℓ-subdivisions of H_d, where ℓ is an appropriate constant (depending on k, c). We will prove the following strengthening of 1.2:

2.2 *For all integers $k, d, c \geq 0$ there exist $\ell_1, \ell_2 \geq 0$, with the following property. Let G be a graph that contains no ℓ_1-subdivision of H_d as a subgraph, and let $S, T \subseteq V(G)$. Then either*

- *there are $k+1$ paths between S, T, pairwise at distance greater than c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex with distance at most ℓ_2 from some member of X.*

We remark that here we are asking for paths with distance $> c$ rather than $\geq c$ as in 1.2; we find this form slightly more convenient.

3 A key lemma

If we contract an edge of a graph, then distances do not change by much, but if we delete an edge or a vertex, they might change considerably. In this section, we prove lemmas that allow us to bypass this problem to some extent, in graphs excluding subdivisions of some H_d. The proofs of these lemmas are the only places in the paper where we use the hypothesis about subdivisions of H_d.

If X is a vertex of a graph G, or a subset of the vertex set of G, or a subgraph of G, and the same for Y, then $\text{dist}_G(X, Y)$ denotes the distance in G between X, Y, that is, the number of edges in the shortest path of G with one end in X and the other in Y. (If no path exists we set $\text{dist}_G(X, Y) = \infty$.)

Here is the first such lemma:

3.1 *Let $d, \ell \geq 2$, and let G be a graph such that no subgraph is an $(\ell - 1)$-subdivision of H_d. Let $Z \subseteq V(G)$. Then there exists $Y \supseteq Z$ with the following properties:*

- *every vertex in Y has distance at most $(d - 2)(\ell - 1)$ from Z;*
- *there is no path P of G of length at least two and at most ℓ, such that the ends u, v of P belong to Y, the interior of P is disjoint from Y, and $\text{dist}_{G[Y]}(u, v) > 2(d - 2)(\ell - 1)$.*

Proof. If $Y \subseteq V(G)$, let us say a path P of G is a *bite for Y* if P has length at least two and at most ℓ, the ends u, v of P belong to Y, the interior of P is disjoint from Y, and $\text{dist}_{G[Y]}(u, v) > 2(d-2)(\ell-1)$. Define $Z_0 = Z$, and inductively for $i \geq 1$, having defined Z_{i-1}, if there is a bite for Z_{i-1}, choose some such bite P and let $Z_i = Z_{i-1} \cup V(P)$. Since the graph is finite, and each bite has nonempty interior, this sequence is finite: let Y be its final term. Thus there is no bite for Y. For $x, y \in Y$, let us say that y is *later* than x if for some i, $x \in Z_i$ and $y \notin Z_i$.

(1) *For each $v \in Y$, and $2 \leq m \leq d$, if $\text{dist}_G(v, Z) > (\ell - 1)(m - 2)$, then there is a subgraph H of $G[Y]$ that is an $(\ell - 1)$-subdivision of the uniform binary tree H_m, with root v, such that none of its vertices are later than v.*

We proceed by induction on $m \geq 2$. Since $\text{dist}_G(v, Z) > (\ell - 1)(m - 2)$ and $\ell, m \geq 2$, it follows that $v \notin Z$. Choose i minimum such that $v \in Z_i$, and let P be a bite for Z_{i-1} with $Z_i = Z_{i-1} \cup V(P)$, with ends u_1, u_2. Thus, $i \geq 1$, and $u_1, u_2 \in Z_{i-1}$, and v belongs to the interior of P, and the latter equals $Z_i \setminus Z_{i-1}$. If $m = 2$, then P (rooted at v) is an $(\ell - 1)$-subdivision of H_2, as required, so we assume that $m \geq 3$. Since the subpaths of P between v and u_1, u_2 both have length at most $\ell - 1$, it follows that $\text{dist}_G(u_j, Z) > (\ell - 1)(m - 3)$ for $j = 1, 2$.

We apply the inductive hypothesis to u_1, u_2, and deduce that for $j = 1, 2$, there is a subgraph L_j of G that is an $(\ell - 1)$-subdivision of the uniform binary tree H_{m-1}, with root u_j, such that none of its vertices are later than u_j. Since

$$\text{dist}_{G[Z_{i-1}]}(u_1, u_2) > 2(d - 2)(\ell - 1) \geq 2(m - 2)(\ell - 1)$$

and every vertex of L_j has distance in L_j at most $(m - 2)(\ell - 1)$ from its root u_j, it follows that L_1, L_2 are vertex-disjoint. Moreover, they are both vertex-disjoint from the interior of P, since the latter is disjoint from Z_{i-1}. Consequently $L_1 \cup L_2 \cup P$ (rooted at v) is an $(\ell - 1)$-subdivision of H_m. This proves (1).

Since there is no subgraph that is an $(\ell - 1)$-subdivision of the uniform binary tree H_d, it follows from (1) that $\text{dist}_G(v, Z) \leq (\ell - 1)(d - 2)$ for each $v \in Y$. This proves 3.1. ∎

We deduce:

3.2 *Let $d, \ell \geq 2$, and let G be a graph such that no subgraph is an $(\ell-1)$-subdivision of H_d. Let $A \subseteq V(G)$. Then there exists $B \subseteq A$ such that:*

- *every vertex in $A \setminus B$ has distance at most $(d-2)(\ell-1)$ from $V(G) \setminus A$;*
- *there is no path P of G of length at most ℓ, such that the ends u, v of P are distinct and nonadjacent, and belong to $V(G) \setminus B$, the interior of P is included in B, and $\text{dist}_{G \setminus B}(u, v) > 2(d-2)(\ell-1)$.*
- *for all $u, v \in V(G) \setminus B$, if $\text{dist}_G(u, v) \leq \ell$, then $\text{dist}_{G \setminus B}(u, v) \leq (d-2)\ell(\ell-1)$.*

Proof. If $d = 2$ we may take $B = A$, so we assume that $d \geq 3$. Let $Z = V(G) \setminus A$, let Y be as in 3.1, and let $B = V(G) \setminus Y$. Thus, the first bullet is satisfied, and there is no path P of G of length at most ℓ, such that the ends u, v of P are distinct and belong to $V(G) \setminus B$, the interior of P is included in B, and $\text{dist}_{G \setminus B}(u, v) > 2(d-2)(\ell-1)$.

To see the second bullet, let $u, v \in V(G) \setminus B$, and assume that P is a path between u, v in G, of length at most ℓ. An *excursion* is a subpath Q of P such that Q has length at least two; its ends are not in B; and all its internal vertices are in B. It follows that the excursions in P are pairwise edge-disjoint, although two excursions might have a common end. Let $Q_1, \ldots, Q_t$ be the excursions, and for $1 \leq i \leq t$ let Q_i have ends u_i, v_i. From the choice of Y, since Q_i has length at most ℓ, it follows that there is a path P_i of $G \setminus B$ between u_i, v_i of length at most $2(d-2)(\ell-1)$. Let there be s edges of P that do not belong to excursions: then $s + 2t \leq \ell$, since each excursion has length at least two. Moreover, the union of $P_1, \ldots, P_t$ and the s edges of P not in excursions is a connected subgraph of $G \setminus B'$ containing u, v. Consequently

$$\text{dist}_{G \setminus B}(u, v) \leq s + 2(d-2)(\ell-1)t \leq s + (d-2)(\ell-1)(\ell-s) \leq (d-2)\ell(\ell-1)$$

(since $d \geq 3$). This proves 3.2. ∎

We also need:

3.3 *Let G be a graph with no subgraph that is an $(\ell-1)$-subdivision of H_d. Let $Z \subseteq V(G)$, and suppose that $M_1, M_2, \ldots, M_t$ are paths of G, each of length at most ℓ. Let $Z_i = Z \cup V(M_1 \cup \cdots \cup M_i)$ for $0 \leq i \leq t$; and suppose in addition that for each $i \geq 1$, the ends of M_i lie in different components of $G[Z_{i-1}]$, and none of its internal vertices lie in Z_{i-1}. Then for each $v \in Z_t \setminus Z$, either v lies in the interior of some M_i with both ends in Z, or there are at least three components C of $G[Z]$ such that v is joined to C by a path in $G[Z_{i-1}]$ of length at most $d(\ell-1)$.*

Proof. We say the *height* of each vertex in Z is zero; and inductively, for $1 \leq i \leq n$, let us say that for each vertex in the interior of M_i, its *height* is one more than the minimum of the heights of u_1, u_2, where u_1, u_2 are the ends of M_i. Then:

(1) *For each $i \geq 0$ and each $v \in Z_i$ with height at least $h \geq 1$, there is a subgraph of $G[Z_i]$ that is an $(\ell-1)$-subdivision of H_{h+1} rooted at v. Consequently, every vertex has height at most $d-2$.*

We use induction on h. The statement is clear if $h = 1$, so we assume $h \geq 2$. We may assume that i is minimum such that $v \in Z_i$, and consequently v belongs to the interior of M_i. Let u_1, u_2 be the ends of M_i, joining components C_1, C_2 of $G[Z_{i-1}]$. Thus, u_1, u_2 have height at least $h-1$. From the inductive hypothesis there is a subgraph L_j of C_j rooted at u_j that is an $(\ell-1)$-subdivision of H_h. But L_1, L_2 are disjoint, since they belong to different components of $G[Z_{i-1}]$; and disjoint from the interior of M_i, since the latter is disjoint from Z_{i-1}. But then $L_1 \cup L_2 \cup M_i$ (rooted at v) is the desired $(\ell-1)$-subdivision of H_{h+1}. This proves the first statement of (1). It follows that every vertex has height at most $d-2$, since no subgraph is an $(\ell-1)$-subdivision of H_d, and this proves (1).

(2) *For each $i \geq 0$ and each $v \in Z_i$ with height $h \geq 0$, v is joined to Z by a path in $G[Z_i]$ of length at most $h(\ell-1)$.*

We prove this by induction on $h \geq 0$. If $h = 0$, the statement is clear, so we assume that $h \geq 1$. Choose i minimum with $v \in Z_i$. Then v is joined to a vertex u of height $h-1$ by a path of $G[Z_i]$ of length at most $\ell - 1$ (a subpath of M_i); and from the inductive hypothesis, u is joined to Z by a path in $G[Z_{i-1}]$ (and hence of $G[Z_i]$) of length at most $(h-1)(\ell-1)$. Consequently v is joined to Z by a path in $G[Z_i]$ of length at most $h(\ell-1)$. This proves (2).

