

# SURFACES HAVE (ASYMPTOTIC) DIMENSION 2

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ABSTRACT. The asymptotic dimension is an invariant of metric spaces introduced by Gromov in the context of geometric group theory. When restricted to graphs and their shortest paths metric, the asymptotic dimension can be seen as a large scale version of weak diameter colorings (also known as weak diameter network decompositions), i.e. colorings in which each monochromatic component has small weak diameter.

In this paper, we prove that for any  $p$ , the class of graphs excluding  $K_{3,p}$  as a minor has asymptotic dimension at most 2. This implies that the class of all graphs embeddable on any fixed surface (and in particular the class of planar graphs) has asymptotic dimension 2, which gives a positive answer to a recent question of Fujiwara and Papasoglu. Our result extends from graphs to Riemannian surfaces. We also prove that graphs of bounded pathwidth have asymptotic dimension at most 1 and graphs of bounded layered pathwidth have asymptotic dimension at most 2. We give some applications of our techniques to graph classes defined in a topological or geometrical way, and to graph classes of polynomial growth. Finally we prove that the class of bounded degree graphs from any fixed proper minor-closed class has asymptotic dimension at most 2. This can be seen as a large scale generalization of the result that bounded degree graphs from any fixed proper minor-closed class are 3-colorable with monochromatic components of bounded size. This also implies that (infinite) Cayley graphs avoiding some minor have asymptotic dimension at most 2, which solves a problem raised by Ostrovskii and Rosenthal.

## 1. INTRODUCTION

1.1. **Asymptotic dimension.** Let  $(X, d)$  be a metric space, and let  $\mathcal{U}$  be a family of subsets of  $X$ . We say that  $\mathcal{U}$  is  $D$ -bounded if each set  $U \in \mathcal{U}$  has diameter at most  $D$ . We say that  $\mathcal{U}$  is  $r$ -disjoint if for any  $a, b$  belonging to different elements of  $\mathcal{U}$  we have  $d(a, b) > r$ .

We say that  $D_X : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an  $n$ -dimensional control function for  $X$  if for any  $r > 0$ ,  $X$  has a cover  $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}_i$ , such that each  $\mathcal{U}_i$  is  $r$ -disjoint and each element of  $\mathcal{U}$  is  $D_X(r)$ -bounded. The *asymptotic dimension* of  $X$ , denoted by  $\text{asdim } X$ , is the least integer  $n$  such that  $X$  has an  $n$ -dimensional control function. If no such integer  $n$  exists, then the asymptotic dimension is infinite. This notion was introduced by Gromov [19] in the context of geometric group theory. The reader is referred to [6] for a survey on asymptotic dimension and its group theoretic applications, and to the lecture notes of Roe [29] on coarse geometry for more detailed proofs of some results of [19].

As the asymptotic dimension of a bounded space is 0, in the context of finite metric spaces we are more interested in the asymptotic dimension of (infinite) *classes* of metric spaces. We define the *asymptotic dimension of a family*  $\mathcal{X}$  of metric spaces as the least  $n$  for which there exists a function  $D_{\mathcal{X}} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  which is an  $n$ -dimensional control function for each  $X \in \mathcal{X}$ .

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**1.2. Graphs as metric spaces.** Given a graph  $G$  and a collection of positive reals  $\ell = (\ell_e)_{e \in E(G)}$ , the *weighted graph*  $(G, \ell)$  is the discrete metric space whose points are the vertices of  $G$ , and whose metric coincides with the weighted shortest path metric in  $G$  (where each edge  $e$  is considered with its weight  $\ell_e$ ). The distance between two vertices  $u$  and  $v$  in this metric is denoted by  $d_{(G, \ell)}(u, v)$  (or simply  $d_G(u, v)$  if the weights are clear from the context). Note that if all weights are equal to 1,  $d_G(u, v)$  is the number of edges of a shortest path between  $u$  and  $v$  in  $G$ . If all the weights of a weighted graph  $G$  lie in some real interval  $I$ , we say that  $G$  is an  *$I$ -weighted graph*. In this paper, it will be convenient to restrict ourselves to  $(0, 1]$ -weighted graphs, but the value 1 could be replaced with any positive constant without changing our results.

What can be said about the asymptotic dimension of graphs (weighted or not)? As observed above, a finite graph has asymptotic dimension 0, so this question is only interesting for infinite graphs, or for infinite classes of (finite or infinite) graphs.

The class of trees has asymptotic dimension 1 [6]. Gromov [19] observed that  $d$ -dimensional Euclidean spaces have asymptotic dimension  $d$ , and it can easily be deduced from this that for any  $d \geq 1$ , the class of  $d$ -dimensional grids (with or without diagonals) has asymptotic dimension  $d$ . On the other hand, it was proved that any infinite family of bounded degree expanders (in particular cubic expanders) has unbounded asymptotic dimension [17]. Another example of a class of graphs with bounded degree and infinite asymptotic dimension is the class of lamplighter graphs of binary trees [4] (these graphs have maximum degree 4). This implies that bounding the degree is not enough to bound the asymptotic dimension.

**1.3. Main contribution.** It was proved in [28] that the class of graphs excluding the complete graph  $K_t$  as a minor has asymptotic dimension at most  $4^t$ . Recently, Fujiwara and Papasoglu [16] proved that cacti have asymptotic dimension 1. They used this to prove that the class of planar graphs has asymptotic dimension at most 3, and asked whether this can be improved to 2.

We give a positive answer to their question.

**Theorem 1.1.** *For any fixed  $p \geq 2$ , the class of graphs excluding the complete bipartite graph  $K_{3,p}$  as a minor has asymptotic dimension at most 2.*

Since planar graphs exclude  $K_{3,3}$  as a minor, this implies that the class of planar graphs has asymptotic dimension 2. Note that the fact that the asymptotic dimension is at least 2 for this class follows from the example of 2-dimensional grids. For any fixed integer  $g \geq 0$ , the class of graphs embeddable on a surface of Euler genus  $g$  excludes  $K_{3,p}$  as a minor, for some fixed integer  $p$ , so our result immediately implies the following result.

**Corollary 1.2.** *For any  $g \geq 0$ , the class of graphs embeddable on a surface of Euler genus  $g$  has asymptotic dimension 2.*

Fujiwara and Papasoglu [16] showed that their result on planar graphs extends to any geodesic Riemannian plane (i.e. any geodesic Riemannian surface homeomorphic to  $\mathbb{R}^2$ ). We prove that our result extends to this setting, and to the class of compact Riemannian surfaces of Euler genus  $g$ , for any fixed  $g \geq 0$ .

**Theorem 1.3.** *Any geodesic Riemannian plane has asymptotic dimension 2 and for any  $g \geq 0$ , the class of compact Riemannian surfaces of Euler genus  $g$  has asymptotic dimension 2.*

Shortly after we posted our manuscript on the arXiv repository, we were informed by Jørgensen and Lang that they had independently proved a version of the results above for planar graphs and geodesic Riemannian planes in [21], using essentially the same tools.

**1.4. Weak diameter coloring and clustered coloring.** The *weak diameter* of a subset  $S$  of vertices of a graph  $G$  is the maximum distance (in  $G$ ) between two vertices of  $S$ . A graph  $G$  is  *$k$ -colorable with weak diameter  $d$*  if each vertex of  $G$  can be assigned a color from  $\{1, \dots, k\}$  in such a way that all monochromatic components (i.e. connected components of the subgraph induced by a color class) have weak diameter at most  $d$ . This notion is also studied under the name of *weak diameter network decomposition* in distributed computing (see [2]), although in this context  $k$  and  $d$  usually depend on  $|V(G)|$  (they are typically of order  $\log |V(G)|$ ), while here we will only consider the case where  $k$  and  $d$  are constant. Observe that the case  $d = 0$  corresponds to the usual notion of (proper) coloring. Note also that this property should not be confused with the stronger property that the subgraph induced by each monochromatic component has bounded diameter, see for instance [24, Theorem 4.1]. We say that a class of graphs  $\mathcal{G}$  has *weak diameter chromatic number* at most  $k$  if there is a constant  $d$  such that every graph of  $\mathcal{G}$  is  $k$ -colorable with weak diameter  $d$ .

**Observation 1.4.** *If a class of graphs  $\mathcal{G}$  has asymptotic dimension at most  $k$ , then  $\mathcal{G}$  has weak diameter chromatic number at most  $k + 1$ .*

*Proof.* Taking  $r = 1$  in the definition of asymptotic dimension, we obtain that there is a constant  $D$  such that any graph  $G \in \mathcal{G}$  has a cover by  $k + 1$  1-disjoint families of  $D$ -bounded sets. This cover can be reduced to a partition of the vertex set of  $G$  (while maintaining the property that the  $k + 1$  families are 1-disjoint and all their elements are  $D$ -bounded). We can now consider each of the  $k + 1$  families as a distinct color class. Since each family is 1-disjoint, each monochromatic component is included in a single element of the partition, and is therefore  $D$ -bounded. This implies that  $\mathcal{G}$  has weak diameter chromatic number at most  $k + 1$ .  $\square$

As a consequence, Theorem 1.1 has the following immediate corollary.

**Corollary 1.5.** *For any fixed  $p \geq 2$ , the class of graphs with no  $K_{3,p}$ -minor (and in particular planar graphs and any class of graphs embeddable on a fixed surface) has weak diameter chromatic number at most 3.*

A graph  $G$  is  *$k$ -colorable with clustering  $c$*  if each vertex of  $G$  can be assigned a color from  $\{1, \dots, k\}$  in such a way that all monochromatic components have size at most  $c$ . The case  $c = 1$  corresponds to the usual notion of (proper) coloring, and there is a large body of work on the case where  $c$  is a fixed constant, see [33] for a recent survey. In this context, we say that a class of graphs  $\mathcal{G}$  has *clustered chromatic number* at most  $k$  if there is a constant  $c$  such that every graph of  $\mathcal{G}$  is  $k$ -colorable with clustering  $c$ .

**Observation 1.6.** *If a class of graphs  $\mathcal{G}$  has maximum degree at most  $\Delta$  and asymptotic dimension at most  $k$ , then  $\mathcal{G}$  has clustered chromatic number at most  $k + 1$ .*

*Proof.* By Observation 1.4, there is a constant  $d$  such that any graph  $G \in \mathcal{G}$  has a  $(k + 1)$ -coloring with weak diameter at most  $d$ . Consider a graph  $G \in \mathcal{G}$  and such a coloring. Since  $G$  has maximum degree  $\Delta$ , each monochromatic component has size at most  $\Delta^d$ . This implies that  $\mathcal{G}$  has clustered chromatic number at most  $k + 1$ .  $\square$

It is known that for every graph  $H$  and integer  $\Delta$ , the class of  $H$ -minor free graphs of maximum degree  $\Delta$  has clustered chromatic number at most 3 [24, 12]. We will prove the following more general result.

**Theorem 1.7.** *For any integer  $\Delta$  and graph  $H$ , the class of  $H$ -minor free graphs of maximum degree at most  $\Delta$  has asymptotic dimension at most 2.*

Note that Observation 1.6 readily implies that the class of  $H$ -minor free graphs of maximum degree  $\Delta$  has clustered chromatic number at most 3, and in fact Theorem 1.7 can be seen as a large scale generalization of results on clustered coloring.

Given a finitely generated group  $G$  and a finite generating set  $S$  (assumed to be symmetric, in the sense that  $s \in S$  if and only if  $s^{-1} \in S$ ), the *Cayley graph*  $\text{Cay}(G, S)$  is the graph with vertex set  $G$ , with an edge between two elements  $u, v \in G$  if and only if  $u = vs$  for some  $s \in S$ . As observed by Gromov [19], the asymptotic dimension of  $\text{Cay}(G, S)$  is independent of the choice of the finite generating set  $S$ , and thus the asymptotic dimension is a group invariant (this was the main motivation for introducing this invariant).

We say that  $\text{Cay}(G, S)$  is *minor excluded* if it excludes some finite graph  $H$  as a minor. The following problem was raised by Ostrovskii and Rosenthal [28].

**Question 1** (Problem 4.1 in [28]). Let  $G$  be a finitely generated group and  $S$  a finite generating set such that  $\text{Cay}(G, S)$  is minor excluded. Does it follow that  $G$  has asymptotic dimension at most 2?

A simple compactness argument (see for instance [18]) shows that the asymptotic dimension of an infinite graph is at most the asymptotic dimension of the class of its finite induced subgraphs. If  $\text{Cay}(G, S)$  excludes some minor  $H$ , then since  $S$  is finite, all induced subgraphs of  $\text{Cay}(G, S)$  have degree at most  $|S|$  and exclude  $H$  as a minor. We thus obtain the following positive answer to Question 1 as an immediate consequence of Theorem 1.7.