(3) *For each $i \geq 0$ and each $v \in Z_i$ with height $h \geq 1$, there are at least two components C of $G[Z]$ such that v is joined to C by a path in $G[Z_i]$ of length at most $(h+1)(\ell-1)$.*

Again we use induction on h. Choose i minimum with $v \in Z_i$. Thus, v belongs to the interior of M_i; let M_i have ends u_1, u_2. Both u_1, u_2 have height at least $h-1$, and the claim follows from (2) applied to u_1 and to u_2. This proves (3).

In particular, for each $v \in V(M_1 \cup \cdots \cup M_n) \setminus Z$, v has height at least one; choose i minimum with $v \in Z_i$. Thus, v belongs to the interior of M_i; let M_i have ends u_1, u_2. If u_1, u_2 both have height zero then M_i has both ends in Z and the theorem holds; so we assume that u_1 has height at least one. By (1), u_1 has height at most $d-2$, so by (3) applied to u_1, there are at least two components C of $G[Z]$ such that u_1 is joined to C by a path in $G[Z_{i-1}]$ of length at most $(d-1)(\ell-1)$; and by (2), there is a third component C of $G[Z]$ such that u_2 is joined to C by a path in $G[Z_{i-1}]$ of length at most $(d-2)(\ell-1)$. Consequently there are at least three components C of $G[Z]$ such that v is joined to C by a path in $G[Z_{i-1}]$ of length at most $d(\ell-1)$. This proves 3.3. ∎

4 Augmenting paths

Let us extend the definition of $\text{dist}_G(u, v)$ a little, to accommodate vertices $u, v \notin V(G)$: if one of $u, v \notin V(G)$ then $\text{dist}_G(u, v) = \infty$.

Some more notation: if P is a path and $u, v \in V(P)$, $P[u, v]$ denotes the subpath between u, v. If $\mathcal{P}$ is a set of vertex-disjoint paths of a graph G, we denote $P_1 \cup \cdots \cup P_k$ by $U\mathcal{P}$, and its vertex set by $V\mathcal{P}$. Let G be a graph, let $S, T \subseteq V(G)$ be disjoint, and let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a set of k vertex-disjoint S-T paths, with

$$V\mathcal{P} \cup S \cup T = V(G),$$

such that for $1 \le h < k$, no proper subpath of P_h is an S-T path. Let P_h have ends $s_h \in S$ and $t_h \in T$. If $u, v \in V(P_h)$ are distinct, and v belongs to $P_h[u, t_h]$, we say that v is *later than* u *in* P_h, and u *is earlier than* v *in* P_h.

It is an elementary theorem (a special case of the theory of augmenting paths) that:

4.1 *Given G, S, T and $\mathcal{P} = \{P_1, \dots, P_k\}$ as above, the following are equivalent:*

1. *For every choice of $v_i \in V(P_i)$ for $1 \le i \le k$, there is an edge ab of G with $a, b \notin \{v_1, \dots, v_k\}$, such that*
   - *either $a \in S \setminus V\mathcal{P}$ or for some $h \in \{1, \dots, k\}$, $a \in V(P_h)$, and a is earlier than v_h in P_h, and*
   - *either $b \in T \setminus V\mathcal{P}$ or for some $h \in \{1, \dots, k\}$, $b \in V(P_h)$ and b is later than v_h in P_h.*
2. *There is a sequence $a_1b_1, a_2b_2, \dots, a_nb_n$ of oriented edges of G, not in $E(P_1 \cup \dots \cup P_k)$, such that*
   - *$a_1 \in S \setminus V\mathcal{P}$, and $b_n \in T \setminus V\mathcal{P}$;*
   - *for $1 \le i < n$, b_i, a_{i+1} belong to the same path P_h say (where $1 \le h \le k$), and a_{i+1} is earlier than b_i in P_h.*
3. *There is a sequence $a_1b_1, a_2b_2, \dots, a_nb_n$ as above, satisfying in addition that for $1 \le h \le k$, and $1 \le i < j \le n$, if $u \in \{a_i, b_i\} \cap V(P_h)$ and $v \in \{a_j, b_j\} \cap V(P_h)$, then either*
   - *u is earlier than v in P_h, or*
   - *$b_i = u = v = a_j$; or*
   - *$b_i = u$ and $a_j = v$ and $j = i + 1$.*
4. *There are $k + 1$ vertex-disjoint S-T paths in G.*

We do not actually need this theorem, and we mention it just for comparison with the more complicated results that we will need.

Let S, T be disjoint sets, and let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a set of k vertex-disjoint S-T paths, each with no internal vertex in $S \cup T$. We call $(S, T, \mathcal{P})$ a *setting*. Let F_0 be the set of all ordered pairs of distinct vertices ab with $a, b \in V\mathcal{P} \cup S \cup T$.

Let us fix some setting $(S, T, \mathcal{P})$ where $\mathcal{P} = \{P_1, \dots, P_k\}$. Let $c \ge 0$ be an integer. A *c-barrier* (in the setting) is a k-tuple $Q_1, \dots, Q_k$, where Q_h is a subpath of P_h of length at most c. We say $ab \in F_0$ *jumps* a c-barrier $Q_1, \dots, Q_k$ (in the setting) if $a, b \notin V(Q_1 \cup \dots \cup Q_k)$, and

- either $a \in S \setminus V\mathcal{P}$ or for some $h \in \{1, \dots, k\}$, $a \in V(P_h)$, and a is earlier than each vertex of Q_h in P_h; and
- either $b \in T \setminus V\mathcal{P}$ or for some $h \in \{1, \dots, k\}$, $b \in V(P_h)$ and b is later than each vertex of Q_h in P_h.

Let us say a set $F \subseteq F_0$ is *c-jumping* (in the setting $(S, T, \mathcal{P})$) if for every c-barrier, some member of F jumps the barrier.

A *partial c-augmenting sequence to* b_n is a sequence $a_1b_1, a_2b_2, \dots, a_nb_n$ of elements of F_0, such that

- $a_1 \in S \setminus V\mathcal{P}$;
- for $1 \le i < t$, b_i, a_{i+1} belong to the same path P_h say (where $1 \le h \le k$), and a_{i+1} is earlier than b_i in P_h, and $P_h[a_{i+1}, b_i]$ has length at least $c+1$.

If in addition $b_n \in T \setminus V\mathcal{P}$, we call such a sequence a *c-augmenting sequence*. Thus, if $a_i \in S \setminus V\mathcal{P}$ and $b_i \in T \setminus T \setminus V\mathcal{P}$ then $i = 1 = n$. For $F \subseteq F_0$, the sequence is *in* F if $a_ib_i \in F$ for $1 \le i \le n$.

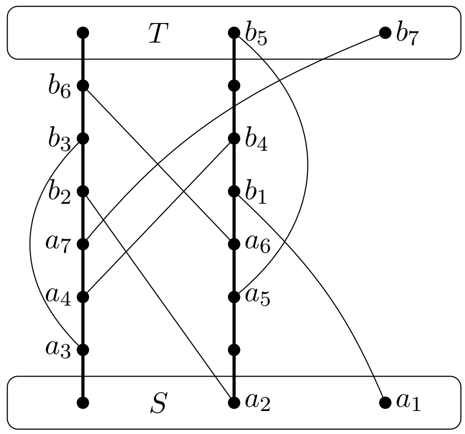

Figure 1: P_1, P_2 are the two paths of thick edges. With $k = 2$, the sequence $a_1b_1, \ldots, a_7b_7$ is minimal 2-augmenting, but it is not 1-separated. For every choice of three vertex-disjoint S-T paths, some edge of $P_1 \cup P_2$ joins two of them.

We begin with:

4.2 *Let $(S, T, \mathcal{P})$ be a setting, with $\mathcal{P} = \{P_1, \ldots, P_k\}$, and let $c \ge 0$ be an integer. With F_0 as before, let $F \subseteq F_0$. Then the following are equivalent:*

- *F is c-jumping;*
- *there is a c-augmenting sequence of elements of F.*

Proof. We show first that the second statement implies the first. To see this, assume that the sequence $a_1b_1, a_2b_2, \ldots, a_nb_n$ of pairs in F is c-augmenting, and let $Q_1, \ldots, Q_k$ be a c-barrier. Choose i maximum such that either $a_i \in S \setminus V\mathcal{P}$, or for some $h \in \{1, \ldots, k\}$, $a_i \in V(P_h) \setminus V(Q_h)$ and a_i is earlier in P_h than each vertex of Q_h. If $b_i \in T \setminus V\mathcal{P}$ then a_ib_i jumps the barrier, so we assume that $b_i \in V(P_j)$ for some $j \in \{1, \ldots, k\}$. Consequently $i < n$, and $a_{i+1} \in V(P_j)$, earlier than b_i in P_j. From the maximality of i, there exists $q \in V(Q_j)$ such that a_{i+1} is not earlier than q in P_j. Since $P_j[a_{i+1}, b_i]$ has length at least $c+1$, it follows that b_i is later than q in P_j, and $P_j[q, b_i]$ has length at least $c+1$. Since Q_j has length at most c, it follows that b_i is later in P_j than every vertex of Q_j; and so a_ib_i jumps the barrier. This proves the first statement.

To show the converse, suppose that F is c-jumping, and for $1 \le h \le k$, choose $v_h \in V(P_h)$ with $P_h[s_h, v_h]$ maximal such that either $v_h = s_h$ or there is a partial c-augmenting sequence to v_h in F. For $1 \le h \le k$, let Q_h be the maximal subpath of $P_h[s_h, v_h]$ with length at most c, such that one of its ends is v_h. Thus, $Q_1, \ldots, Q_k$ is a barrier, and so, since F is c-jumping, some $ab \in F$ jumps this barrier. Suppose first that $a \in S \setminus V\mathcal{P}$. If $b \in T \setminus V\mathcal{P}$, then ab is a c-augmenting sequence, so we assume that $b \in V(P_h)$ for some $h \in \{1, \dots, k\}$. Since ab jumps the barrier, it follows that b is later than v_h in P_h, contradicting the choice of v_h, since ab is a partial c-augmenting sequence to b. Thus, we may assume that for some $h \in \{1, \dots, k\}$, $a \in V(P_h)$, and a is earlier than each vertex of Q_h in P_h. Since $a \notin V(Q_h)$, it follows from the maximality of Q_h that Q_h has length exactly c, and therefore $P_h[a, v_h]$ has length at least $c+1$. Let $a_1b_1, \dots, a_sb_s$ be a partial c-augmenting sequence to v_h in F. Consequently $a_1b_1, \dots, a_sb_s, ab$ is a partial c-augmenting sequence to b in F. If $b \notin T \setminus V\mathcal{P}$, then, since ab jumps the barrier, there exists $h' \in \{1, \dots, k\}$ such that $b \in P_{h'}[v_{h'}, t_{h'}]$ and $b \neq v_{h'}$; but this contradicts the definition of $v_{h'}$. Thus, $b \in T \setminus V\mathcal{P}$, and so $a_1b_1, \dots, a_sb_s, ab$ is a c-augmenting sequence in F. This proves 4.2. ∎

This provides an analogue of the first two bullets of 4.1, and the next result gives an analogue of the third bullet.

4.3 *Let $(S, T, \mathcal{P})$ be a setting, with $\mathcal{P} = \{P_1, \dots, P_k\}$, and let $c \geq 0$ be an integer. Let $F \subseteq F_0$ be c-jumping, and choose a c-augmenting sequence $a_1b_1, \dots, a_nb_n$ of elements of F, with n minimum. For $1 \leq h \leq k$, and $1 \leq i < j \leq n$, if $u \in \{a_i, b_i\} \cap V(P_h)$ and $v \in \{a_j, b_j\} \cap V(P_h)$, then either*

- *u is earlier than v in P_h; or*
- *$b_i = u$ and $v = a_j$ and $P_h[u, v]$ has length at most c; or*
- *$b_i = u$ and $v = a_j$ and $j = i + 1$.*

Proof. Suppose that $1 \leq h \leq k$, and $1 \leq i < j \leq n$, and $u \in \{a_i, b_i\} \cap V(P_h)$ and $v \in \{a_j, b_j\} \cap V(P_h)$, and u is not earlier than v in P_h. If $u = a_i$ and $v = a_j$, then $i \geq 2$ and

$$a_1b_1, \dots, a_{i-1}b_{i-1}, a_jb_j, \dots, a_nb_n$$

is a c-augmenting sequence in F, contrary to the minimality of n. Similarly, if $u = b_i$ and $v = b_j$, then

$$a_1b_1, \dots, a_ib_i, a_{j+1}b_{j+1}, \dots, a_nb_n$$

is a c-augmenting sequence, a contradiction; and if $u = a_i$ and $v = b_j$, then $i \geq 2$ and $j \leq n-1$ and

$$a_1b_1, \dots, a_{i-1}b_{i-1}, a_{j+1}b_{j+1}, \dots, a_nb_n$$

is a c-augmenting sequence, a contradiction. Thus, we assume that $u = b_i$ and $v = a_j$. If $P_h[u, v]$ has length at least $c+1$, then

$$a_1b_1, \dots, a_ib_i, a_jb_j, \dots, a_nb_n$$

is a c-augmenting sequence, and so $j = i+1$; and otherwise $P_h[u, v]$ has length at most c. In either case the result holds. This proves 4.3. ∎

The results 4.2 and 4.3 do not quite provide an analogue of 4.1, because we have no counterpart to the fourth statement of 4.1, the existence of $k+1$ vertex-disjoint S-T paths. One might hope that

- *In the graph obtained from $U\mathcal{P}$ by adding the remainder of $S \cup T$ as extra vertices and the pairs in F as edges, there exist $k+1$ S-T paths, such that no two of them are joined by a path of $U\mathcal{P}$ of length at most c.*

could be added to the list of equivalent statements given by 4.2 and 4.3 to give an analogue of the fourth statement of 4.1, but that is wrong. This statement does imply the statements of 4.2, but the reverse implication does not hold. For instance, the graph of Figure 1 with $k = 2$ gives a 2-augmenting sequence, and yet for every three vertex-disjoint S-T paths, some two of them are joined by one of the edges of $P_1 \cup P_2$, which is more than we needed for a counterexample.

The property given by 4.3 implies that for each $h \in \{1, \dots, k\}$, the vertices a_i that lie in P_h are all distinct and in order in P_h, but it does not imply that they are far apart in P_h. For instance, if $k = 1$ and P_1 has vertices $s_1 = p_1\text{-}\cdots\text{-}p_n = t_1$, and $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$, and F is the union of $\{s_2v_{c+2}, v_{n-c-1}t_2\}$ and the pairs v_iv_{i+c+2} for $1 \le i \le n - c - 2$, then the only c-augmenting sequence in F uses all of F. Nevertheless, we can arrange that the a_i's are far apart, and the b_j's are far apart, by sacrificing some of the jumping power. We show this in two steps: first we arrange that the b_j's are far apart, in the following.

We recall that $\text{dist}_{U\mathcal{P}}(b, b') = \infty$ unless $b, b' \in V\mathcal{P}$ and b, b' belong to the same component of $U\mathcal{P}$.

4.4 *Let $p, q \ge 0$ be integers, and let $F \subseteq F_0$ be $(p + q)$-jumping. Then there exists $D \subseteq F$ that is p-jumping, such that if $ab, a'b' \in D$ are distinct then* $\text{dist}_{U\mathcal{P}}(b, b') > q$.

Proof. We will use a modified version of the second half of the proof of 4.2. We say a partial p-augmenting sequence $a_1b_1, \dots, a_sb_s$ is *end-separated* if $\text{dist}_{U\mathcal{P}}(b_i, b_j) > q$ for all distinct $i, j \in \{1, \dots, s\}$. By 4.2 it suffices to show that there is an end-separated p-augmenting sequence in F.

For $1 \le h \le k$, choose $v_h \in V(P_h)$ with $P_h[s_h, v_h]$ maximal such that either $v_h = s_h$ or there is an end-separated partial p-augmenting sequence to v_h in F. For $1 \le h \le k$, let Q_h be the maximal subpath of P_h containing v_h, such that $Q_h \cap P_h[s_h, v_h]$ has length at most p, and $Q_h \cap P_h[v_h, t_h]$ has length at most q. Thus, $Q_1, \dots, Q_k$ is a $(p + q)$-barrier, and so, since F is $(p + q)$-jumping, some $ab \in F$ jumps this barrier. Suppose first that $a \in S \setminus V\mathcal{P}$. If $b \in T \setminus V\mathcal{P}$, then ab is an end-separated p-augmenting sequence, so we assume that $b \in V(P_h)$ for some $h \in \{1, \dots, k\}$. Since ab jumps the barrier, it follows that b is later than v_h in P_h, contradicting the choice of v_h, since ab is an end-separated partial p-augmenting sequence to b in F.

Thus, we may assume that for some $h \in \{1, \dots, k\}$, $a \in V(P_h)$, and a is earlier than each vertex of Q_h in P_h. Let $a_1b_1, \dots, a_sb_s$ be an end-separated partial p-augmenting sequence to v_h in F. Since $a \notin V(Q_h)$, it follows that $Q_h \cap P_h[s_h, v_h]$ has length exactly p, and $P_h[a, v_h]$ has length at least $p+1$. Consequently $a_1b_1, \dots, a_sb_s, ab$ is a partial p-augmenting sequence to b in F. If $b \notin T \setminus V\mathcal{P}$, then, since ab jumps the barrier, there exists $h' \in \{1, \dots, k\}$ such that $b \in P_{h'}[v_{h'}, t_{h'}]$ and $b \ne V(Q_{h'})$; but then $Q_{h'} \cap P_{h'}[v_{h'}, t_{h'}]$ has length exactly q, and so $P_{h'}[v_{h'}, b]$ has length $> q$. Since each b_i in $V(P_{h'})$ belongs to $P_{h'}[s_{h'}, v_{h'}]$ from the definition of $v_{h'}$, it follows that $a_1b_1, \dots, a_sb_s, ab$ is an end-separated p-augmenting sequence to b in F, contrary to the definition of $v_{h'}$. Thus, $b \in T \setminus V\mathcal{P}$, and so $a_1b_1, \dots, a_sb_s, ab$ is an end-separated p-augmenting sequence in F. This proves 4.4. ▮

Let us say a subset $D \subseteq F_0$ is *ℓ-separated* if $\text{dist}_{U\mathcal{P}}(a, a') > \ell$ and $\text{dist}_{U\mathcal{P}}(b, b') > \ell$ for all distinct $ab, a'b' \in D$. We deduce:

4.5 *In the same notation, let $c \ge 0$ be an integer, and let $F \subseteq F_0$ be $5c$-jumping. Then there exists $D \subseteq F$ that is c-jumping and $2c$-separated.*

Proof. This follows from two applications of 4.4: first, to F with $(p, q) = (3c, 2c)$, giving some $3c$-jumping set F'; and then to F' with S, T exchanged and $(p, q) = (c, 2c)$. This proves 4.5. ▮

Now we can obtain something like an analogue of the fourth statement of 4.1:

4.6 *In the same notation, let $c \geq 0$ be an integer, and let $F \subseteq F_0$ be c-jumping and $2c$-separated. Let H be obtained from $U\mathcal{P}$ by adding the remainder of $S \cup T$ as vertices, and the pairs in F as edges. Then there exist $k+1$ vertex-disjoint S-T paths in H, such that no two of them are joined by a path of $U\mathcal{P}$ of length at most c.*

Proof. By 4.3, there is a c-augmenting sequence $a_1b_1, \ldots, a_nb_n$ in F such that:

(1) *For $1 \leq h \leq k$, and $1 \leq i < j \leq n$, if $u \in \{a_i, b_i\} \cap V(P_h)$ and $v \in \{a_j, b_j\} \cap V(P_h)$, then either*

- *u is earlier than v in P_h; or*
- *$b_i = u$ and $v = a_j$ and $P_h[a_j, b_i]$ has length at most c; or*
- *$b_i = u$ and $v = a_j$ and $j = i+1$.*

We deduce:

(2) *Let $1 \leq h \leq k$, and $1 \leq i \leq n$ with $b_i \in V(P_h)$ (and hence $a_{i+1} \in V(P_h)$); then for $1 \leq j \leq n$, if a_j belongs to $P_h[a_{i+1}, b_i]$ then either $j = i+1$, or $j > i+1$ and $P_h[a_j, b_{i+1}]$ has length at most c. Consequently there is at most one value of $j \neq i+1$ with $a_j \in V(P_h[a_{i+1}, b_i])$, and any such j satisfies $j \geq i+2$. Similarly there is at most one value of $j \neq i$ with $b_j \in V(P_h[a_{i+1}, b_i])$, and any such j satisfies $j \leq i-1$.*

By (1), $a_1, \ldots, a_n$ are all distinct, and $b_1, \ldots, b_n$ are all distinct. Suppose that a_j belongs to $P_h[a_{i+1}, b_i]$, and $j \neq i+1$. Thus, a_i is earlier than a_j in P_h. If $j < i$ then setting $u = a_j$ and $v = a_i$ in (1) yields a contradiction; so $i < j$, and hence $j \geq i+2$. By (1) with $u = b_{i+1}$ and $v = a_j$, it follows that $P_h[a_j, b_{i+1}]$ has length at most c. Consequently, if $j' \neq j$ also satisfies that $a_{j'}$ belongs to $P_h[a_{i+1}, b_i]$, and $j' \neq i+1$, then $P_h[a_j, a_{j'}]$ has length at most c, contradicting that F is $2c$-separated. This proves the first assertion of (2), and the second follows from the symmetry. This proves (2).