**Corollary 1.8.** *Let  $G$  be a finitely generated group and  $S$  a finite generating set such that  $\text{Cay}(G, S)$  is minor excluded. Then  $G$  has asymptotic dimension at most 2.*

Fujiwara and Papasoglu [16] asked the following more general question.

**Question 2** (Question 5.2 in [16]). Is there a constant  $k$  such that for any graph  $H$ , the class of  $H$ -minor free graphs has asymptotic dimension at most  $k$ ? Can we take  $k = 2$ ?

Theorem 1.1 proves that Question 2 has a positive answer when  $H = K_{3,p}$ , for any constant  $p$ , and we will prove in Corollary 7.6 that the result also holds when  $H$  is an apex-forest (i.e. when  $H$  contains a vertex  $v$  such that  $H - v$  is a forest). This will be deduced from the result that any class of graphs of bounded pathwidth has asymptotic dimension 1 (Theorem 7.1). In a different direction, Theorem 1.7 gives a positive answer to Question 2 when  $H$  is arbitrary but we restrict ourselves to  $H$ -minor free graphs of bounded degree.

**1.5. Polynomial growth.** For some function  $f$ , a graph  $G$  has *growth* at most  $f$  if for any integer  $r$ , any vertex  $v \in V(G)$  has at most  $f(r)$  vertices at distance at most  $r$ . It is known that vertex-transitive graphs of polynomial growth have bounded asymptotic dimension, while some classes of graphs of exponential growth have unbounded asymptotic dimension [17].

We first prove the following result.

**Theorem 1.9.** *For any  $d \geq 1$ , the class of (non necessarily induced) subgraphs of the  $d$ -dimensional grid has asymptotic dimension at most  $d$ .*

Using a result of Krauthgamer and Lee [22], we then deduce the following.

**Corollary 1.10.** *Any class of graphs of polynomial growth has bounded asymptotic dimension*

We also prove that this is sharp: for any superpolynomial function  $f$  we construct a class of graphs of growth at most  $f$  with unbounded asymptotic dimension.

**1.6. Linear type and Assouad-Nagata dimension.** Gromov [19] noticed that the notion of asymptotic dimension of a metric space can be refined by restricting the growth rate of the control function  $D(r)$  in its definition. Although this function can be chosen to be linear in many cases, its complexity can be significantly worse in general, as there exist (Cayley) graphs of asymptotic dimension  $n$  for which any  $n$ -dimensional control function  $D(r)$  grows as fast as  $\Omega(\exp \exp \cdots \exp r^k)$ , for any given height of the tower of exponentials [27]. A control function  $D_X$  for a metric space  $X$  is said to be *linear* if there is a constant  $c > 0$  such that  $D_X(r) \leq cr + c$  for any  $r > 0$ . We say that a metric space  $(X, d)$  has *asymptotic dimension at most  $n$  of linear type* if  $X$  has a linear  $n$ -dimensional control function. The definition extends to families of metric spaces in a natural way. This notion is sometimes called *asymptotic dimension with Higson property* [9]. Nowak [27] proved that the asymptotic dimension of linear type is not bounded by any function of the asymptotic dimension, by constructing (Cayley) graphs of asymptotic dimension 2 and infinite asymptotic dimension of linear type. It turns out that some of our main results hold for the asymptotic dimension of linear type, i.e. the diameter bound that we obtain is linear in the disjointness parameter  $r$ .

Asymptotic dimension of linear type is related to the following notion. A control function  $D_X$  for a metric space  $X$  is said to be a *dilation* if there is a constant  $c > 0$  such that  $D_X(r) \leq cr$ , for any  $r > 0$ . We say that a metric space  $(X, d)$  has *Assouad-Nagata dimension at most  $n$*  if  $X$  has an  $n$ -dimensional control function which is a dilation. The Assouad-Nagata dimension was introduced by Assouad [1] (see [23] for more results on this notion). The main difference between this notion and the asymptotic dimension of linear type is that the latter is a large-scale dimension (we can assume that  $r$  is arbitrarily large by tuning the constant  $c$  if necessary), while the former is a dimension at all scales: it sheds some light on the geometry of the space as  $r \rightarrow \infty$  (large-scale), but also as  $r \rightarrow 0$  (microscopic scale). Note that the two notions are equivalent for *uniformly discrete metric spaces*, i.e. metric spaces such that all pairs of distinct points are at distance at least  $\epsilon > 0$  apart, for some universal  $\epsilon > 0$ . This is the case for graphs (two distinct vertices are at distance at least 1 apart), and for weighted graphs with a uniform lower bound on the edge weights. It follows that in these cases, some of our main results hold for the Assouad-Nagata dimension as well (on the other hand our results on Riemannian surfaces do not extend to the Assouad-Nagata dimension).

**1.7. Sparse partitions.** A *ball of radius  $r$*  (or  *$r$ -ball*) centered in a point  $x$ , denoted by  $B_r(x)$ , is the set of points of  $X$  at distance at most  $r$  from  $x$ . For a real  $r \geq 0$  and an integer  $n \geq 0$ , the family  $\mathcal{U}$  has  *$r$ -multiplicity at most  $n$*  if each  $r$ -ball in  $X$  intersects at most  $n$  sets of  $\mathcal{U}$ . It is not difficult to see that if  $D_X(r)$  is an  $n$ -dimensional control function for a metric space  $X$ , then for any  $r > 0$ ,  $X$  has a  $D_X(2r)$ -bounded cover of  $r$ -multiplicity at most  $n + 1$ . Gromov [19] proved that a converse of this result also holds, in the sense that the asymptotic dimension of  $X$  is exactly the least integer  $n$  such that for any real number  $r \geq 0$ , there is a real number  $D'_X(r)$  such that  $X$  has a  $D'_X(r)$ -bounded cover of  $r$ -multiplicity at most  $n + 1$ . Moreover, the function  $D'_X$  has the same type as the  $n$ -dimensional control function of  $D_X$  of  $X$ :  $D'_X$  is linear if and only if  $D_X$  is linear, and  $D'_X$  is a dilation if and only if  $D_X$  is a dilation.

As a consequence, the notions of Assouad-Nagata dimension and asymptotic dimension of linear type are closely related to the well-studied notions of sparse covers and sparse partitions in theoretical computer science. A weighted graph  $G$  admits a  $(\sigma, \tau)$ -*weak sparse partition scheme* if for any  $r \geq 0$ , the vertex set of  $G$  has a partition into  $(\sigma \cdot r)$ -bounded sets of  $r$ -multiplicity at most  $\tau$ , and such a partition can be computed in polynomial time. As before, we say that a family of graphs admits a  $(\sigma, \tau)$ -weak sparse partition scheme if all graphs in the family admit a  $(\sigma, \tau)$ -weak sparse partition scheme. This definition was

introduced in [20], and is equivalent to the notion of weak sparse cover scheme of Awerbuch and Peleg [3] (see [15]). Note that if a family of graphs admits a  $(\sigma, \tau)$ -weak sparse partition scheme then its Assouad-Nagata dimension is at most  $\tau - 1$ . Conversely, if a family of graphs has Assouad-Nagata dimension at most  $d$  and the covers can be computed efficiently, then the family admits a  $(\sigma, d + 1)$ -weak sparse partition scheme, for some constant  $\sigma$ .

While the two notions are almost equivalent, it should be noted that the emphasis is on different parameters. In the case of the Assouad-Nagata dimension, the goal is to minimize  $d$  (or equivalently  $\tau$  in the sparse partition scheme), while in the  $(\sigma, \tau)$ -weak sparse partition scheme, the goal is usually to minimize a function of  $\sigma$  and  $\tau$  which depends on the application. As an example, it was proved in [20] that if an  $n$ -vertex graph admits a  $(\sigma, \tau)$ -weak sparse partition scheme, then the graph has a *universal steiner tree* with stretch  $O(\sigma^2 \tau \log_\tau n)$ , so in this case the goal is to minimize  $\sigma^2 \cdot \frac{\tau}{\log \tau}$ .

All our proofs are constructive and give polynomial-time algorithms to compute the covers. The following is a sample of our results (translated into the terminology of sparse partition schemes).

- trees (and more generally chordal graphs) admit  $(O(1), 2)$ -weak partition schemes (this actually follows directly from the results of [16]);
- graphs excluding  $K_{3,p}$  as a minor admit  $(O(p^2), 3)$ -weak partition schemes, while using the original approach of [16], it can be shown that they admit  $(O(p), 4)$ -weak partition schemes;
- in particular, graphs of Euler genus  $g \geq 0$  admit  $(O(g^2), 3)$ -weak partition schemes and  $(O(g), 4)$ -weak partition schemes.

**1.8. Outline of the paper.** In Section 2, we introduce some terminology and describe a result of Brodskiy, Dydak, Levin and Mitra [7] (Theorem 2.1, part of which is not explicitly stated in [7] and is proved in the appendix for completeness). We then show how to deduce that planar graphs and Riemannian planes have asymptotic dimension 2 (Theorem 2.4). After that, we explain how to deduce our main tool from Theorem 2.1 (Theorem 2.6), which states informally that if all graphs in a class  $\mathcal{G}$  have a layering in which any constant number of layers induce a graph from a class of asymptotic dimension at most  $k$ , then  $\mathcal{G}$  has asymptotic dimension at most  $k + 1$ .

In Section 3 we use Theorem 2.6 to prove that graphs excluding  $K_{3,p}$  as a minor have asymptotic dimension 2 (Theorem 3.5). In Section 4 we show that this implies that any class of graphs embeddable on a fixed surface has asymptotic dimension at most 2 (Corollary 4.1). The fact that this result actually holds for weighted graphs is then used to deduce that for any integer  $g \geq 0$ , the class of (compact) Riemannian surfaces of Euler genus  $g$  has asymptotic dimension 2.

In Section 5, we prove that classes of graphs of bounded degree and bounded treewidth have asymptotic dimension 1 (Theorem 5.5) and deduce that classes of graphs of bounded degree and bounded layered treewidth have asymptotic dimension 2 (Theorem 5.6). In particular this implies Theorem 1.7, stating that graphs of bounded degree from any proper minor-closed class have asymptotic dimension at most 2.

In Section 6 we explore some consequences of Theorem 2.6 for classes of graphs with a low-dimensional representation. We prove for instance that (not necessarily induced) subgraphs of the  $d$ -dimensional grid have asymptotic dimension  $d$  (Theorem 6.1) and deduce that graphs of polynomial growth have bounded asymptotic dimension (Corollary 6.4). The assumption on the growth turns out to be sharp. We also make some observations about a characterisation of classes of bounded asymptotic dimension by forbidden (induced) subgraphs.

In Section 7 we prove that classes of graphs of bounded pathwidth have asymptotic dimension at most 1 (Theorem 7.1) and deduce that classes of graphs of bounded layered pathwidth have asymptotic dimension at most 2 (Corollary 7.5). This shows that any class of graphs excluding an apex-forest as a minor has asymptotic dimension at most 2 (Corollary 7.6).

We conclude with some open problems in Section 8.

## 2. CONTROL FUNCTIONS AND LAYERINGS

Most of our results make use of a result of Brodskiy, Dydak, Levin and Mitra [7], extending a result of Bell and Dranishnikov [5]. In this section we introduce their notation and state a quantitative version of (a special case of) their result. We then explain the consequences of the result for graph layerings.

**2.1.  $r$ -components and  $(r, s)$ -components.** Let  $(X, d)$  be a metric space. Recall that a subset  $S \subseteq X$  is  $r$ -bounded if for any  $x, x' \in S$ ,  $d(x, x') \leq r$ . Given a subset  $A \subseteq X$ , two points  $x, x' \in A$  are  $r$ -connected in  $A$  if there are points  $x = x_1, \dots, x_\ell = x'$  in  $A$  such that for any  $1 \leq i \leq \ell - 1$ ,  $\{x_i, x_{i+1}\}$  is  $r$ -bounded. A maximal set of  $r$ -connected points in  $A$  is called an  $r$ -component of  $A$  (note that these components form a partition of  $A$ ). Observe that in an unweighted graph  $G$ , the 1-components of a subset  $U \subseteq V(G)$  of vertices are exactly the connected components of  $G[U]$ , the subgraph of  $G$  induced by  $U$  (see Figure 1, left, for an example of 2-components of a subset  $A$  of vertices).

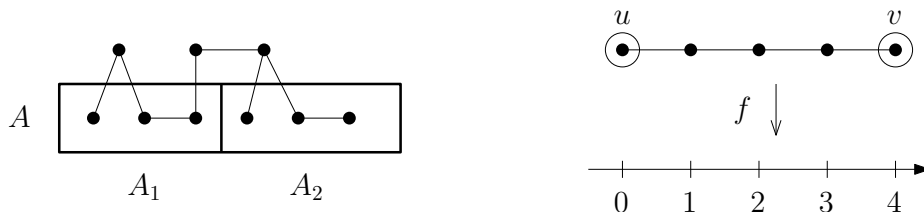


FIGURE 1. The two 2-components  $A_1$  and  $A_2$  of a subset  $A$  (left), and an example of a set  $B = \{u, v\}$  which has one 4-component but two  $(4, 3)$ -components (right).

Recall that  $D_X : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an  $n$ -dimensional control function for  $X$  if for any  $r > 0$ ,  $X$  has a cover  $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}_i$ , such that each  $\mathcal{U}_i$  is  $r$ -disjoint and each element of  $\mathcal{U}$  is  $D_X(r)$ -bounded. Observe that, equivalently (using the terminology introduced above),  $D_X : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an  $n$ -dimensional control function for  $X$  if for any  $r > 0$ ,  $X$  has a cover by  $n + 1$  sets whose  $r$ -components are  $D_X(r)$ -bounded. In this section it will be convenient to work with this definition of control functions.

Given a metric space  $(X, d)$ , and a real  $c > 0$ , a function  $f : X \rightarrow \mathbb{R}$  is  $c$ -Lipschitz if for any  $x, y \in X$ ,  $|f(x) - f(y)| \leq c \cdot d(x, y)$  (such functions can be defined between any two metric spaces, but here we will only consider  $\mathbb{R}$  as the codomain). A 1-Lipschitz function  $f : X \rightarrow \mathbb{R}$  is called a *real projection* of  $X$ . Let  $(X, d)$  be a metric space and let  $f : X \rightarrow \mathbb{R}$  be a real projection of  $X$ . A subset  $A \subseteq V(G)$  is said to be  $(r, s)$ -bounded with respect to  $f$  if for all  $x, x' \in A$  we have  $d(x, x') \leq r$  and  $|f(x) - f(x')| \leq s$  (When  $f$  is clear from the context we often omit “with respect to  $f$ ”). Two vertices  $x, x'$  of  $A$  are  $(r, s)$ -connected in  $A$  if there are vertices  $x = x_1, \dots, x_\ell = x'$  in  $A$  such that for any  $1 \leq i \leq \ell - 1$ ,  $\{x_i, x_{i+1}\}$  is  $(r, s)$ -bounded as above. A maximal set of  $(r, s)$ -connected vertices in  $A$  is called an  $(r, s)$ -component of  $A$  (note that these components form a partition of  $A$ ).

Note that by the definition of a real projection, any  $r$ -bounded set is also  $(r, r)$ -bounded, and similarly being  $r$ -connected is equivalent to being  $(r, r)$ -connected, and an  $r$ -component is the same as an  $(r, r)$ -component. Observe that in Figure 1, right, the vertices  $u$  and  $v$  are 4-connected in  $\{u, v\}$  but they are not  $(4, 3)$ -connected in  $\{u, v\}$  with respect to  $f$ , so this shows that the notions of  $r$ -components and  $(r, s)$ -components differ when  $s < r$ .

**2.2. Control functions for real projections.** We have seen above that  $D_X : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an  $n$ -dimensional control function for  $X$  if for any  $r > 0$ ,  $X$  has a cover by  $n + 1$  sets whose  $r$ -components are  $D_X(r)$ -bounded. It will be convenient to extend this definition to real projections, as follows. For a metric space  $X$  and a real projection  $f : X \rightarrow \mathbb{R}$ , we say that  $D_f : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an  $n$ -dimensional control function for  $f$  if for any  $r, S > 0$ , any  $(\infty, S)$ -bounded subset  $A \subseteq X$  has a cover by  $n + 1$  sets whose  $r$ -components are  $D_f(r, S)$ -bounded. We say that the control function is *linear* if there are constants  $a, b, c > 0$  such that  $D_f(r, S) \leq ar + bS + c$  for any  $r, S > 0$ . We say that the control function is a *dilation* if there are constants  $a, b > 0$  such that  $D_f(r, S) \leq ar + bS$  for any  $r, S > 0$ .

The following is a special case of a result of Brodskiy, Dydak, Levin and Mitra [7]. We will need a more precise version of their result (which follows from their proof, but is not stated explicitly in their paper), so for completeness we give a proof of Theorem 2.1 in the appendix.

**Theorem 2.1** (Theorem 4.9 in [7]). *Let  $X$  be a metric space and  $f : X \rightarrow \mathbb{R}$  be a real projection of  $X$ . If  $f$  has an  $n$ -dimensional control function  $D_f$ , then  $X$  has an  $(n + 1)$ -dimensional control function  $D_X$  depending only on  $D_f$  and  $n$ . Moreover, if  $D_f$  is linear then  $D_X$  is linear and if  $D_f(r, S) \leq ar + bS$ , for some constants  $a \geq 1, b \geq 1$  (i.e.  $D_f$  is a dilation) then  $D_X(r) \leq 20a(6a + b(n + 4)) \cdot r$  (i.e.  $D_X$  is also a dilation).*

**2.3. Layerings and annuli.** In the context of graphs, an interesting example of real projections comes from *layerings*. A layering  $L = (L_0, L_1, \dots)$  of a graph  $G$  is a partition of  $V(G)$  into sets (called the *layers*), such that for any edge  $uv$  of  $G$ ,  $u$  and  $v$  lie in the same layer or in two consecutive layers. Note that a layering can also be seen as a function  $L : V(G) \rightarrow \mathbb{N}$  such that for any edge  $uv$ ,  $|L(u) - L(v)| \leq 1$ . In particular a layering is a real projection.

A related example comes from rooted real projections. Given a metric space  $(X, d)$ , and a point  $x \in X$ , the *real projection rooted at  $x$*  is the application  $y \in X \mapsto d(x, y)$ . Note that the triangle inequality implies that this function is 1-Lipschitz, and thus a real projection. In the context of unweighted graphs, each layer in the rooted real projection (i.e. the preimages of each element of  $\mathbb{N}$  under the rooted real projection) corresponds to a level of a Breadth-First Search (BFS) tree rooted in  $x$  (this type of layering is also known as a *BFS-layering*). For general metric spaces (and weighted graphs) it will be convenient to consider the following continuous analogue of layers: given a metric space  $(X, d)$ , a base point  $x \in X$ , and two reals  $0 < a < b$  we denote by  $A(a, b)$  the set of points  $y$  such that  $a \leq d(x, y) < b$ . The sets  $A(a, b)$  are called *annuli*.

We give the first applications of Theorem 2.1 for planar sets and planar graphs. A *planar set* is a geodesic metric set  $(P, d_P)$  with an injective continuous map  $\phi : P \rightarrow \mathbb{R}^2$ . Fujiwara and Papasoglu [16] proved that any annulus  $A(a, b)$  in a rooted real projection of a planar set or planar graph can be covered by two sets whose  $r$ -components are  $(10^5 r + 2(b - a))$ -bounded. Note that each annulus  $A(a, b)$  is precisely an  $(\infty, b - a)$ -bounded set (with respect to the rooted real projection), so the result of Fujiwara and Papasoglu [16] can be rephrased as follows.



**Lemma 2.2** (Lemma 4.4 in [16]). *Suppose  $(P, d_P)$  is a planar set or a connected planar graph, and let  $L$  be a rooted real projection of  $(P, d_P)$ . Then  $L$  has a 1-dimensional control function  $D_L$  with  $D_L(r, S) \leq 10^5 r + 2S$ .*

Observe the following.

**Observation 2.3.** *A function  $D$  is an  $n$ -dimensional control function for a graph  $G$  if and only if  $D$  is an  $n$ -dimensional control function for each of the connected components of  $G$ .*

Using Lemma 2.2, Fujiwara and Papasoglu [16] proved that planar sets and planar graphs have Assouad-Nagata dimension at most 3. Applying Theorem 2.1 and Observation 2.3 together with Lemma 2.2, we obtain the following immediate improvement over their result.

**Theorem 2.4.** *Planar sets and planar graphs have Assouad-Nagata dimension at most 2.*

This gives a positive answer to Question 5.1 in [16]. Note that this result is best possible, as the class of 2-dimensional grids has asymptotic dimension 2. As mentioned in the introduction, Theorem 2.4 was also proved independently by Jørgensen and Lang in [21].

In Section 3 we will extend this result from planar graphs to graphs avoiding  $K_{3,p}$  as a minor. This requires additional work, some of which will be crucial to the remainder of the paper. In Section 4 we will explain the consequences for graphs embeddable on surfaces of bounded genus (and for compact Riemannian surfaces). For all the remaining results, it will be convenient to work not in the metric induced by the original metric space  $X$ , but in the intrinsic metric in the subspace defined by an annulus. We now explain how to prove a variant of Theorem 2.1 in this setting.

**2.4. Intrinsic control of real projections.** From now on, all the metric spaces under consideration are finite weighted or unweighted graphs<sup>1</sup>, together with their shortest path metric. When considering an induced subgraph  $H$  of  $G$ , we will call the shortest path metric of  $H$  *intrinsic*, and denote it  $(H, d_H)$ , to make a clear distinction with the space  $(H, d_G)$  where the metric under consideration is the shortest path metric of  $G$ .

Let  $G$  be a (weighted or unweighted) graph and  $L : V(G) \rightarrow \mathbb{R}^+$  be a real projection of  $G$  (in the remainder  $L$  will always be a layering in the case of unweighted graphs, and a rooted real projection, in the case of weighted graphs).

A function  $D_L : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an  $n$ -dimensional *intrinsic control function* for  $L$  if for all  $r, S > 0$ , the graph  $H$  induced by any  $(\infty, S)$ -bounded set of  $G$  (with respect to  $L$  and in the metric  $d_G$ ) can be covered by  $n + 1$  sets whose  $r$ -components are  $D_L(r, S)$ -bounded (where the definition of  $r$ -components and  $D_L(r, S)$ -bounded are in the intrinsic metric  $d_H$ ).

As before, we say that an intrinsic control function  $D_L$  for a real projection  $L$  is *linear* if it is of the form  $D_L(r, S) = ar + bS + c$ , for some constants  $a, b, c > 0$ . We also say that  $L$  is a *dilation* if it has an intrinsic control function of form  $D_L(r, S) = ar + bS$ , for some constants  $a, b > 0$ .

We now prove that intrinsic control functions can be transformed into (classical) control functions.

**Lemma 2.5.** *Let  $G$  be a (weighted or unweighted) graph and  $L : V(G) \rightarrow \mathbb{R}^+$  be a real projection of  $G$ , such that  $L$  admits an  $n$ -dimensional intrinsic control function  $D$ . Then  $D'(r, S) := D(r, S + 2r)$  is an  $n$ -dimensional control function for  $L$ . In particular, if  $D$  is linear, then  $D'$  is also linear, and if  $D$  is a dilation, then  $D'$  is a dilation.*

<sup>1</sup>We will also consider compact Riemannian surfaces but all results about these metric spaces will be deduced from our results on finite weighted graphs.

*Proof.* Let  $H = G[X]$  be the subgraph induced by some  $(\infty, S)$ -bounded subset  $X$  of vertices of  $G$ . It follows that there is some  $a > 0$  such that for any  $x \in X$ ,  $L(x) \in [a, a + S]$ . Fix some  $r > 0$ , and denote by  $X^+ \supseteq X$  the preimage of the interval  $[a - r, a + S + r]$  under  $L$ . Let  $H := G[X^+]$  be the subgraph of  $G$  induced by  $X^+$ . Then by definition of  $D$ ,  $(H, d_H)$  has a cover by  $n + 1$  sets  $(U_i^+)_{i=1}^{n+1}$ , whose  $r$ -components are  $D(r, S + 2r)$ -bounded. For each  $1 \leq i \leq n + 1$ , let  $U_i$  be the restriction of the family  $U_i^+$  to  $X$ . It follows that the sets  $(U_i)_{i=1}^{n+1}$  cover  $X$ . Consider now an  $r$ -component  $C$  of some set  $U_i$ , for  $1 \leq i \leq n + 1$  (where the distances in the definition of  $r$ -component are with respect to the metric  $d_G$ ). For any  $u, v \in C$ , there are  $u = u_0, u_1, \dots, u_t = v$  in  $U_i$  such for any  $0 \leq j \leq t - 1$ ,  $d_G(u_j, u_{j+1}) \leq r$ . It follows that  $u_j$  and  $u_{j+1}$  are connected by a path (of length at most  $r$ ) in  $G$ , thus also in  $H = G[X^+]$  by definition of  $X^+$ . Hence,  $u_j$  and  $u_{j+1}$  lie in the same  $r$ -component of  $U_i^+$  (where all the distances involved in the definitions are with respect to  $d_H$ ). Since all  $r$ -components of  $U_i^+$  (with respect to  $d_H$ ) are  $D(r, S + 2r)$ -bounded with respect to  $d_H$ , they are also  $D(r, S + 2r)$ -bounded with respect to  $d_G$ . This shows that  $D'(r, S) := D(r, S + 2r)$  is an  $n$ -dimensional control function for  $L$ .  $\square$

**2.5. Layerability.** Given a (weighted or unweighted) graph class  $\mathcal{C}$  and a sequence of (weighted or unweighted) graph classes  $\mathcal{L} = (\mathcal{L}_i)_{i \in \mathbb{N}}$  with  $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \dots$ , we say that  $\mathcal{C}$  is  $\mathcal{L}$ -layerable if there is a function  $f : \mathbb{R}^+ \rightarrow \mathbb{N}$  such that any graph  $G \in \mathcal{C}$  has a real projection  $L : V(G) \rightarrow \mathbb{R}$  such that for any  $S > 0$ , any  $(\infty, S)$ -bounded set in  $G$  (with respect to  $L$ ) induces a graph from  $\mathcal{L}_{f(S)}$ . If there is a constant  $c > 0$  such that  $f(S) \leq cS$  for any  $S > 0$ , we say that  $\mathcal{C}$  is  $c$ -linearly  $\mathcal{L}$ -layerable.