For $1 \leq i < n$, b_i and a_{i+1} both belong to the same member of $\mathcal{P}$, say P_h; let $R_i = P_h[a_{i+1}, b_i]$.

(3) *Every vertex in $V\mathcal{P}$ belongs to at most two of $R_1, \ldots, R_{n-1}$.*

Suppose that some vertex w of P_h belongs to $R_i, R_{i'}, R_{i''}$, where $i < i' < i''$. Thus,

$$a_i, a_{i'}, a_{i''}, b_i, b_{i'}, b_{i''}$$

are in order in P_h (and are all distinct except possibly $a_{i''} = b_i$), and $w \in P_h[a_{i''}, b_i]$. By (1) with $u = b_i, v = a_{i'}$, $P_h[a_{i'}, b_i]$ has length at most c. But it includes $P_h[a_{i'}, a_{i''}]$ as a subpath, and this has length at least $2c+1$ since F is $2c$-separated, a contradiction. This proves (3).

For $i = 0, 1, 2$, let X_i be the set of edges of $U\mathcal{P}$ that belong to exactly i of $R_1, \ldots, R_{n-1}$. Let H' be the digraph obtained from H as follows:

- direct the edges of $U\mathcal{P}$ such that each $P \in \mathcal{P}$ is a directed path from S to T;
- reverse the direction of all edges in X_2;
- for $1 \leq i \leq n$, direct the edge a_ib_i of H from a_i to b_i; and
- delete all edges in X_1.

We claim:

(4) *Every vertex of H' has outdegree one and indegree one, except for $a_1, s_1, \ldots, s_k$, which have outdegree one and indegree zero, and $t_1, \ldots, t_k, b_n$, which have indegree one and outdegree zero.*

Let $v \in V(H')$. The claim is true for v if $v \in \{a_1, b_n\}$, so we may assume that $v \in V\mathcal{P}$. Let $v \in V(P_h)$ where $1 \leq h \leq k$. Suppose that $v = s_h$, and let e be the edge of P_h incident with v. It follows that $v \neq b_1, \ldots, b_n$, and either $v \neq a_1, \ldots, a_n$ (and then $e \in X_0 \subseteq E(H')$) or $v = a_i$ for some $i < n$ (and then $e \in E(R_i)$, and so $e \in X_1$ and $e \notin E(H')$). In either case v has outdegree one and indegree zero in H'. Thus we may assume that $v \neq s_h$ and similarly $v \neq t_h$, and so v is an internal vertex of P_h. Let e_1, e_2 be the two edges of P_h incident with v, where e_1 is between s_h and v.

If $v \neq \{a_2, \ldots, a_n, b_1, \ldots, b_{n-1}\}$ then v is an internal vertex of the (at most two) paths R_i that contain v, and so v has indegree one and outdegree one in H'. Thus from the symmetry, we may assume that $v \in \{a_2, \ldots, a_n\}$; let $v = a_i$ say. Thus $i \geq 2$ and v is an end of R_{i-1}, and $e_2 \in E(R_{i-1})$.

Suppose next that $v \neq b_1, \ldots, b_{n-1}$. If $e_2 \in X_1$ then $e_2 \notin E(H')$, and $e_1 \in X_0 \subseteq E(H')$, and v therefore has indegree and outdegree one in H'; so we may assume that $e_2 \in X_2$. Let $e_2 \in E(R_j)$ say, where $1 \leq j \leq n-1$ and $j \neq i$. Since $a_j, b_j \neq v$, it follows that $e_1 \in E(R_j)$, and so $e_1 \in X_1$ by (3) applied to v; but then $e_2 \in E(H'), e_1 \notin E(H')$, and the claim is true for v (because v is the head of the directed edge e_2, since $e_2 \in X_2$).

So we may assume that $v \in \{b_1, \ldots, b_{n-1}\}$; let $v = b_j$. Hence $i \neq j$, and v is an end of both R_i, R_j, and both $e_1, e_2 \in X_1$ by (3). Hence $e_1, e_2 \notin E(H')$, and again the claim holds for v. This proves (4).

From (4), each component of $H \setminus X_1$ is either an S-T path or a cycle; and $a_1, s_1, \ldots, s_k$ all belong to different components of $H \setminus X_1$. Consequently there are $k+1$ vertex-disjoint S-T paths $P'_1, \ldots, P'_{k+1}$ in H, each a component of $H \setminus X_1$. It remains to show that no two of these paths are joined by a path of $U\mathcal{P}$ with length at most c. Suppose that Q is such a path; and we can assume that no internal vertex of Q belongs to any of $P'_1, \ldots, P'_{k+1}$. Consequently the first and last edges of Q are not edges of $H \setminus X_1$, and so they belong to X_1. Choose $h \in \{1, \ldots, k\}$ such that Q is a subpath of P_h, with ends u, v say, where u is earlier than v in P_h. Consequently $u, v \in \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$. Let $u \in \{a_i, b_i\}$ and $v \in \{a_j, b_j\}$. Since F is $2c$-separated, not both $u = a_i$ and $v = a_j$, and similarly not both $u = b_i$ and $v = b_j$. So either $u = a_i$ and $v = b_j$, or $u = b_i$ and $v = a_j$.

Suppose first that $u = a_i$ and $v = b_j$. Since $Q = P_h[a_i, b_j]$ has length at most c, and $R_j = P_h[a_{j+1}, b_j]$ has length at least $c+1$, and both a_i, a_{j+1} are earlier than b_j, it follows that R_j contains Q, and similarly R_{i-1} contains Q. In particular, both R_j, R_{i-1} contain the end-edges of Q, which are in X_1, and so $j = i-1$. But then $Q = R_j$ and so has length more than c, a contradiction.

Finally, suppose that $u = b_i$ and $v = a_j$. Since $u \neq v$ and F is $2c$-separated, it follows that $v \notin \{b_1, \ldots, b_n\}$, and so a_jb_j is the only edge in F incident with v. Let e be the edge of $P_h[u, v]$

incident with v. Since $e \in X_1$, there exists $i' \in \{1, \ldots, n\}$ such that $e \in E(R_{i'})$, and therefore $i' \neq j-1$. Consequently $b_{i'}$ is in P_h and later than $v = a_j$ (and therefore later than b_i) in P_h, and so $i' > i$. Moreover, $a_{i'+1}$ is in P_h and earlier than v in P_h. Since F is $2c$-separated, it follows that $P_h[a_{i'+1}, a_j]$ has length more than $2c$, and so $a_{i'+1}$ is also earlier than b_i, and $P_h[a_{i'+1}, b_i]$ has length more than c since $Q = P[b_i, a_j]$ has length at most c. Since $i' \geq i+1$, this contradicts (1) (taking u, v of (1) to be $b_i, a_{i'+1}$ respectively). This proves 4.6. ∎

Finally, here is another lemma we will need:

4.7 *In the same notation, let $a_1b_1, \ldots, a_nb_n$ be a c-augmenting sequence, and let $\mathcal{J}$ be a partition of $\{1, \ldots, n\}$. Then there is a c-augmenting sequence $a'_1b'_1, a'_2b'_2, \ldots, a'_mb'_m$ such that*

- *for $1 \leq i' \leq m$, there exist $J \in \mathcal{J}$ and $i, j \in J$ such that $a'_{i'} = a_i$ and $b'_{i'} = b_j$;*
- *for each $J \in \mathcal{J}$ there is at most one $i \in J$ such that $a_i \in \{a'_1, \ldots, a'_m\}$, and (therefore) at most one $j \in J$ such that $b_j \in \{b'_1, \ldots, b'_m\}$.*

Proof. We proceed by induction on n. We may assume all members of $\mathcal{J}$ are nonempty. If they are all of size one, the result is true, so we may assume that $J_1 \in \mathcal{J}$ has size at least two. Choose $i, j \in J_1$ respectively minimum and maximum; then

$$a_1b_1, \ldots, a_{i-1}b_{i-1}, a_ib_j, a_{j+1}b_{j+1}, \ldots, a_nb_n$$

is a c-augmenting sequence. Let $m = n + i - j$, and define:

$$\begin{aligned}
a'_h &= a_h \text{ for } 1 \leq h \leq i \\
b'_h &= b_h \text{ for } 1 \leq h \leq i-1 \\
a'_h &= a_{h+j-i} \text{ for } i+1 \leq h \leq m \\
b'_h &= b_{h+j-i} \text{ for } i \leq h \leq m
\end{aligned}$$

Define $f(J_1) = \{i\}$, and for each $J \in \mathcal{J} \setminus \{J_1\}$, define

$$f(J) = \{h : 1 \leq h \leq i-1 \text{ and } h \in J\} \cup \{h : i+1 \leq h \leq m \text{ and } h+j-i \in J\}.$$