**Theorem 2.6.** *Suppose that all the classes in a sequence  $\mathcal{L} = \mathcal{L}_1, \mathcal{L}_2, \dots$  of (weighted or unweighted) graph classes have asymptotic dimension at most  $n$ , and let  $\mathcal{C}$  be an  $\mathcal{L}$ -layerable class of (weighted or unweighted) graphs. Then  $\mathcal{C}$  has asymptotic dimension at most  $n + 1$ . Let  $a, b, c \geq 1$  and  $d \geq 0$  be some constants, and assume moreover that the class  $\mathcal{C}$  is  $c$ -linearly  $\mathcal{L}$ -layerable. If each class  $\mathcal{L}_i$  has an  $n$ -dimensional control function  $D_i(r) \leq ar + bi + d$ , then  $\mathcal{C}$  has asymptotic dimension at most  $n + 1$  of linear type. If moreover  $d = 0$ , then  $\mathcal{C}$  has Assouad-Nagata dimension at most  $n + 1$ , with an  $(n + 1)$ -dimensional control function  $D_{\mathcal{C}}(r) \leq 20(6(a + 2bc)^2 + (a + 2bc)bc(n + 4)) \cdot r$ .*

*Proof.* For any  $i \geq 1$ , Let  $D_i$  be an  $n$ -dimensional control function for the graphs of  $\mathcal{L}_i$ . Let  $G \in \mathcal{C}$ , and let  $L : V(G) \rightarrow \mathbb{R}$  be a real projection of  $G$  such that for any  $S > 0$ , any  $(\infty, S)$ -bounded set in  $G$  (with respect to  $L$ ) induces a graph of  $\mathcal{L}_{f(S)}$ . Note that  $D(r, S) := D_{f(S)}(r)$  naturally defines an  $n$ -dimensional intrinsic control function for  $L$ . By Lemma 2.5,  $D'(r, S) := D(r, S + 2r)$  is an  $n$ -dimensional control function for  $L$ . By Theorem 2.1,  $G$  admits an  $(n + 1)$ -dimensional control function  $D_G$  such that  $D_G(r)$  only depends on  $r$ ,  $D$  and  $n$ . This shows that  $\mathcal{C}$  has asymptotic dimension at most  $n + 1$ .

If the class  $\mathcal{C}$  is  $c$ -linearly  $\mathcal{L}$ -layerable, then it holds that  $D'(r, S) = D(r, S + 2r) = D_{c(S+2r)}(r)$ . If moreover  $D_i(r) \leq ar + bi + d$ , then

$$D'(r, S) \leq ar + b(c(S + 2r)) + d = (a + 2bc)r + bcS + d,$$

which means that  $D'$  is linear. In this case it follows from Theorem 2.1 that  $D_G$  is linear, and thus  $\mathcal{C}$  has asymptotic dimension at most  $n + 1$  of linear type.

If moreover  $d = 0$ , we have  $D'(r, S) \leq (a + 2bc)r + bcS$ . By Theorem 2.1,  $D_G(r) \leq 20(6(a + 2bc)^2 + (a + 2bc)bc(n + 4)) \cdot r$  (i.e.  $D_G$  is also linear). In particular,  $\mathcal{C}$  has Assouad-Nagata dimension at most  $n + 1$ .  $\square$

Informally, if all the graphs in a class  $\mathcal{C}$  have a layering such that any constant number of layers induce a graph from a class of asymptotic dimension  $n$ , then  $\mathcal{C}$  has asymptotic dimension at most  $n + 1$ . The following immediate corollary is a simple rephrasing of Theorem 2.6, avoiding the terminology of layerability.

**Corollary 2.7.** *Let  $\mathcal{C}$  be a class of (weighted or unweighted) graphs, and let  $a \geq 1, b \geq 1$  and  $d \geq 0$  be some real numbers. Suppose that there exists an integer  $n$  such that any graph  $G$  of  $\mathcal{C}$  has a real projection  $L : V(G) \rightarrow \mathbb{R}$  such that any  $(\infty, S)$ -bounded set (with respect to  $L$ ) induces a graph with an  $n$ -dimensional control function  $D_{G,S}(r) \leq ar + bS + d$  for any  $r, S > 0$ . Then  $\mathcal{C}$  has asymptotic dimension at most  $n + 1$  of linear type. If moreover  $d = 0$ , then  $\mathcal{C}$  has Assouad-Nagata dimension at most  $n + 1$ , with an  $(n + 1)$ -dimensional control function  $D_{\mathcal{C}}(r) \leq 20(6(a + 2b)^2 + (a + 2b)b(n + 4)) \cdot r$ .*

*Proof.* Simply define  $\mathcal{L}_i$  as the class of graphs having an  $n$ -dimensional control function  $D_i(r) \leq ar + bi + d$ . Then the class  $\mathcal{C}$  is 1-linearly  $\mathcal{L}$ -layerable, with  $\mathcal{L} := (\mathcal{L}_i)_{i \geq 0}$ . The result thus follows from Theorem 2.6 by taking  $c = 1$ .  $\square$

### 3. $K_{3,p}$ -MINOR FREE GRAPHS

**3.1. Terminology.** We recall that in this section all graphs are finite. For two subsets  $A, B$  of vertices of  $G$ , we define

$$d_G(A, B) := \min\{d_G(u, v) \mid (u, v) \in A \times B\}.$$

The *diameter* of a subset  $U$  of vertices of  $G$ , denoted by  $\text{diam}_G(U)$ , is the maximum over the distances in  $G$  between two vertices of  $U$  (we have called it the *weak diameter* in Section 1.4, which is the terminology in theoretical computer science, but in this section we will refer to it simply as the diameter, which is the usual terminology in the theory of metric spaces).

A path between two vertices  $u$  and  $v$  in a weighted graph  $G$  is *geodesic* if its length is precisely  $d_G(u, v)$  (i.e. the path is a (weighted) shortest path between its endpoints).

We say that a subset  $S$  of vertices of  $G$  is *connected* if the subgraph of  $G$  induced by  $S$ , denoted by  $G[S]$ , is connected.

**3.2. Fat bananas.** For two integers  $p$  and  $q$ , a  $q$ -fat  $p$ -banana in a weighted graph  $G$  is a pair of connected subsets  $A$  and  $B$  of vertices of  $G$  with  $d_G(A, B) \geq q$ , together with  $p$  geodesic paths between  $A$  and  $B$ , such that  $d_G(x, y) \geq q$  if  $x, y$  belong to different paths (see Figure 2, left). Note that for  $(0, 1]$ -weighted graphs, the assumption that  $A$  and  $B$  are at distance at least  $q$  implies that the geodesic paths contain at least  $q$  vertices.

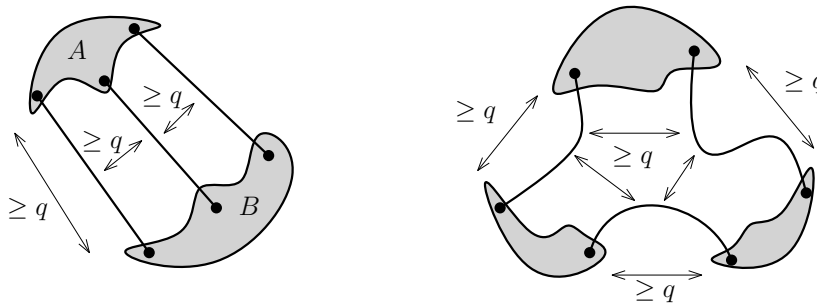


FIGURE 2. A  $q$ -fat 3-banana (left), and a  $q$ -fat  $K_3$ -minor (right).

In this section, we will prove a technical lemma that generalises the following (rephrased) result of Fujiwara and Papasoglu [16].

**Theorem 3.1** (Theorem 3.1 in [16]). *Let positive integers  $q$  and  $r$  be given. Any graph  $G$  with no  $q$ -fat 3-banana has a cover by 2 sets whose  $r$ -components are  $(10^5 r)$ -bounded.*

We now extend their result to graphs with no  $q$ -fat  $p$ -bananas.

**Lemma 3.2.** *Let positive integers  $p \geq 2$ ,  $q \geq 1$  and a real number  $r > 0$  be given. Then, any weighted graph  $G$  with no  $q$ -fat  $p$ -banana has a cover by 2 sets whose  $r$ -components are  $(5r + 3q)p$ -bounded.*

*Proof.* Let  $G$  be a weighted graph with no  $q$ -fat  $p$ -banana. We assume that the graph  $G$  is connected, otherwise we can consider each connected component separately. We fix a root vertex  $v \in G$ , and recall that an *annulus* is a vertex set of the form  $A(a, b) = \{u \mid a \leq d_G(u, v) < b\}$ , for some  $0 < a < b$ . For any integer  $k \geq 1$ , let  $A_k := A(kr, (k+1)r)$ . For any integer  $k$ , we say that the annuli  $A_k$  and  $A_{k+1}$  are *consecutive*.

Let  $k_0$  be the smallest integer such that  $k_0 r \geq r + q$ , and assume without loss of generality that  $k_0$  is odd (if not, interchange even and odd in the remainder of the proof). Note that by definition,  $(k_0 - 1)r \leq r + q$ , and thus  $k_0 r \leq 2r + q$ . Let  $A_0$  be the set of vertices at distance less than  $k_0 r$  from  $v$ . Note that  $A_0$  has diameter at most  $2k_0 r \leq 4r + 2q \leq (5r + 3q)p$ . It remains to cover the vertices at distance at least  $k_0 r$  from  $G$ , that is the vertices in  $\bigcup_{k \geq k_0} A_k$ .

We define the first covering set as  $C_0 := A_0 \cup \{A_k \mid k \geq k_0, k \text{ even}\}$  and the second covering set as  $C_1 := \{A_k \mid k \geq k_0, k \text{ odd}\}$ . Note that these two sets clearly cover  $V(G)$ , so we only need to show that each  $r$ -component in  $C_0$  or  $C_1$  is  $(5r + 3q)p$ -bounded. By the definition of the annuli  $A_k$ , observe that each  $r$ -component of  $C_0$  or  $C_1$  is contained in  $A_0$  or in some annulus  $A_k$  with  $k \geq k_0$ . Since  $A_0$  is  $(5r + 3q)p$ -bounded, it thus suffices to show that for any  $k \geq k_0$ , each  $r$ -component of  $A_k$  is also  $(5r + 3q)p$ -bounded.

Fix some  $k \geq k_0$ , and let  $C$  be an  $r$ -component of  $A_k$ . Assume for the sake of contradiction that there exist  $x$  and  $y$  in  $C \subseteq A_k$ , with  $d_G(x, y) \geq (5r + 3q)p$ . By definition of  $C$ , we may find  $x_1, \dots, x_\ell \in C$ , where  $d_G(x_i, x_{i+1}) \leq r$ ,  $x_1 = x$  and  $x_\ell = y$ .

We now define an increasing function  $\iota : \{1, \dots, p\} \mapsto \mathbb{N}$ , as follows:  $\iota(1) = 1$ ,  $\iota(p) = \ell$ , and for any  $1 \leq i \leq p - 2$ ,  $\iota(i + 1)$  is one plus the largest integer  $j \geq \iota(i)$  such that  $d_G(x_{\iota(i)}, x_j) \leq 4r + 3q$ . By the definition of  $\iota$ , we have  $d_G(x_{\iota(i)}, x_{\iota(j)}) > 4r + 3q$  for any  $1 \leq i < j \leq p - 1$ . For  $j = i + 1$  the property follows from the definition and for  $j > i + 1$  it follows from  $\iota(j) \geq \iota(i + 1)$ .

We claim that we also have  $d_G(x_{\iota(i)}, x_{\iota(p)}) > 4r + 3q$  for any  $1 \leq i \leq p - 1$ . Since for any  $1 \leq i \leq p - 2$  we have

$$\begin{aligned} d_G(x_{\iota(i+1)}, x_{\iota(i)}) &\leq d_G(x_{\iota(i+1)}, x_{\iota(i+1)-1}) + d_G(x_{\iota(i+1)-1}, x_{\iota(i)}) \\ &\leq r + 4r + 3q \\ &\leq 5r + 3q, \end{aligned}$$

we obtain that for any  $1 \leq i \leq p - 1$ ,  $d_G(x_1, x_{\iota(i)}) \leq (5r + 3q)i$ . As a consequence,

$$\begin{aligned} d_G(x_{\iota(i)}, x_\ell) &\geq d_G(x_1, x_\ell) - d_G(x_1, x_{\iota(i)}) \\ &\geq (5r + 3q)p - (5r + 3q)i \geq 5r + 3q > 4r + 3q, \end{aligned}$$

which concludes the proof of our claim, and shows that we have  $d_G(x_{\iota(i)}, x_{\iota(j)}) > 4r + 3q$  for any  $1 \leq i < j \leq p$ .

For each  $1 \leq i \leq p$ , consider a shortest path  $P_i$  from  $x_{\iota(i)}$  to the root  $v$ . We now define a  $q$ -fat  $p$ -banana  $\Theta$  as follows:

- the set  $A$  is the subset of vertices of  $\bigcup_{1 \leq i \leq p} V(P_i)$  at distance at most  $kr - (r + q)$  from  $v$  (note that since  $k \geq k_0$ , we have  $kr \geq r + q$  and the distance  $kr - (r + q)$  is well defined);
- the set  $B$  is the set  $x_1, x_2, \dots, x_\ell$ , together with shortest paths (of length at most  $r$ ) between consecutive vertices  $x_i, x_{i+1}$  for any  $1 \leq i \leq \ell - 1$ ;
- each of the  $p$  geodesic paths of  $\Theta$  is a minimal  $A$ - $B$  path in  $P_i$  (i.e. this subpath contains vertices at distance at least  $kr - (r + q)$  from  $v$ ).

Note that it immediately follows from the definition of  $\Theta$  that  $A$  and  $B$  induce connected subgraphs of  $G$ .

Since the vertices  $x_1, x_2, \dots, x_\ell$  are in  $A_k$ , they are at distance at least  $kr$  from  $v$ , and the vertices on the shortest paths connecting two consecutive  $x_i, x_{i+1}$  are at distance at least  $kr - r$  from  $v$ . It follows that  $B$  is at distance at least  $(k - 1)r$  from  $v$  and thus at distance at least  $(k - 1)r - (kr - (r + q)) = q$  from  $A$ , as desired.

It remains to check that any two vertices  $x, y$  lying on two different paths of  $\Theta$  are at distance at least  $q$  apart in  $G$ . By definition of  $A$ , we may assume that  $d_G(x_{i(i)}, x) \leq (k + 1)r - (kr - (r + q)) \leq 2r + q$  and similarly, for some  $i \neq j$ , that  $d_G(x_{i(j)}, y) \leq 2r + q$ . So if  $d_G(x, y) \leq q$ , we obtain  $d_G(x_{i(i)}, x_{i(j)}) \leq 4r + 3q$ , which is a contradiction.

We have proved that  $\Theta$  is a  $q$ -fat  $p$ -banana, which contradicts our assumption that  $G$  does not contain any  $q$ -fat  $p$ -banana. This shows that each  $r$ -component of  $C_0$  or  $C_1$  is  $(5r + 3q)p$ -bounded, as desired.  $\square$

Note that a graph containing a 2-fat 2-banana also contains a chordless cycle of length at least 4. It follows that chordal graphs have no 2-fat 2-bananas. Setting  $p = 2, q = 2$  in Lemma 3.2, we obtain that for any  $r > 0$ , chordal graphs have a cover by 2 sets whose  $r$ -components are  $(10r + 12)$ -bounded. Recall that any  $r$ -ball intersects at most one  $2r$ -component of a given set, so we obtain a cover of any chordal graph by  $(20r + 12)$ -bounded sets such that any  $r$ -ball intersects at most two sets of the cover. For unweighted chordal graphs, we can assume that  $r \geq 1$  is an integer and thus  $20r + 12 \leq 32r$ . Hence, we obtain a  $(32, 2)$ -weak sparse partitioning scheme for chordal graphs. Note that Filtser [15] obtained a  $(24, 3)$ -weak partition scheme for this class of graphs.

**3.3. Fat minors.** A (weighted or unweighted) graph  $G$  contains a graph  $H$  as a *minor* if  $G$  contains  $|V(H)|$  vertex-disjoint subsets  $\{T_v \mid v \in V(H)\}$ , each inducing a connected subgraph in  $G$ , and such that for every edge  $uv$  in  $H$ ,  $T_u$  and  $T_v$  are connected by an edge in  $G$ . We will also need the following interesting metric variant of minors: for some integer  $q \geq 1$ , a weighted graph  $G$  contains a graph  $H$  as a  *$q$ -fat minor* if  $G$  contains  $|V(H)|$  vertex-disjoint subsets  $\{T_v \mid v \in V(H)\}$ , each inducing a connected subgraph in  $G$ , and such that

- any two sets  $T_u$  and  $T_v$  are at distance at least  $q$  apart in  $G$ ;
- for every edge  $uv$  in  $H$ ,  $T_u$  and  $T_v$  are connected by a path  $P_{uv}$  (of length at least  $q$ ) in  $G$ , such that
  - for any pair of distinct edges  $uv$  and  $xy$  of  $H$  (possibly sharing a vertex), the paths  $P_{uv}$  and  $P_{xy}$  are at distance at least  $q$  in  $G$ , and
  - for any edge  $uv$  in  $H$  and any vertex  $w$  distinct from  $u$  and  $v$ ,  $P_{uv}$  is at distance at least  $q$  from  $T_w$  in  $G$ .

An example of  $q$ -fat  $K_3$ -minor is depicted in Figure 2, right, where the last property (stating that sets  $T_w$  are far from paths  $P_{uv}$ ) is not mentioned explicitly, for the sake of readability.

This will not be needed in the remainder, but we mention nevertheless that for  $(0, 1]$ -weighted graphs, any  $q$ -fat  $H$ -minor can be transformed into a  $(q/3 - 1)$ -fat  $H$  minor in which all paths

are geodesic. Indeed, for an edge  $uv$  of  $H$  we can grow  $T_u$  along  $P_{uv}$  until the first vertex at graph distance at most  $q/3$  from  $T_v$  and then add in a geodesic  $Q_{uv}$  to  $T_v$ . If we do this procedure once for each edge, then the resulting geodesics are still  $q/3$  apart, and the vertices added to  $T_u$  will be at least  $q/3 - 1$  from each vertex in  $T_v$ .

Note that if  $G$  contains  $H$  as a  $q$ -fat minor, then it also contains  $H$  as a minor. Fat minors are related to fat bananas by the following simple lemma (we recall that here and in the remainder, we have chosen the interval  $(0, 1]$  for simplicity but it can be replaced with any interval  $(0, C]$ ,  $C > 0$ , without affecting our results).

**Lemma 3.3.** *Let  $G$  be a  $(0, 1]$ -weighted graph. If  $G$  does not contain the complete bipartite graph  $K_{2,p}$  as a  $q$ -fat minor, then  $G$  contains no  $(3q + 2)$ -fat  $p$ -banana.*

*Proof.* We prove the contrapositive. Consider a  $(3q + 2)$ -fat  $p$ -banana in  $G$ , i.e. connected sets  $A$  and  $B$  at distance at least  $3q + 2$  apart in  $G$ , together with  $p$  geodesic paths  $P_1, \dots, P_p$  connecting  $A$  and  $B$ , such that any pair of paths  $P_i, P_j$  is at distance at least  $3q + 2$  apart in  $G$ .

For any  $1 \leq i \leq p$ , we partition the vertices of each  $P_i$  into three parts  $P_{i,A}, P_{i,0}$  and  $P_{i,B}$ . The subpath  $P_{i,A}$  of  $P_i$  consists of all vertices at distance at most  $q$  from  $A$  and similarly for  $P_{i,B}$ . Since  $3q + 2 > 1$  and since the weight of any edge is at most 1, the set  $P_{i,0}$  is non-empty. Moreover, each  $P_{i,0}$  is at distance at least  $q$  from  $A$  and  $B$ , and the paths  $P_{i,A}$  and  $P_{i,B}$  are at least  $q$  apart as well. Finally, for  $i \neq j$  the lower bound of  $3q + 2$  on the distance between vertices in  $P_i$  and  $P_j$  remains. This implies that the sets  $A, B, P_1^0, P_2^0, \dots, P_p^0$ , together with the paths  $P_1^A, P_1^B, \dots, P_p^A, P_p^B$  define a  $q$ -fat  $K_{2,p}$  minor in  $G$ .  $\square$

Let  $G$  be a graph and  $H$  be an induced subgraph of  $G$ . We recall that in Section 2.4 we have defined the intrinsic shortest path metric  $d_H$  in  $H$ , by opposition with the  $(H, d_G)$ , which is the graph  $H$  equipped with the shortest path metric in  $G$ . We say that  $H$  contains  $\Theta$  as an *intrinsic* fat banana, or contains  $H$  as an *intrinsic* fat minor, if the definition of fat banana or fat minor is with respect to the metric  $d_H$ .

Given a graph  $H$ , the graph  $H^*$  is obtained from  $H$  by adding a universal vertex (i.e. adding a vertex and joining it to all the vertices of  $H$ ). We now prove that if a graph  $G$  avoids  $H^*$  as a minor, then the subgraph of  $G$  induced by any annulus in  $G$  avoids  $H$  as an intrinsic fat minor of comparable or greater length.

**Lemma 3.4.** *Let  $H$  be a graph, and let  $G$  be a weighted graph with no  $H^*$  minor. For any vertex  $v \in V(G)$ , and any real numbers  $0 < s < t$  and  $q \geq 2(t - s)$ , the subgraph  $G'$  of  $G$  induced by the vertices  $\{u \in V(G) \mid s \leq d_G(u, v) < t\}$  does not contain  $H$  as an intrinsic  $q$ -fat minor.*

*Proof.* Let  $A = \{u \in V(G) \mid d_G(u, v) < s\}$  and  $B = \{u \in V(G) \mid s \leq d_G(u, v) < t\}$ . Assume for the sake of contradiction that  $G' = G[B]$  contains  $H$  as an intrinsic  $q$ -fat minor. Let  $\{T_u \mid u \in V(H)\}$  and  $\{P_{uw} \mid uw \in E(H)\}$  be as in the definition of a  $q$ -fat minor. For each  $u \in V(H)$ , let  $P_u$  be a shortest path from  $T_u$  to  $v$  in  $G$  and let  $P_u^+ = P_u \cap B$ . Note that the length of  $P_u^+$  is less than  $t - s$ . For any  $u \in V(H)$ , set  $T_u^+ := T_u \cup V(P_u^+)$ .

Each of the sets  $T_u^+$  is connected and disjoint from  $A$ , and two sets  $T_u^+$  and  $T_w^+$  are at distance greater than  $q - 2(t - s) \geq 0$  apart in  $G'$  (in particular the sets are pairwise disjoint). The set  $A$  is connected and has an edge to  $T_u^+$  for all  $u \in V(H)$ . For any edge  $uw$  in  $H$ , let  $P_{uw}^+$  be a minimum subpath of  $P_{uw}$  between  $T_u^+$  and  $T_w^+$ . Since each path  $P_{uw}$  is at distance at least  $q > t - s$  from all the sets  $T_x$  with  $x \notin \{u, w\}$ , the subpaths  $P_{uw}^+$  are disjoint from all sets  $T_x^+$  with  $x \notin \{u, w\}$ . Moreover, since the paths  $P_{uw}$  are pairwise vertex-disjoint, and

disjoint from  $A$ , the subpaths  $P_{uv}^+$  are also pairwise vertex-disjoint and disjoint from  $A$ . It follows that  $G$  contains  $H^*$  as a minor, which is a contradiction.  $\square$

**3.4.  $K_{3,p}$ -minor free graphs.** Recall that Fujiwara and Papasoglu [16] proved that unweighted planar graphs have Assouad-Nagata dimension at most 3. We now prove that the dimension can be reduced to 2, and the class of graphs can be extended to all graphs avoiding  $K_{3,p}$  as a minor, for some fixed integer  $p \geq 2$ . Moreover the result below holds for weighted graphs (this will be used to derive our result on compact Riemannian surfaces in Section 4).