Then

$$\{f(J) : J \in \mathcal{J} \text{ and } f(J) \neq \emptyset\}$$

is a partition of $\{1, \ldots, m\}$, and the result follows from the inductive hypothesis applied to this partition and $a'_1b'_1, \ldots, a'_mb'_m$. This proves 4.7. ∎

5 The main proof

Now we prove 2.2, which we restate:

5.1 *For all integers $k, c, d \geq 0$ there exist $f(k, c, d), g(k, c, d) \geq 0$, with the following property. Let G be a graph that does not contain a subgraph that is an $f(k, c, d)$-subdivision of the binary tree H_d. Let $S, T \subseteq V(G)$. Then either*

- *there are $k+1$ paths between S, T, pairwise at distance greater than c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex with distance at most $g(k, c, d)$ from some member of X.*

Proof. We proceed by induction on k; the result is trivial for $k = 0$, so we assume that $k \geq 1$, and for all $k' < k$ and all c', the numbers $f(k', c', d), g(k', c', d)$ exist for all nonnegative $k' < k$ and all $c' \geq 0$. (We can keep d fixed.) We could assume that $k \geq 2$ if we wanted, because the result is known to be true for $k = 1$ [1, 5, 8], but there is no need. We are given $c \geq 0$, and we may assume that $c \geq 2$ by increasing c. Choose $c_1, \ldots, c_9$, satisfying:

$$
\begin{aligned}
&c_1 \geq c \\
&c_2 \geq c_1 + c + d(c-1) \\
&c_3 \geq 2(c + c_2) + 2cd \\
&c_4 \geq c_3^2 d \\
&c_5 \geq 5c_4 \\
&c_6 \geq c_3 \\
&c_7 \geq c_6 + 2c_3 d \\
&c_8 \geq \max(c, f(k-1, c^2 d, d), f(k-1, c_7, d)) \\
&c_9 \geq \max(cd, c_2 + c_5, g(k-1, c^2 d, d), g(k-1, c_7, d)).
\end{aligned}
$$

(We suggest that this should be read as just saying that each c_i is much larger than c_{i-1}.) We will show that we may define $f(k, c, d) = c_8$ and $g(k, c, d) = c_9$, and thereby complete the inductive definition.

Now let G be a graph with no subgraph that is a c_8-subdivision of H_d, and let $S, T \subseteq V(G)$. We assume

(1) *There is no X with $|X| \leq k$, such that every path between S, T contains a vertex with distance at most c_9 from some member of X.*

We must therefore show that there are $k+1$ paths between S, T, pairwise at distance more than c. The next step illustrates the power of 3.2.

(2) *We may assume that $S \cap T = \emptyset$.*

Suppose that $r \in S \cap T$. Let A be the set of all vertices with distance at most $c + (d-2)(c-1)$ from r. By 3.2, taking $\ell = c$, there exists $B \subseteq A$ such that

- every vertex in $A \setminus B$ has distance at most $(d-2)(c-1)$ from $V(G) \setminus A$; and consequently every vertex with distance at most c from r belongs to B;
- for all $u, v \in V(G) \setminus B$, if $\text{dist}_G(u, v) \leq c$, then $\text{dist}_{G \setminus B}(u, v) \leq (d-2)c(c-1)$.

From the inductive hypothesis, applied to $G \setminus B$, since $f(k-1, (d-2)c(c-1), d) \leq c_8$, either:

- there are k paths of $G \setminus B$ between S, T, pairwise with distance in $G \setminus B$ more than $(d-2)c(c-1)$; or
- there is a set $X \subseteq V(G) \setminus B$ with $|X| \leq k-1$ such that every path of $G \setminus B$ between S, T contains a vertex with distance in $G \setminus B$ at most $g(k-1, (d-2)c(c-1), d) \leq c_9$ from some member of X.

The second case cannot occur, because otherwise adding r to X gives a set violating (1). Suppose that $P_1, \ldots, P_k$ are paths of $G \setminus B$ as in the first case. Since $\text{dist}_{G \setminus B}(P_i, P_j) > (d-2)c(c-1)$, it follows from the choice of B that $\text{dist}_G(P_i, P_j) > c$ for all distinct $i, j \in \{1, \ldots, k\}$, and adding the one-vertex path with vertex r gives a set of $k+1$ paths satisfying the theorem. This proves (2).

An S-T path P is *near-geodesic* if for all $u, v \in V(P)$, either $\text{dist}_P(u, v) \leq (d-2)c_3(c_3-1)$ or $\text{dist}_G(u, v) > c_3$. We claim that

(3) *There are k S-T paths in G pairwise with distance more than c_6, each near-geodesic.*

From the inductive hypothesis, since $c_8 \geq f(k-1, c_7, d)$ and $c_9 \geq g(k-1, c_7, d)$, there are k S-T paths $P_1, \ldots, P_k$, pairwise with distance more than c_7. Let $A = V(G) \setminus (V(P_1) \cup \cdots \cup V(P_k))$. By 3.2, taking $\ell = c_3$, there exists $B \subseteq A$ such that

- every vertex in $A \setminus B$ has distance at most $(d-2)(c_3-1)$ from $V(G) \setminus A$;
- for all $u, v \in V(G) \setminus B$, if $\text{dist}_G(u, v) \leq c_3$, then $\text{dist}_{G \setminus B}(u, v) \leq (d-2)c_3(c_3-1)$.

For $1 \leq i \leq k$, there is a path in $G \setminus B$ between the ends of P_i, since P_i is such a path. Let P'_i be a shortest such path. If $u, v \in V(P'_i)$ with $\text{dist}_{P'_i}(u, v) > (d-2)c_3(c_3-1)$, then $\text{dist}_{G \setminus B}(u, v) > (d-2)c_3(c_3-1)$, and so $\text{dist}_G(u, v) > c_3$, that is, P'_i is near-geodesic, for $1 \leq i \leq k$.

For each $v \in V(P'_i)$, since $v \notin B$, it follows that v has distance at most $(d-2)(c_3-1)$ from $V(G) \setminus A$, that is, from some P_j, say $Q(v)$. If $u, v \in V(P_i)$ are adjacent, then

$$\text{dist}_G(Q(u), Q(v)) \leq 2(d-2)(c_3-1) + 1 \leq c_7,$$

and so $Q(u) = Q(v)$ since $P_1, \ldots, P_k$ pairwise have distance more than c_7. Since $Q(v) = P_i$ when v is an end of P_i, it follows that $Q(v) = P_i$ for all $v \in V(P_i)$, that is, every vertex in P'_i has distance at most $(d-2)(c_3-1)$ from P_i. Consequently, $P'_1, \ldots, P'_k$ pairwise have distance more than $c_7 - 2(d-2)(c_3-1) \geq c_6$. This proves (3).

Fix S-T paths $P_1, \ldots, P_k$, each near-geodesic and pairwise with distance more than c_6, and we may choose them such that no internal vertex of P_h belongs to $S \cup T$ for $1 \leq h \leq k$. Let $\mathcal{P} = \{P_1, \ldots, P_k\}$. Let P_h have ends $s_h \in S$ and $t_h \in T$.

For $p \geq 1$, let V_p be the set of vertices with distance more than p from $V\mathcal{P}$. Let L be a path of G with ends a, b. We say:

- L is a a *leap of type 1* if $a, b \in V\mathcal{P}$, and there exist $x, y \in V(L)$ with a, x, y, b in order, such that the subpaths $L[a, x], L[b, y]$ have length exactly c_2, and every internal vertex of $L[x, y]$ belongs to V_{c_2}. (It follows that $L[a, x]$ is an $(x, V\mathcal{P})$-geodesic, and $L[b, y]$ is a $(y, V\mathcal{P})$-geodesic.)

- L is a *leap of type 2* if $a \in V\mathcal{P}$, $b \in (S \cup T) \cap V_{c_2}$, and there exists $x \in V(L)$ such that $L[a, x]$ has length c_2, and every internal vertex of $L[x, b]$ belongs to V_{c_2}.
- L is a *leap of type 3* if $a \in V\mathcal{P}$, $b \in (S \cup T) \setminus V_{c_2}$, and L is a $(b, V\mathcal{P})$-geodesic.
- L is a *leap of type 4* if $a \in S$ and $b \in T$ and $V(L) \subseteq V_{c_2}$.

A *leap* is a leap of type 1, 2, 3 or 4.

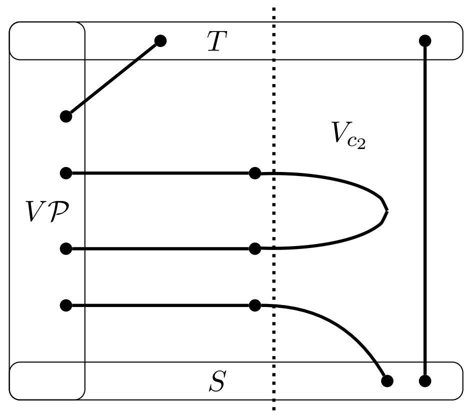

Figure 2: The four types of leaps. (The thick lines represent paths.)

Let F be the set of all ordered pairs uv such that some leap has ends u, v. (Thus, if $ab \in F$ then $ba \in F$.)

(4) *F is c_5-jumping in the setting $(S, T, \mathcal{P} = \{P_1, \dots, P_k\})$.*

For $1 \leq i \leq k$ let Q_i be a subpath of P_i of length at most c_5; thus, $Q_1, \dots, Q_k$ is a c_5-barrier in the stated setting. We may assume (by extending Q_h) that for $1 \leq h \leq k$, either $Q_h = P_h$ or Q_h has length exactly c_5. For $1 \leq h \leq k$, $P_h \setminus V(Q_h)$ has at most two components. If one of them contains s_h, call it A_h, and otherwise let A_h be the null graph; and if one contains t_h call it B_h, and otherwise B_h is null. Choose $q_h \in V(Q_h)$ for $1 \leq h \leq k$. Let X be the set of vertices v of G with $\text{dist}_G(v, A_1 \cup \dots \cup A_k) \leq c_2$ and $\text{dist}_G(v, \{q_1, \dots, q_k\}) > c_9$; and let Y be the set of v with $\text{dist}_G(v, B_1 \cup \dots \cup B_k) \leq c_2$ and $\text{dist}_G(v, \{q_1, \dots, q_k\}) > c_9$.

Suppose that there exists $v \in X \cap Y$; then $\text{dist}_G(A_1 \cup \dots \cup A_k, B_1 \cup \dots \cup B_k) \leq 2c_2$. Choose $i, j \in \{1, \dots, k\}$ such that $\text{dist}_G(A_i, B_j) \leq 2c_2$. Since $\text{dist}_G(P_i, P_j) > c_6 \geq 2c_2$ for all distinct i, j, it follows that $i = j$. Hence there are vertices $u, v \in P_i$, such that $\text{dist}_{P_i}(u, v) \geq c_5 + 2$ and yet $\text{dist}_G(u, v) \leq 2c_2$, contradicting that P_i is near-geodesic, since $2c_2 \leq c_3$ and $c_5 + 2 > (d-2)c_3(c_3 - 1)$. This proves that $X \cap Y = \emptyset$.

We claim that for each $y \in Y$, every $(y, V\mathcal{P})$-geodesic is a $(y, B_1 \cup \dots \cup B_k)$-geodesic. Let J be a $(y, V\mathcal{P})$-geodesic, and let b be its end in $V\mathcal{P}$. Then $b \notin V(A_1 \cup \dots \cup A_k)$ since $y \notin X \cap Y$, and $b \notin V(Q_1 \cup \dots \cup Q_k)$ since $\text{dist}_G(y, \{q_1, \dots, q_k\}) > c_9 \geq c_2 + c_5$ and J has length at most c_2 and $Q_1, \dots, Q_k$ all have length at most c_5. Thus $b \in V(B_1 \cup \dots \cup B_k)$ and so J is a $(y, B_1 \cup \dots \cup B_k)$-geodesic as claimed. Similarly, for each $x \in X$, every $(x, V\mathcal{P})$-geodesic is an $(x, A_1 \cup \dots \cup A_k)$-geodesic.

If $S \cap Y \neq \emptyset$, let $s \in S \cap Y$ and let J be an $(s, V\mathcal{P})$-geodesic, and let $b \in B_1 \cup \cdots \cup B_k$ be the end of J in $V\mathcal{P}$. Then J is a leap of type 3, and so $sb \in F$ jumps the c_5-barrier $Q_1, \ldots, Q_k$. Thus, we may assume that $S \cap Y = \emptyset$, and similarly $T \cap X = \emptyset$. Since $S \cap T = \emptyset$ by (2), this proves that $S \cup X$ is disjoint from $T \cup Y$.

From (1), applied to the set $\{q_1, \ldots, q_k\}$, there is an S-T path P in G such that

$$\mathrm{dist}_G(P, \{q_1, \ldots, q_k\}) > c_9.$$

Consequently, for each vertex $v \in V(P) \setminus (X \cup Y)$, $\mathrm{dist}_G(v, Q_1 \cup \cdots \cup Q_k) > c_9 - c_5 \geq c_2$, and $\mathrm{dist}_G(v, A_1 \cup \cdots \cup A_k) > c_2$ (since $v \notin X$), and similarly $\mathrm{dist}_G(v, B_1 \cup \cdots \cup B_k) > c_2$; so $v \in V_{c_2}$, and therefore $V(P) \subseteq X \cup Y \cup V_{c_2}$. Since P has first vertex in $S \cup X$ and last vertex in $T \cup Y$, there is a subpath Q of P with one end some $x \in S \cup X$, the other end some $y \in T \cup Y$, and with no internal vertex in $X \cup Y \cup S \cup T$. Thus, $x \neq y$, and all internal vertices of Q belong to V_{c_2}. If $x \in S \setminus X$ and $y \in T \setminus Y$, then Q is a leap of type 4 and $xy \in F$ jumps the c_5-barrier; so from the symmetry we may assume that $x \in X$. Let J_x be an $(x, V\mathcal{P})$-geodesic, with ends x and $a \in V(A_1 \cup \cdots \cup A_k)$. Thus, J_x has length c_2, since $x \in X$ and has a neighbour in V_{c_2}. If $y \in T \setminus Y$, then $Q \cup J_x$ is a leap of type 2, and $ay \in F$ jumps the c_5-barrier. Thus, we may assume that $y \in Y$; let J_y be a $(y, V\mathcal{P})$-geodesic, with ends y, b where $b \in V(B_1 \cup \cdots \cup B_k)$. Then $Q \cup J_x \cup J_y$ is a leap of type 1, and $ab \in F$ jumps the c_5-barrier. This proves (4).

From 4.5, and 4.2, there is a c_4-augmenting, $2c_4$-separated, sequence $a_1b_1, \ldots, a_nb_n$ in F. Let $W = \{a_2, \ldots, a_n, b_1, \ldots, b_{n-1}\}$. Thus, $W \subseteq V\mathcal{P}$. Let us say distinct $u, v \in W$ are *mated* if $\mathrm{dist}_{U\mathcal{P}}(u, v) \leq c_4$. It follows that if such u, v are mated, then one of u, v is in $\{a_2, \ldots, a_n\}$ and the other is in $\{b_1, \ldots, b_{n-1}\}$, because $a_1b_1, \ldots, a_nb_n$ is $2c_4$-separated; and for the same reason each vertex in W is mated with at most one other vertex in this set.

(5) *If distinct $u, v \in W$ are not mated, then* $\mathrm{dist}_G(u, v) > c_3$.

Suppose that $u, v \in W$, and $\mathrm{dist}_G(u, v) \leq c_3$. Consequently $u, v \in V(P_h)$ for some $h \in \{1, \ldots, k\}$, since $u, v \in V\mathcal{P}$ and $c_3 \leq c_6$. Since P_h is near-geodesic,

$$\mathrm{dist}_{P_h}(u, v) \leq (d-2)c_3(c_3 - 1) \leq c_4,$$

and so u, v are mated. This proves (5).

Some notation: if P is a path of G and $X \subseteq V(G)$, we write $P[X]$ for $P[V(P) \cap X]$. For $1 \leq i \leq n$ choose a leap L_i with ends a_i, b_i. If some L_i has type 4 (and hence $i = n = 1$), then $P_1, \ldots, P_k, L_i$ are S-T paths satisfying the theorem, since $c_2 \geq c$ and $c_6 \geq c$; so we may assume that each L_i has type 1, 2 or 3. Thus, L_1, L_n have types 2 or 3, and all the others have type 1.

For each $w \in W$, let $S(w)$ be the maximal subpath of L_i with one end w and with length at most c_2. Thus, $S(w)$ has length c_2 unless $w \in \{b_1, a_n\}$. Let $S'(w) = S(w)[V_{c_1}]$ (thus, if $S(w)$ has length at most c_1 then $S'(w)$ is the null graph). Let $S''(w) = S(w) \setminus V_{c_1}$, and let the ends of $S''(w)$ be $w, s(w)$. For $1 \leq i \leq n$, let $R_i = L_i[V_{c_1}]$. Thus, R_i is a path unless L_i is a leap of type 3 and has length at most c_1, and then R_i is null.

We need to be careful with L_1, L_n. There are three possibilities for L_n (and the same for L_1):

- L_n is a leap of type 2;

- L_n is a leap of type 3 and has length more than c_1;
- L_n is a leap of type 3 and has length at most c_1.

(See Figure 3.) Note that, in the second case when L_n has length more than c_1, since L_n is a $(b_n, V\mathcal{P})$-geodesic it follows that $V(L_n) \subseteq V(S(a_n)) \cup V_{c_1}$, and so R_n joins b_n and a neighbour of $s(a_n)$.

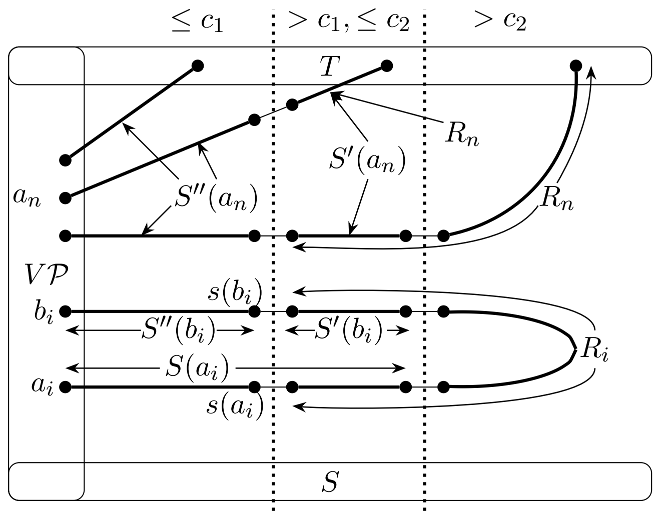

Figure 3: Definitions of $R_i, S(w), S'(w), S''(w)$ and $s(w)$.

For each mated pair $u, v \in W$, if there is a path in G of length at most c between $S'(u), S'(v)$, with all vertices in V_{c_1}, choose some such path and call it T_{uv}. Let

$$Z = \bigcup (V(R_i) : 1 \leq i \leq n) \cup \bigcup (V(T_{uv}) : u, v \in W \text{ are mated}).$$

Thus, $Z \subseteq V_{c_1}$.

Choose t maximum such that there is a sequence of paths $M_1, \ldots, M_t$ satisfying, for each $i \geq 1$:

- M_i has length at most c;
- the ends of M_i lie in different components of $G[Z \cup V(M_1 \cup \cdots \cup M_{i-1})]$, and none of its internal vertices lie in this set; and
- M_i intersects at most one of the paths $S'(w)$ $(w \in W)$.

We claim:

(6) *Every vertex in $M_1 \cup \cdots \cup M_t$ has distance at most $d(c-1)$ from V_{c_2}.*

Let $x \in V(M_1 \cup \cdots \cup M_t)$, and suppose that $\text{dist}_G(x, V_{c_2}) > d(c-1)+1$. Thus, x has distance at most one from a vertex $x' \in V(M_1 \cup \cdots \cup M_t) \setminus Z$, and by 3.3, taking $\ell = c$, either x' is in the interior of some M_i with both ends in Z, or there are three components of $G[Z]$ such that x' has distance at most $d(c-1)$ from each of them.