**Theorem 3.5.** *For any integer  $p \geq 2$ , any  $(0, 1]$ -weighted graph with no  $K_{3,p}$ -minor has a linear 2-dimensional control function  $D_p$ . Moreover, if  $G$  is  $[\epsilon, 1]$ -weighted, for some  $\epsilon > 0$ , then  $D_p(r) \leq c \cdot p^2 r$ , for some constant  $c = c(\epsilon) > 0$ . In particular, the class of  $(0, 1]$ -weighted graphs with no  $K_{3,p}$ -minor has asymptotic dimension at most 2 of linear type, and the class of unweighted graphs with no  $K_{3,p}$ -minor has Assouad-Nagata dimension at most 2.*

*Proof.* By Observation 2.3, it suffices to prove the result for connected graphs. Consider a  $(0, 1]$ -weighted connected graph  $G$  with no  $K_{3,p}$ -minor, and let  $L : V(G) \rightarrow \mathbb{R}$  be a rooted real projection of  $G$ , i.e. we fix a vertex  $v$  and for any  $u \in V(G)$  we set  $L(u) := d_G(u, v)$ . We fix  $S > 0$  and we consider a subgraph  $H$  of  $G$  induced by some  $(\infty, S)$ -bounded subset of  $V(G)$  (with respect to  $L$ ).

If  $v \in V(H)$ , then  $(H, d_H)$  is  $2S$ -bounded (since all the vertices of  $V(H)$  are at distance at most  $S$  from  $v$  in  $H$ ), and thus all  $r$ -components (with respect to  $d_H$ ) of  $V(H)$  are  $2S$ -bounded in  $d_H$ . If  $v \notin V(H)$  then by Lemma 3.4,  $H$  does not contain  $K_{2,p}$  as an intrinsic  $2S$ -fat minor. As a consequence, it follows from Lemma 3.3 that  $H$  does not contain any intrinsic  $(6S + 2)$ -fat  $p$ -banana. By Lemma 3.2,  $(H, d_H)$  has a 1-dimensional function  $D_H$  with  $D_H(r) \leq (5r + 18S + 6)p$ .

We obtain that in any case, any  $(\infty, S)$ -bounded subset of  $V(G)$  (with respect to  $L$ ) induces a graph with a 1-dimensional function  $D_S$  with  $D_S(r) \leq (5r + 18S + 6)p$  for any  $r > 0$ . By Corollary 2.7,  $G$  has a linear 2-dimensional control  $D_G$  depending only on  $p$ , and thus the class of  $(0, 1]$ -weighted graphs with no  $K_{3,p}$ -minor has asymptotic dimension at most 2 of linear type.

If  $G$  is  $[\epsilon, 1]$ -weighted, for  $\epsilon > 0$ , then we can assume that  $r \geq \epsilon$  and thus  $D_S(r) \leq (5r + 18S + 6)p \leq (5 + 6/\epsilon)pr + 18pS$ . By Corollary 2.7, the class of  $K_{3,p}$ -minor free  $[\epsilon, 1]$ -weighted graphs has Assouad-Nagata dimension at most 2, with a 2-dimensional control function  $D_p(r) \leq c \cdot p^2 r$ , for some constant  $c = c(\epsilon) > 0$ . The case of unweighted graphs corresponds to  $\epsilon = 1$ .  $\square$

## 4. SURFACES

**4.1. Graphs on surfaces.** In this section, a *surface* is a non-null compact connected 2-dimensional manifold without boundary. A surface can be orientable or non-orientable. The *orientable surface of genus  $h$*  is obtained by adding  $h \geq 0$  *handles* to the sphere; while the *non-orientable surface of genus  $k$*  is formed by adding  $k \geq 1$  *cross-caps* to the sphere. By the Surface classification theorem, any surface is of one of these two types (up to homeomorphism). The *Euler genus* of a surface  $\Sigma$  is defined as twice its genus if  $\Sigma$  is orientable, and as its non-orientable genus otherwise.

It is well known that any graph embeddable in the plane (or equivalently the sphere) excludes  $K_{3,3}$  as a minor. More generally, for any integer  $g \geq 0$ , no graph embeddable on a surface of

Euler genus  $g$  contains  $K_{3,2g+3}$  as a minor<sup>2</sup>. We thus obtain the result below as an immediate consequence of Theorem 3.5.

**Corollary 4.1.** *For any integer  $g \geq 0$ , any  $(0, 1]$ -weighted graph embeddable on a surface of Euler genus  $g$  has a linear 2-dimensional control function  $D_g$  depending only on  $g$ . Moreover, if  $G$  is  $[\epsilon, 1]$ -weighted, for some real  $\epsilon > 0$ , then  $D_g(r) \leq c \cdot g^2 r$ , for some constant  $c = c(\epsilon) > 0$ . In particular, the class of  $(0, 1]$ -weighted graphs embeddable on a surface of Euler genus  $g$  has asymptotic dimension at most 2 of linear type, and the class of unweighted graphs embeddable on a surface of Euler genus  $g$  has Assouad-Nagata dimension at most 2.*

**4.2. From graphs on surfaces to Riemannian surfaces.** A *Riemannian surface*  $(S, m)$  is a surface  $S$  (which is assumed to be compact, as defined above) together with a metric  $m$ , defined by a scalar product on the tangent space of every point. The only property that we will need is that for any point  $p \in S$ , there is a small open neighborhood  $N$  containing  $p$  that is strongly convex, i.e. any two points in  $N$  are joined by a unique shortest path. For more background on Riemannian surfaces, the interested reader is referred to the standard textbook [31].

The following result appears to be well known in the area. For instance it can be deduced from the work of Saucan [30]. Here we include a simple proof (suggested to us by Gaël Meignez) in dimension 2.

**Lemma 4.2.** *Let  $(S, m)$  be a Riemannian surface. Then there is a  $(0, 1]$ -weighted graph  $G$ , embedded in  $S$ , such that any point of  $S$  is at distance at most 2 from a vertex of  $G$  in  $S$  and for any vertices  $x, y \in V(G)$ ,  $d_m(x, y) \leq d_G(x, y) \leq 5d_m(x, y) + 3$ .*

*Proof.* We consider an inclusion-wise maximal set  $P$  of points of  $S$  that are pairwise at distance at least  $\frac{1}{5}$  apart in  $S$ . By maximality of  $P$ , the open balls of radius  $\frac{2}{5}$  centered in  $P$  cover  $S$ . Let  $G'$  be the graph with vertex set  $P$ , in which two points  $p, q \in P$  are adjacent if their balls of radius  $\frac{1}{2}$  intersect. For each such pair  $p, q$ , we join  $p$  to  $q$  by a shortest path (of length  $d_m(p, q) \leq 1$ ) on the surface  $S$ . Note that any two such shortest paths intersect in a finite number of points and segments (since otherwise we could find an arbitrarily small neighborhood containing two points joined by two distinct shortest paths). For each intersection point between two paths (and each end of an intersecting segment between the paths), we add a new vertex to  $G'$ . Let  $G$  be the resulting graph (where two vertices of  $G$  are adjacent if they are consecutive on some shortest path between vertices of  $G'$ ). By definition,  $G$  is properly embedded in  $S$ . Note that each edge  $e$  of  $G$  corresponds to a shortest path between the two endpoints of  $e$  in  $S$  (we denote the length of this shortest path by  $\ell_e$ ). From now on, we consider  $G$  as a weighted graph with weights  $(\ell_e)_{e \in E(G)}$ , and all path lengths and distances in  $G$  refer to the weighted shortest path metric induced by  $(\ell_e)_{e \in E(G)}$ . Note that by definition, all the weights are in the interval  $(0, 1]$ . For any two vertices  $p, q$  in  $G$ , we clearly have  $d_m(p, q) \leq d_G(p, q)$ . Consider now a length-minimizing geodesic  $\gamma$  (of length  $\ell = d_m(p, q)$ ) between  $p$  and  $q$  in  $S$ , and take  $k \leq 5\ell + 2$  points  $p_1, \dots, p_k$  (in this order) on  $\gamma$ , with  $p_1 = p$ ,  $p_k = q$ , and such that any two consecutive points  $p_i, p_{i+1}$  are at distance at most  $\frac{1}{5}$  apart in  $S$ . Recall that each point of  $S$  is at distance at most  $\frac{2}{5}$  from a point of  $P$ . It follows that any segment  $[p_i, p_{i+1}]$  of  $\gamma$  is contained in a ball of radius  $\frac{1}{10} + \frac{2}{5} = \frac{1}{2}$  centered in a point  $r_i$  of  $P$  (which is a vertex of  $G'$ ). By definition of  $G'$ , for any  $1 \leq i \leq k - 1$ ,  $r_i$  and  $r_{i+1}$  coincide or

<sup>2</sup>Since the class of graphs embeddable in a surface of Euler genus  $g$  is closed under taking minors, it suffices to show that  $K_{3,2g+3}$  cannot be embedded on a surface of Euler genus  $g$ . This is a simple consequence of Euler's Formula (see Proposition 4.4.4 in [25]).



are adjacent in  $G'$ . Note that  $p = p_1$  is on some edge of length at most  $2 \cdot \frac{1}{2} = 1$  between two vertices of  $V(G') = P$ , and thus  $p$  is at distance at most  $\frac{1}{2}$  from one of these points (call it  $r_0$ ) in  $G$ . In particular, by definition of  $G'$ ,  $r_0$  coincides or is adjacent with  $r_1$  in  $G'$ . Similarly,  $q = p_k$  is at distance at most  $\frac{1}{2}$  in  $G$  from a vertex  $r_k \in V(G') = P$  that coincides or is adjacent to  $r_{k-1}$  in  $G'$ . As a consequence, there is a path of length at most  $2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} \cdot k \leq 5\ell + 3$  between  $p$  and  $q$  in  $G$ .  $\square$

**4.3. Quasi-isometry.** Two metric spaces  $(X, d_X)$  and  $(Y, d_Y)$  are *quasi-isometric* if there is a map  $f : X \rightarrow Y$  and constants  $\epsilon \geq 0$ ,  $\lambda \geq 1$ , and  $C \geq 0$  such that  $Y \subseteq N_C(f(X))$  (recall that  $N_C(S)$  denotes the  $C$ -neighborhood of  $S$ , so this condition means that for any  $y \in Y$  there is  $x \in X$  such that  $d_Y(y, f(x)) \leq C$ ), and for every  $x_1, x_2 \in X$ ,

$$\frac{1}{\lambda}d_X(x_1, x_2) - \epsilon \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + \epsilon.$$

It is not difficult to check that the definition is symmetric. Moreover, if for every  $r \geq 0$ ,  $X$  has a cover by  $n$  sets whose  $r$ -components are  $D_X(r)$ -bounded and there exist a map  $f : X \rightarrow Y$  as above, then for every  $r \geq 0$ ,  $Y$  has a cover by  $n$ -sets whose  $r$ -components are  $D_Y(r)$ -bounded, where  $D_Y$  only depends on  $D_X$  and the constants  $\lambda$ ,  $\epsilon$ , and  $C$  in the definition of  $f$ . Moreover,  $D_X$  is linear if and only if  $D_Y$  is linear. This implies that asymptotic dimension (of linear type) is invariant under quasi-isometry. Moreover, if all members of a family  $\mathcal{X} = X_1, X_2, \dots$  of metric spaces are quasi-isometric to some metric space  $Y$ , with uniformly bounded constants  $\lambda$ ,  $\epsilon$ , and  $C$  in the definition of the quasi-isometry map, then  $\text{asdim } \mathcal{X} \leq \text{asdim } Y$ , and the same holds for the asymptotic dimension of linear type.

Combining Corollary 4.1 with Lemma 4.2 and the remarks above on the invariance of asymptotic dimension of linear type under quasi-isometry, we obtain the following result.

**Theorem 4.3.** *For any integer  $g \geq 0$ , the class of compact Riemannian surfaces of Euler genus  $g$  has asymptotic dimension at most 2 of linear type.*

*Proof.* By Lemma 4.2, every compact Riemannian surface  $(S, m)$  of genus  $g$  is quasi-isometric to some  $(0, 1]$ -weighted graph embeddable on  $S$ , with constants in the quasi-isometry that are uniform (in fact the constants are even independent of  $g$ ). By Corollary 4.1, such graphs have asymptotic dimension at most 2 of linear type (uniformly, i.e. with constants that only depend on  $g$ ), and thus the class of compact Riemannian surfaces of Euler genus  $g$  has asymptotic dimension at most 2 of linear type.  $\square$

## 5. TOPOLOGICAL GRAPHS AND $H$ -MINOR FREE GRAPHS OF BOUNDED DEGREE

For a graph  $G$ , a *tree-decomposition* of  $G$  consists of a tree  $T$  and collection  $\mathcal{B} = (B_x : x \in V(T))$  of subsets of  $V(G)$ , called *bags*, indexed by the nodes of  $T$ , such that:

- for every vertex  $v$  of  $G$ , the set  $\{x \in V(T) : v \in B_x\}$  induces a non-empty subtree of  $T$ , and
- for every edge  $vw$  of  $G$ , there is a vertex  $x \in V(T)$  for which  $v, w \in B_x$ .

The *width* of such a tree-decomposition is  $\max\{|B_x| : x \in V(T)\} - 1$ . The *treewidth* of a graph  $G$  is the minimum width of a tree-decomposition of  $G$ .