In the first case, let M_i have ends x_1, x_2. Then $V(M_i) \cap V_{c_2} = \emptyset$, since M_i has length at most c and $\text{dist}_G(x', V_{c_2}) > d(c-1) \geq c$; but then, for $j = 1, 2$, x_j belongs either to $S'(w)$ for some $w \in W$, or to T_{vw} for some mated pair $v, w \in W$. Consequently, for $j = 1, 2$, there exists $w_j \in W$ with distance at most $c + c_2$ from x_j, such that $S'(w_1), S'(w_2)$ belong to different components of Z. In particular $w_1 \neq w_2$, and since M_i has length at most c, it follows that

$$\text{dist}_G(w_1, w_2) \leq 2(c + c_2) + c \leq c_3,$$

and so w_1, w_2 are mated by (5). If there exists $w \in W$ such that w, w_1 are mated and x_1 belongs to $T_{w_1 w}$, then $w = w_2$ since w_1, w_2 are mated, contradicting that $S'(w_1), S'(w_2)$ belong to different components of Z. Hence $x_1 \in S'(w_1)$, and similarly $x_2 \in S'(w_2)$, contrary to the assumption that M_i intersects at most one of the paths $S'(w)$ $(w \in W)$.

Thus, x' is not in the interior of some M_i with both ends in Z; and so there are three components C_1, C_2, C_3 of $G[Z]$ such that x' has distance at most $d(c-1)$ from each of them. For $i = 1, 2, 3$, let N_i be a path from x' to C_i of length at most $d(c-1)$, and let x_i be the end of N_i in C_i. Since $\text{dist}_G(x', V_{c_2}) > d(c-1)$, each of these paths is disjoint from V_{c_2}. In particular, for $i = 1, 2, 3$, there exists $w_i \in W$ with distance at most $c + c_2$ from x_i, such that $S'(w_1), S'(w_2), S'(w_3)$ all belong to different components of $G[Z]$. Therefore, some two of w_1, w_2, w_3 are not mated, say w_1, w_2; but

$$\text{dist}_G(w_1, w_2) \leq 2(c + c_2) + 2d(c-1) \leq c_3,$$

contrary to (5). This proves (6).

Let $\mathcal{D}$ be the set of components of $G[Z \cup V(M_1 \cup \cdots \cup M_t)]$. For each $D \in \mathcal{D}$, let W_D be the union of the sets $\{a_i, b_i\} \cap W$, over all $i \in \{1, \ldots, n\}$ such that R_i is a non-null subgraph of D. The sets W_D $(D \in \mathcal{D})$ are nonempty and pairwise disjoint, and their union includes $W \setminus \{b_1, a_n\}$, and it might include b_1, a_n as well. If $D \in \mathcal{D}$, let D^+ be the union of D and the paths $S(w)$ with $w \in W_D$. (Incidentally, even if $D_1, D_2 \in \mathcal{D}$ are distinct and therefore disjoint, it is possible that D_1^+, D_2^+ might intersect, because there might exist $w_i \in W_{D_i}$ for $i = 1, 2$ such that $S''(w_1), S''(w_2)$ intersect. But then w_1, w_2 would be mated.)

For $D \in \mathcal{D}$, we say $v \in V(D^+)$ is *innocuous in* D^+ if $v \in V(S(w))$ for some $w \in W_D$, and $S(w)[v, w]$ has length at most $c_1 + c$, and for each vertex y of $S(w)[v, w]$, all edges of D^+ incident with y belong to $S(w)$. We claim:

(7) *If $D_1, D_2 \in \mathcal{D}$ are different, and M is a path of length at most c in G between D_1^+, D_2^+, then for $i = 1, 2$, the end of M in D_i^+ is innocuous in D_i^+.*

Let M have ends $x_i \in D_i^+$ for $i = 1, 2$. Suppose first that $\text{dist}_G(M, V\mathcal{P}) > c_1$, and let M' be a minimal subpath of M that has nonempty intersection with two of the graphs D^+ $(D \in \mathcal{D})$ (and therefore with two members of $\mathcal{D}$). Let the ends of M' be $x'_1 \in D'_1$ and $x'_2 \in D'_2$. From the maximality of t in the definition of $M_1, \ldots, M_t$, it follows that M' intersects at least two of the paths $S'(w)$ $(w \in W)$, and therefore, for $i = 1, 2$, there exists $w_i \in W_{D'_i}$ such that x'_i belongs to $S'(w_i)$. Thus, $\text{dist}_G(w_1, w_2) \leq 2c_2 + c \leq c_3$, and so w_1, w_2 are mated by (5). Since $D'_1 \neq D'_2$, and $\text{dist}_G(M, V\mathcal{P}) > c_1$, it follows that $T_{w_1 w_2}$ exists, contradicting that $D'_1 \neq D'_2$.

Consequently, $\mathrm{dist}_G(M, V\mathcal{P}) \le c_1$, and so $\mathrm{dist}_G(x_i, V\mathcal{P}) \le c_1 + c$, and therefore $\mathrm{dist}_G(x_i, V_{c_2}) > d(c-1)$ since $c_2 \ge c_1 + c + d(c-1)$. for $i = 1, 2$. By (6), x_i is in none of $M_1, \ldots, M_t$, and so either $x_i \notin D_i$, or there exists $w_i \in W_{D_i}$ with $x_i \in V(S(w_i))$, or there is a mated pair $w_i, w_i' \in W_{D_i}$ with $x_i \in T_{w_i w_i'}$. In each case, it follows that for some $w_i \in W_{D_i}$, either $x_i \in S(w_i)$, or $x_i \in V(T_{w_i w_i'})$ for some $w_i' \in W_{D_i}$ such that w_i, w_i' are mated, for $i = 1, 2$. Since $\mathrm{dist}_G(x_i, w_i) \le c + c_2$, it follows that $\mathrm{dist}_G(w_1, w_2) \le 3c + 2c_2 \le c_3$, and so w_1, w_2 are mated by (5). Thus, w_1', w_2' do not exist, and so $x_1 \in S(w_1)$ and $x_2 \in S(w_2)$.

Since $\mathrm{dist}_G(x_i, V\mathcal{P}) \le c_1 + c$, it follows that the subpath of $S(w_i)$ between x_i, w_i has length at most $c_1 + c$, because it is an $(x_i, V\mathcal{P})$-geodesic. Let y be a vertex of this subpath, and let e be an edge of D_i^+ incident with y. To show that x_i is innocuous in D_i^+, it remains to show that e is an edge of $S(w_i)$, for all such y, e. We may assume that $i = 1$. Since $\mathrm{dist}_G(y, w_1) \le c_1 + c$, it follows that $\mathrm{dist}_G(y, V_{c_2}) > c_2 - c_1 - c \ge d(c-1)$, and so y belongs to none of $M_1, \ldots, M_t$, by (6). Thus, e is an edge of $D_i^+[Z]$, and so either

- there exists $w \in W_{D_1}$ with $e \in E(S(w))$; or
- there is a mated pair $w, w' \in W_{D_1}$ such that $T_{ww'}$ exists and e is an edge of $T_{ww'}$.

In either case $w \in W_{D_1}$, and $\mathrm{dist}_G(w, w_1) \le (c_1 + c) + (c_2 + c) \le c_3$. Now $w \ne w_2$ since $w_2 \notin W_{D_1}$, and yet w, w_1 are not mated since w_1, w_2 are mated. By (5), $w = w_1$; and therefore w' does not exist, and so $e \in E(S(w_1))$. This proves that x_1 is innocuous, and so proves (7).

Each $D \in \mathcal{D}$ includes at least one of the paths R_i. For each $D \in \mathcal{D}$, let J_D be the set of $i \in \{1, \ldots, n\}$ such that $R_i \subseteq D$. We would like to apply 4.7 to the set of sets $\{J_D : D \in \mathcal{D}\}$, but it might not be a partition of $\{1, \ldots, n\}$. Certainly its union contains $\{2, \ldots, n-1\}$, but we have to be careful about $1, n$. There is no $D \in \mathcal{D}$ with $1 \in J_D$ if and only if L_1 is a leap of type 3 of length at most c_1; and the same for n, L_n. Let $\mathcal{J}$ be the partition of J formed by the sets $\{J_D : D \in \mathcal{D}\}$, together with $\{1\}$ if L_1 is a leap of type 3 of length at most c_1, and $\{n\}$ if L_n is a leap of type 3 of length at most c_1. The sequence $a_1b_1, \ldots, a_nb_n$ is c_4-augmenting and $2c_4$-separated; and by applying 4.7 to the partition $\mathcal{J}$ and this sequence, we deduce that there is a c_4-augmenting, $2c_4$-separated sequence $p_1q_1, \ldots, p_mq_m$ such that:

- $p_1, \ldots, p_m \in \{a_1, \ldots, a_n\}$ and $q_1, \ldots, q_m \in \{b_1, \ldots, b_n\}$;
- for $1 \le i \le m$, either:
  - $S'(p_i) \cup S'(q_i)$ is non-null, and there exists $D \in \mathcal{D}$ such that $S'(p_i) \cup S'(q_i) \subseteq D$; or
  - $i = 1$, and L_1 is a leap of type 3 with length at most c_1, and $(p_1, q_1) = (a_1, b_1)$, or
  - $i = m$, and L_n is a leap of type 3 with length at most c_1, and $(p_m, q_m) = (a_n, b_n)$;

  and
- $D_2, \ldots, D_{m-1}$ and (if they exist) D_1, D_m are all different.

To see this, observe that $p_1 \in S \setminus V\mathcal{P}$, and $p_1 \in \{a_1, \ldots, a_n\}$, and therefore $p_1 = a_1$, and so if $\{1\} \in \mathcal{J}$ then $(p_1, q_1) = (a_1, b_1)$; and similarly if $\{m\} \in \mathcal{J}$ then $(p_m, q_m) = (a_n, b_n)$.

We recall that for $w \in W$, $S''(w)$ is the subpath of $S(w)$ between w and $s(w)$, of length c_1 unless $S(w)$ has length less than c_1. For $w \in \{a_1, b_n\}$ let us define $S''(w)$ to be the one-vertex path with vertex w, and $s(w) = w$. For $1 \le i \le m$, if D_i exists (which it does unless $i \in \{1, m\}$) let Q_i be a path between $s(p_i), s(q_i)$ with interior in $V(D_i)$. If D_1 does not exist, let $Q_1 = L_1$ (in this case, L_1 has length at most c_1 and joins $s(p_1) = a_1$ and $s(q_1) = b_1$). Similarly if D_m does not exist let $Q_m = L_n$.