A graph  $G$  has *layered treewidth* at most  $t$  if it has a tree-decomposition and a layering  $L_0, L_1, \dots$  such that each bag of the tree-decomposition intersects each layer in at most  $t$  vertices. Note that if such a layering exists, then any  $\ell$  consecutive layers induce a graph of treewidth at most  $t\ell$  (it suffices to restrict the tree-decomposition of  $G$  to the  $\ell$  layers under

consideration, and thus each bag contains at most  $t\ell$  vertices). It follows that families of graphs of bounded layered treewidth are layerable by families of bounded treewidth.

Using the terminology of layerable classes, these results can be rephrased as follows.

**Theorem 5.1.** *Let  $\mathcal{T} = (\mathcal{T}_i)_{i \in \mathbb{N}}$ , where  $\mathcal{T}_i$  is the class of graphs of treewidth at most  $i$ . Then for any  $t$ , the class of graphs of layered treewidth at most  $t$  is  $t$ -linearly  $\mathcal{T}$ -layerable.*

Theorems 2.6 and 5.1 thus have the following immediate consequence.

**Theorem 5.2.** *Assume that there exists  $k$ , such that for any  $t$ , the class of graphs of treewidth at most  $t$  has asymptotic dimension at most  $k$ . Then any class of bounded layered treewidth has asymptotic dimension at most  $t + 1$ . The analogous statement holds for the asymptotic dimension of linear type and the Assouad-Nagata dimension.*

A graph is  $(g, k)$ -planar if it can be drawn in a surface of Euler genus  $g$  with at most  $k$  crossings per edge. It was proved by Dujmović, Eppstein, and Wood [11] that for any  $g$  and  $k$  the class of  $(g, k)$ -planar graphs has bounded layered treewidth. An *apex graph* is a graph  $G$  such that there is a vertex  $v$  with the property that  $G - v$  is planar. It is well known that excluding a planar graph as a minor is equivalent to having bounded treewidth. Dujmović, Morin, and Wood [13] proved that for any apex graph  $H$ , any graph excluding  $H$  as a minor has a bounded layered treewidth (they also prove the converse for minor-closed classes). We thus obtain the following immediate corollary of Theorem 5.2.

**Corollary 5.3.** *Assume that there exists  $k$ , such that for any  $t$ , the class of graphs of treewidth at most  $t$  has asymptotic dimension at most  $k$ . Then*

- *for any fixed integers  $\ell$  and  $g$ , the class of  $(g, \ell)$ -planar graphs, and*
- *for any apex graph  $H$ , the class of  $H$ -minor free graphs*

*have asymptotic dimension at most  $k + 1$ . The analogous result holds for the asymptotic dimension of linear type and the Assouad-Nagata dimension.*

Recall that Fujiwara and Papasoglu asked whether there is a constant  $k$  (possibly  $k = 2$ ) such that for any proper minor-closed class of graphs has asymptotic dimension at most  $k$  (Question 2). What Corollary 5.3 shows is that in order to prove that this holds for graphs excluding an apex-minor, it is enough to prove it for classes of graphs of bounded treewidth.

We now prove that the question of Fujiwara and Papasoglu has a positive answer when we restrict ourselves to graphs of bounded degree. Recall that since any infinite family of cubic expanders has unbounded asymptotic dimension [17], restricting the maximum degree only is not enough to imply that a class of graphs has bounded asymptotic dimension.

In [8] (see also [32]), it is proved that if a graph  $G$  has treewidth  $t \geq 1$  and maximum degree  $\Delta \geq 1$ , then  $G$  is a subgraph of the strong product  $T \boxtimes K_{24t\Delta}$ , where  $T$  is a tree (an equivalent formulation is that  $G$  can be obtained as a subgraph of a tree  $T$  in which each vertex has been replaced with a clique on  $24t\Delta$  vertices and each edge by a complete bipartite graph between the corresponding cliques). This implies the following layerability result.

**Theorem 5.4.** *Let  $\mathcal{S} = (\mathcal{S}_i)_{i \in \mathbb{N}}$ , where  $\mathcal{S}_i$  is the class of graphs whose connected components have size at most  $i$ . Then for any integers  $t$  and  $\Delta$ , the class of graphs of treewidth at most  $t$  and maximum degree at most  $\Delta$  is  $\mathcal{S}$ -layerable.*

*Proof.* Let  $G$  be a graph of treewidth at most  $t$  and maximum degree at most  $\Delta$ , and let  $T$  be a tree such that  $G$  is a subgraph of  $T \boxtimes K_{24t\Delta}$ . Note that since  $G$  has maximum degree  $\Delta$ , and is a subgraph of  $T \boxtimes K_{24t\Delta}$ , we can assume that  $T$  has maximum degree at most  $24t\Delta \cdot \Delta = 24t\Delta^2$ .

We consider a BFS-layering of  $T$  (see Section 2.3), and the induced layering of  $G$  (i.e. a vertex of  $G$  is in layer  $i$  if the corresponding vertex of  $T$  is in layer  $i$  of the BFS-layering of  $T$ ). Call  $L_0, L_1, \dots$  this layering of  $G$  (it can be checked that it is indeed a layering, since any edge of  $G$  is either inside some  $K_{24t\Delta}$  or between two such cliques). Consider  $\ell$  consecutive layers in this layering. In  $T$ , the corresponding layers are also consecutive and since  $T$  is a tree of maximum degree at most  $24t\Delta^2$ , each component of the subgraph of  $T$  induced by these  $\ell$  layers is a tree of at most  $(24t\Delta^2)^\ell$  vertices. It follows that any  $\ell$  consecutive layers in  $G$  induce a graph whose connected components have size at most  $24t\Delta(24t\Delta^2)^\ell$ . This implies that the class of graphs of treewidth at most  $t$  and maximum degree at most  $\Delta$  is  $\mathcal{S}$ -layerable.  $\square$

Since for any  $i$ , the class  $\mathcal{S}_i$  of graphs whose connected components have size at most  $i$  has asymptotic dimension 0, we obtain the following as a direct consequence of Theorems 2.6 and 5.4.

**Theorem 5.5.** *For any integers  $t$  and  $\Delta$ , the class of graphs of treewidth at most  $t$  and maximum degree at most  $\Delta$  has asymptotic dimension at most 1.*

Theorems 5.5 and 5.2 now imply the following.

**Theorem 5.6.** *For any integers  $t$  and  $\Delta$ , the class of graphs of layered treewidth at most  $t$  and maximum degree at most  $\Delta$  has asymptotic dimension at most 2.*

In [12], it is proved that graphs of bounded degree from a proper minor-closed class have bounded layered treewidth (this is a direct consequence of Theorem 19 and Lemma 12 in the paper). We thus obtain Theorem 1.7 as a direct consequence of Theorem 5.6.

Using that  $(g, k)$ -planar graphs have bounded layered treewidth in combination with Theorem 5.6, we also obtain the following unconditional counterpart of Corollary 5.3.

**Theorem 5.7.** *For any integers  $g \geq 0$ ,  $k \geq 0$  and  $\Delta \geq 1$ , the class of  $(g, k)$ -planar graphs of maximum degree at most  $\Delta$  has asymptotic dimension at most 2.*

The hypothesis of Corollary 5.3 is that there exists  $k$ , such that for any  $t$ , the class of graphs of treewidth at most  $t$  has asymptotic dimension at most  $k$ . A natural question is whether this might hold not only for the asymptotic dimension, but also for the asymptotic dimension of linear type, or equivalently (for unweighted graphs) for the Assouad-Nagata dimension.

We now prove that the answer to this stronger question is negative. In particular this implies that the asymptotic dimension cannot be replaced with the asymptotic dimension of linear type or the Assouad-Nagata dimension in Question 2.

Given a graph  $G$  and an integer  $k$ , the  $k$ -subdivision of  $G$ , denoted by  $G^{(k)}$ , is obtained from  $G$  by replacing each edge of  $G$  by a path on  $k + 1$  edges. We start with the following simple observation.

**Observation 5.8.** *For every integers  $k$  and  $n$ , if  $G^{(k)}$  admits an  $n$ -dimensional control function  $D$  which is a dilation, then  $D$  is also an  $n$ -dimensional control function for  $G$ .*

*Proof.* Let  $D(r) = c \cdot r$ , for some constant  $c > 0$ . Consider the graph  $G$  and fix some  $r > 0$ . Let  $U_1, \dots, U_{n+1}$  be a cover of  $G^{(k)}$  by  $n + 1$  sets whose  $(k + 1)r$ -components are  $D((k + 1)r)$ -bounded, and consider the restriction  $U'_1, \dots, U'_{n+1}$  of this cover to  $V(G)$ . In  $(G, d_G)$ , each  $r$ -component of  $U'_i$  (for some  $1 \leq i \leq n + 1$ ) is precisely the restriction of a  $(k + 1)r$ -component of  $U_i$  (in  $G^{(k)}$  and the associated graph metric) to  $V(G)$ , and thus each  $r$ -component of  $U'_i$  is  $D((k + 1)r)$ -bounded with respect to the shortest path metric in  $G^{(k)}$ . Note that  $D((k + 1)r) = c \cdot (k + 1)r$  and for any two vertices  $u$  and  $v$  in  $G$ , we have

$d_{G^{(k)}}(u, v) = (k + 1) \cdot d_G(u, v)$ . It follows that for any  $1 \leq i \leq n + 1$ , each  $r$ -component of  $U'_i$  is  $cr$ -bounded, and thus  $D(r)$ -bounded, as desired.  $\square$

Recall that Corollary 5.3 shows that if there was an integer  $k$ , such that for any  $t$ , the class of graphs of treewidth at most  $t$  has Assouad-Nagata dimension at most  $k$ , then the class of 1-planar graphs would have Assouad-Nagata dimension at most  $k + 1$ .

Now, take any family  $\mathcal{F}$  of graphs of unbounded Assouad-Nagata dimension (for instance, grids of increasing size and dimension), and for each graph  $G \in \mathcal{F}$ , let  $G^* := G^{(|E(G)|)}$  (i.e. subdivide each of  $G$  as many times as the number of edges of  $G$ ). Note that  $G^*$  is 1-planar (this can be seen by consider a drawing of  $G$  in the plane and subdividing each edge at least once between any two consecutive crossings involving this edge). Define  $\mathcal{F}^* := \{G^* \mid G \in \mathcal{F}\}$ , and note that all the graphs of  $\mathcal{F}^*$  are 1-planar. If the class of 1-planar graphs had bounded Assouad-Nagata dimension, then by Observation 5.8,  $\mathcal{F}$  would have bounded Assouad-Nagata dimension, which is a contradiction. We obtain the following.

**Theorem 5.9.** *There is no integer  $k$  such that for any  $t \geq 0$ , the class of graphs of treewidth at most  $t$  has Assouad-Nagata dimension at most  $k$ .*

As we have seen in the introduction, Assouad-Nagata dimension and asymptotic dimension of linear type are equivalent in unweighted graphs, so the analogous result holds for the asymptotic dimension of linear type for unweighted graphs.

## 6. GEOMETRIC GRAPHS AND GRAPHS OF POLYNOMIAL GROWTH

We now explain some consequences of Theorem 2.6 for geometric graph classes and graph classes of polynomial growth.

For an integer  $d \geq 1$ , and some real  $C \geq 1$ , let  $\mathcal{D}^d(C)$  be the class of graphs  $G$  whose vertices can be mapped to points of  $\mathbb{R}^d$  such that

- any two vertices of  $G$  are mapped to points at (Euclidean) distance at least 1 apart, and
- any two adjacent vertices of  $G$  are mapped to points at distance at most  $C$  apart.

For technical reasons it will be convenient to consider also a sequence of subclasses of  $\mathcal{D}^d(C)$ . For  $1 \leq i \leq d$ , and some real  $K > 0$ , we let  $\mathcal{D}_i^d(C, K)$  be the set of graphs of  $\mathcal{D}^d(C)$  such that the points in the definition are restricted to the subspace  $\mathbb{R}^i \times [0, K]^{d-i}$  of  $\mathbb{R}^d$ . Note that the box  $[0, K]^{d-i}$  can be replaced with any translation of  $[0, K]^{d-i}$  without changing the definition of  $\mathcal{D}_i^d(C, K)$ . Note also that  $\mathcal{D}_0^d(C, K) \subseteq \mathcal{D}_1^d(C, K) \subseteq \dots \subseteq \mathcal{D}_d^d(C, K) = \mathcal{D}^d(C)$ .