(8) *For all distinct $i, j \in \{1, \dots, m\}$, if the distance in G between $S''(p_i) \cup Q_i \cup S''(q_i)$ and $S''(p_j) \cup Q_j \cup S''(q_j)$ is at most c, then one of $\{p_i, q_i\}$ is mated with one of $\{p_j, q_j\}$.*

Let M be a path of length at most c with ends x_i, x_j, where $x_i \in V(S''(p_i) \cup Q_i \cup S''(q_i))$ and $x_j \in V(S''(p_j) \cup Q_j \cup S''(q_j))$. By (7), x_i is innocuous in D_i^+ and x_j is innocuous in D_j^+. Choose $w_i \in W_{D_i}$ with $x_i \in V(S(w_i))$, and choose $w_j \in W_{D_j}$ similarly. Since $S''(p_i) \cup Q_i \cup S''(q_i)$ is a path in D_i^+ containing x_i with both ends in $W \cup \{a_1, b_n\}$, and for each vertex y of $S(w_i)[x_i, w_i]$, all edges of D_i^+ incident with y belong to $S(w_i)$, it follows that $S(w_i)[x_i, w_i]$ is a subpath of $S''(p_i) \cup Q_i \cup S''(q_i)$, and therefore w_i is one of p_i, q_i. Since $S(w_i)[x_i, w_i]$ has length at most $c_1 + c$, and the same for x_j, and therefore $\text{dist}_G(w_i, w_j) \le 2c_1 + 3c \le c_3$, it follows from (5) that w_i, w_j are mated. This proves (8).

But now the result follows from 4.6 applied to $p_1q_1, \dots, p_mq_m$, replacing each pair p_iq_i in the resulting paths by $S''(p_i) \cup Q_i \cup S''(q_i)$. Let us see this in more detail. Let $F = \{p_1q_1, \dots, p_mq_m\}$, and let H be the graph obtained from $U\mathcal{P}$ by adding the remainder of $S \cup T$ as vertices, and the ordered pairs in F as (undirected) edges. Since F is c_4-jumping (by 4.2) and $2c_4$-separated, we deduce from 4.6 that there exist $k+1$ vertex-disjoint S-T paths $Z_1, \dots, Z_{k+1}$ in H, such that no two of them are joined by a subpath of $U\mathcal{P}$ of length at most c_4. Each Z_s is a concatenation of subpaths of $U\mathcal{P}$ and edges p_iq_i.

For $1 \le s \le k+1$, let F_s be the set of pairs in F that are edges of Z_s. Thus, $Z_s \setminus F_s$ is a subgraph of G, and each of its components is a subpath of a member of $\mathcal{P}$.

(9) *If $s, t \in \{1, \dots, k+1\}$ are distinct, then $\text{dist}_G(V(Z_s), V(Z_t)) > c_3$.*

Suppose not; then there exist $x \in V(Z_s)$ and $y \in V(Z_t)$ with $\text{dist}_G(x, y) \le c_3$. Since $x, y \in V\mathcal{P}$, with distance at most c_3, and $c_3 \le c_6$, both x, y belong to the same member of $\mathcal{P}$, say P_h. Since $\text{dist}_G(x, y) \le c \le c_3$ and P_h is near-geodesic, it follows that $\text{dist}_{P_h}(x, y) \le (d-2)c_3(c_3 - 1) \le c_4$. But Z_s, Z_t are not joined by a subpath of $U\mathcal{P}$ of length at most c_4, a contradiction. This proves (9).

For each $s \in \{1, \dots, k+1\}$, let Y_s be the union of $Z_s \setminus F_s$ and the path $S''(p_i) \cup Q_i \cup S''(q_i)$ for each pair $p_iq_i \in F_s$. Then Y_s is a connected subgraph of G, containing a vertex in S and a vertex in T.

(10) *$Y_1, \dots, Y_{k+1}$ pairwise have distance more than c.*

Suppose that $s, t \in \{1, \dots, k+1\}$ are distinct, and there exist $x \in V(Y_s)$ and $y \in V(Y_t)$ such that $\text{dist}_G(x, y) \le c$. By (9), it is not the case that $x \in V(Z_s \setminus F_s)$ and $y \in V(Z_t \setminus F_t)$, so we may assume that $y \notin V(Z_t \setminus F_t)$. Choose $p_jq_j \in F_t$ such that $y \in V(S''(p_j) \cup Q_j \cup S''(q_j))$.

Suppose that $x \notin V(Z_s \setminus F_s)$. Then $x \in V(S''(p_i) \cup Q_i \cup S''(q_i))$ for some $p_iq_i \in F_s$. From (8), some $w_i \in \{p_i, q_i\}$ is mated with some $w_j \in \{p_j, q_j\}$. Hence w_i, w_j belong to the same member of $\mathcal{P}$, say P_h, and $\text{dist}_{P_h}(w_i, w_j) \le c_4$. Yet $w_i \in V(Z_s)$ and $w_j \in V(Z_t)$, contradicting that Z_s, Z_t are not joined by a subpath of $U\mathcal{P}$ of length at most c_4.

So $x \in V(Z_s \setminus F_s)$. Since $x \in V\mathcal{P}$, $\text{dist}_G(x, Q_j) > c_1$ and so $y \notin V(Q_j)$; and so $y \in V(S''(w))$ for some $w \in \{p_j, q_j\}$. Since $S''(w)$ is a $(w, V\mathcal{P})$-geodesic, and $x, w \in V\mathcal{P}$ and $w \in V(S''(w))$, it follows that

$$\text{dist}_G(y, w) = \text{dist}_G(y, V\mathcal{P}) \leq \text{dist}_G(y, x) \leq c,$$

and therefore $\text{dist}_G(x, w) \leq 2c \leq c_3$; but $x \in V(Z_s)$ and $w \in V(Z_t)$, contrary to (9). This proves (10).

From (10), this proves 5.1. ∎

6 Concluding remarks

In the form given in 5.1, our main result involves two functions $f(k, c, d)$ and $g(k, c, d)$. How do they depend on c? The counterexamples in [8] contain large uniform binary trees, not subdivided at all, so one might even hope that taking $f(k, d, c) = 1$ works. The last is false, because subdividing every edge of one of our counterexamples gives another counterexample with k, d the same but c doubled, and now there is no large binary tree as a subgraph.

In fact $f(k, d, c), g(k, d, c)$ must both be at least linear in c. This is a little vague, because there are two functions f, g involved, and we can trade off between them, but we can make it precise as follows. Let us fix $k, d \geq 2$, and say a triple (p, q, c) of integers *works* if for every graph G with no subgraph that is a p-subdivision of the binary tree H_d, and all $S, T \subseteq V(G)$, either

- there are $k + 1$ paths between S, T, pairwise at distance greater than c; or
- there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex with distance at most q from some member of X.

For each c, let $p(c)$ be minimum such that $(p(c), q, c)$ works for some q. Suppose that $p(nc) < np(c)$ for some integer $n \geq 1$, and choose q such that $(p(nc), q, nc)$ works. From the minimality of $p(c)$, $(p(c) - 1, q, c)$ does not work, and so there is a graph G and S, T that show that $(p(c) - 1, q, c)$ does not work. If we replace every edge of G by a path of length n, we obtain a graph G' and S, T, that show that $(np(c) - 1, q, nc)$ does not work, a contradiction. So $p(nc) \geq np(c)$ for all $n \geq 1$. Similarly, if we define $q(c)$ to be minimum such that $(p, q(c), c)$ works for some p, the same construction shows that $q(nc) > n(q(c) - 1)$ for all integers $n \geq 1$.

One would think that $f(k, c, d)$ and $g(k, c, d)$ should be linear in c. Our proof gives functions $f(k, c, d), g(k, c, d)$ that are both non-linear in c; polynomial, but at least something like c^{2^k}, because of the condition $c_4 \geq c_3^2 d$, which is iterated every time we increase k by 1. We only use that condition to apply 3.2, and if we could find a linear way through 3.2, the rest of the proof would show that $f(k, c, d), g(k, c, d)$ are both linear in c.

What about infinite graphs? We assumed that all our graphs were finite at the start of the paper, but augmenting path arguments work fine in infinite graphs (provided we only want some finite number of paths), and the only place in the proof that we used finiteness was in the section on the "key lemma", where we had to show that the process of adding bites stopped; and similarly, in the choice of $M_1, \ldots, M_t$ with t maximum just before step (6) of the main proof. An easy application of

Zorn's lemma would do instead, so in fact our theorem works for infinite graphs. (And "path-width" needs to be replaced by "line-width" for infinte graphs: see [10] for example.)

And for free, we can get a strengthening to graphs with "bounded coarse path-width". A (p, q)-*path-decomposition* of G is a family $(B_t : t \in L)$ of subsets of $V(G)$, where L is a linearly ordered set, such that

- $\bigcup_{t \in L} G[B_t] = G$;
- for all $t_1, t_2, t_3 \in T$, if $t_1 \leq t_2 \leq t_3$ (where $\leq$ is the linear order on L) then $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$; and
- for each $t \in L$, B_t is the union of at most p subsets each with diameter in G at most q.

A class of graphs has *bounded coarse path-width* if there are p, q such that every graph in the class has a (p, q)-path-decomposition (see [11] for a coarse structural characterization of graphs with bounded coarse path-width).

We showed in [9] that for all p, q, there exist ℓ, c such that every graph that admits a (p, q)-path-decomposition also admits an (ℓ, c)-quasi-isometry to a graph of path-width at most p. (See [9] for definitions.) So we can strengthen our theorem, since its conclusion is invariant under taking quasi-isometries, and obtain that the coarse Menger conjecture is true for graphs in any class with bounded coarse path-width:

6.1 *Let $k \geq 0$ and $c, p, q \geq 1$ be integers. Then there exists $\ell \geq 0$, such that for every graph G with a (p, q)-path-decomposition, and all $S, T \subseteq V(G)$, either:*

- *there are $k + 1$ paths between S, T, pairwise at distance at least c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex with distance at most ℓ from some member of X.*

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