Given a graph  $G \in \mathcal{D}^d(C)$  and a mapping of its vertices in  $\mathbb{R}^d$  as in the definition, a natural layering of  $G$  can be obtained as follows: fix some dimension  $1 \leq j \leq d$ , and for any  $k \in \mathbb{Z}$ , let  $L_k$  be the set of vertices of  $G$  whose  $j$ -th coordinate lie in the interval  $[kC, (k + 1)C)$ . By definition of  $\mathcal{D}^d(C)$  this is clearly a layering, and any  $\ell$  consecutive layers induce a graph of  $\mathcal{D}_{d-1}^d(C, \ell C)$ . More generally, if  $i \geq 1$  and  $G \in \mathcal{D}_i^d(C, K)$ , and the dimension chosen for the definition of the layering of  $G$  is  $j \leq i$ , then any  $\ell$  consecutive layers in the layering induce a graph of  $\mathcal{D}_{i-1}^d(C, \max(K, \ell C))$ .

With these observations in mind, the following result is now a fairly direct consequence of Theorem 2.6.

**Theorem 6.1.** *For any integer  $d \geq 1$  and real  $C \geq 1$ , the class  $\mathcal{D}^d(C)$  has asymptotic dimension at most  $d$ .*

*Proof.* We will prove by induction on  $0 \leq i \leq d$  that for any fixed  $K$ , the class  $\mathcal{D}_i^d(C, K)$  has asymptotic dimension at most  $i$ . Since for every real  $K > 0$ ,  $\mathcal{D}_d^d(C, K) = \mathcal{D}^d(C)$ , this will prove the desired result.

We start with the base case  $i = 0$ . Then for any representation of a graph  $G \in \mathcal{D}_0^d(C, K)$  in  $\mathbb{R}^d$ , all the vertices of  $G$  are mapped to  $[0, K]^d$ . Since the points are pairwise at distance at least 1 apart, a simple volume computation shows that  $G$  contains a bounded number of vertices, and thus  $\mathcal{D}_0^d(C, K)$  has asymptotic dimension 0.

Assume now that  $1 \leq i \leq d$ , and for any fixed  $K'$ , the class  $\mathcal{D}_{i-1}^d(C, K')$  has asymptotic dimension at most  $i - 1$ . Consider a real  $K > 0$  and a graph  $G \in \mathcal{D}_i^d(C, K)$ , together with a representation of  $G$  in  $\mathbb{R}^i \times [0, K]^{d-i}$ . Construct the natural layering of  $G$  (defined above) in dimension  $i$ . As observed above, any  $\ell$  layers in this layering induce a graph of  $\mathcal{D}_{i-1}^d(C, \max(K, \ell C))$ . By defining the sequence  $\mathcal{B} = (\mathcal{D}_{i-1}^d(C, \max(K, \ell C)))_{\ell \in \mathbb{N}}$ , we obtain that the class  $\mathcal{D}_i^d(C, K)$  is  $\mathcal{B}$ -layerable. By the induction hypothesis, every class of  $\mathcal{B}$  has asymptotic dimension at most  $i - 1$ , thus it follows from Theorem 2.6 that  $\mathcal{D}_i^d(C, K)$  has asymptotic dimension at most  $i - 1 + 1 = i$ , as desired.  $\square$

We now explore some consequences of Theorem 6.1.

Consider first the class of *subgraphs* of contact graphs of unit balls in  $\mathbb{R}^d$ . A graph  $G$  is in this class if the vertices of  $G$  can be mapped to points of  $\mathbb{R}^d$  such that any two vertices are mapped to points at distance at least 1 apart, while adjacent vertices are mapped to points at distance exactly 1 apart. This class is precisely  $\mathcal{D}^d(1)$ , and thus Theorem 6.1 implies that it has asymptotic dimension at most  $d$  (note that this class is a natural generalization of  $d$ -dimensional grids, whose asymptotic dimension is  $d$ ).

**Corollary 6.2.** *For any  $d \geq 1$ , the class of subgraphs of contact graphs of unit balls in  $\mathbb{R}^d$  has asymptotic dimension at most  $d$ .*

Note that since  $d$ -dimensional grids are contact graphs of unit balls in  $\mathbb{R}^d$ , this immediately implies Theorem 1.9.

We note that the property that pairs of vertices are mapped to points at distance at least 1 apart in the definition of  $\mathcal{D}^d(C)$  is only used in the volume argument showing that any bounded region contains a bounded number of points. So we could replace it by the condition that any unit  $d$ -cube (the cartesian product of  $d$  unit intervals) in  $\mathbb{R}^d$  contains at most  $k$  points, for some universal constant  $k$ , without changing the results.

Note that all the unconditional results that have been obtained in this section and the previous one concern graphs of bounded degree, which in particular are metric spaces of *bounded geometry* (i.e. metric spaces for which there is a function  $f$  such any  $r$ -ball contains at most  $f(r)$  points). We now give a consequence of Theorem 2.6 for a graph class with unbounded degree. A *unit ball graph* in  $\mathbb{R}^d$  is the intersection graph of a set of unit balls in  $\mathbb{R}^d$ . In other words, the vertices of a unit ball graph  $G$  can be mapped to points of  $\mathbb{R}^d$  such that any two vertices are adjacent if and only if the corresponding points are at (Euclidean) distance at most 1 apart. It should be mentioned that this is very different from the previous application on (subgraphs of) contacts of unit balls. Here we really need to consider the intersection graph itself, not some of its subgraphs.

**Theorem 6.3.** *For any integer  $d$ , the class of unit ball graphs in  $\mathbb{R}^d$  has asymptotic dimension at most  $d$ .*

*Proof.* We denote the class of unit ball graphs in  $\mathbb{R}^d$  by  $\mathcal{B}^d$ . As in the proof of Theorem 6.1, for any real  $K > 0$  and  $0 \leq i \leq d$  we consider the class  $\mathcal{B}_i^d(K)$  of unit ball graphs whose points

are restricted to the subspace  $\mathbb{R}^i \times [0, K]^{d-i}$ . We again note that  $[0, K]^{d-i}$  can be replaced with the cartesian product of  $d-i$  intervals of length  $K$  without changing the definition of the class, and  $\mathcal{B}_0^d(K) \subseteq \mathcal{B}_1^d(K) \subseteq \dots \subseteq \mathcal{B}_d^d(K) = \mathcal{B}^d$ .

We will prove by induction on  $0 \leq i \leq d$  that for any fixed  $K$ , the class  $\mathcal{B}_i^d(K)$  has asymptotic dimension at most  $i$ . Since for every real  $K > 0$ ,  $\mathcal{B}_d^d(K) = \mathcal{B}^d$ , this will prove the desired result.

We start with the base case  $i = 0$ . Then for any representation of a graph  $G \in \mathcal{B}_0^d(K)$  in  $\mathbb{R}^d$ , all the vertices of  $G$  are mapped to  $[0, K]^d$ . We claim that each connected component of  $G$  has diameter at most  $2(\sqrt{d}K)^d$ . To see this, take a shortest path  $P$  of length more than  $2(\sqrt{d}K)^d$  in  $G$ . Observe that  $[0, K]^d$  can be partitioned into  $(\sqrt{d}K)^d$  cubes of diameter at most 1, and that by the pigeonhole principle some of these cubes contains at least 3 vertices of  $P$ . Since this cube has diameter at most 1, the three corresponding vertices are pairwise adjacent in  $G$ , which contradicts the fact that  $P$  is a shortest path (since a shortest path is an induced path). Hence, all the connected components of  $\mathcal{B}_0^d(K)$  are  $2(\sqrt{d}K)^d$ -bounded, and thus it follows that  $\mathcal{B}_0^d(K)$  has asymptotic dimension 0.

The rest of the proof is now exactly the same as that of Theorem 6.1. Assume that  $1 \leq i \leq d$ , and for any fixed  $K'$ , the class  $\mathcal{B}_{i-1}^d(K')$  has asymptotic dimension at most  $i-1$ . Consider a real  $K > 0$  and a graph  $G \in \mathcal{B}_i^d(K)$ , together with a representation of  $G$  in  $\mathbb{R}^i \times [0, K]^{d-i}$ . Construct the natural layering of  $G$  (defined before Theorem 6.1) in dimension  $i$ . As before, any  $\ell$  layers in this layering induce a graph of  $\mathcal{B}_{i-1}^d(\max(K, \ell))$ . By defining the sequence  $\mathcal{B} = (\mathcal{B}_{i-1}^d(\max(K, \ell)))_{\ell \in \mathbb{N}}$ , we obtain that the class  $\mathcal{B}_i^d(K)$  is  $\mathcal{B}$ -layerable. By the induction hypothesis, every class of  $\mathcal{B}$  has asymptotic dimension at most  $i-1$ , thus it follows from Theorem 2.6 that  $\mathcal{B}_i^d(K)$  has asymptotic dimension at most  $i-1+1=i$ , as desired.  $\square$

As before, we have restricted ourselves to unit balls, but the only property that has been used is that the ratio between the largest radius and the smallest radius of a ball is bounded uniformly. It follows that our results hold in this wider setting as well.

A graph has *growth* at most  $f$  if every  $r$ -ball contains at most  $f(r)$  vertices. A graph has *growth rate* at most  $d$  if it has growth at most  $f(r) = r^d$  (for  $r > 1$ ). Classes of graphs whose growth rate is uniformly bounded are also known as *classes of polynomial growth*. In [22] it is proved that any graph  $G$  with growth rate at most  $d$  is in the class  $\mathcal{D}^{O(d \log d)}(2)$ . We thus obtain the following result as a direct corollary of Theorem 6.1.

**Corollary 6.4.** *For any real  $d > 1$ , the class of graphs of growth rate at most  $d$  has asymptotic dimension  $O(d \log d)$ . In particular, classes of graphs of polynomial growth have bounded asymptotic dimension.*

Again, this is a fairly natural extension of the fact that  $d$ -dimensional grids have bounded asymptotic dimension ( $d$ -dimensional grids form the basic example of graphs of polynomial growth). It seems that previously, it was only known that *vertex-transitive* graphs of polynomial growth have bounded asymptotic dimension [17].

Let us now argue why the assumption that the growth is polynomial in Corollary 6.4 cannot be weakened. We say that a function  $f$  is *superpolynomial* if it can be written as  $f(r) = r^{g(r)}$  with  $g(r) \rightarrow \infty$  when  $r \rightarrow \infty$ .

**Theorem 6.5.** *For any superpolynomial function  $f$  that satisfies  $f(r) \geq 3r + 1$  for every  $r \in \mathbb{N}$ , the class of graphs with growth at most  $f$  has unbounded asymptotic dimension.*

*Proof.* Given  $p \geq 0$ , a  $p$ -subdivided 3-regular tree is obtained from a tree in which all internal vertices have degree 3 by subdividing each edge  $p$  times. Given two integers  $k, p$  and a graph

$G$ , we say that a graph  $G'$  is a  $(k, p)$ -stretch of  $G$  if it is obtained by subdividing every edge of  $G$   $k$  times, and replacing every vertex  $v \in V(G)$  with a  $p$ -subdivided 3-regular tree  $T_v$  with  $d_G(v)$  leaves, and with minimal radius with respect to this conditions, in an arbitrary way (see Figure 3 for an illustration). For any set  $S$  of vertices of  $G'$ , the *projection* of  $S$  in  $G$  is the set of vertices  $v$  of  $G$  such that  $T_v \cap S \neq \emptyset$ .

For every  $d \geq 2$ , let  $\mathcal{G}_d$  be the class of  $d$ -dimensional grids. Recall from Section 1 that  $\mathcal{G}_d$  has asymptotic dimension  $d$ . We now define  $\mathcal{G}_d^{k,p}$  to be the class of all  $(k, p)$ -stretches of graphs from  $\mathcal{G}_d$ . Note that all vertices  $v$  of a graph of  $\mathcal{G}_d$  have degree at most  $2d$ , and thus each tree  $T_v$  as above has radius at most  $\lceil \log_2(2d) \rceil \leq 4 \log_2 d$  and diameter at most  $2 \lceil \log_2(2d) \rceil \leq 8 \log_2 d$ .

We first show that  $\mathcal{G}_d^{k,p}$  has asymptotic dimension at least  $d$ , and then that the growth of  $\mathcal{G}_d^{k,p}$  can be controlled by tuning the parameters  $k$  and  $p$ .

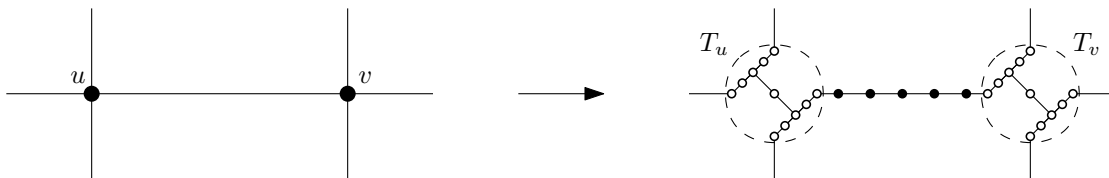


FIGURE 3. A local view of the  $(k, p)$ -stretch of a graph, for  $p = 1$  and  $k = 5$ .

**Claim.** For every  $d, k, p$ , the class  $\mathcal{G}_d^{k,p}$  has asymptotic dimension at least  $d$ .

Assume for a contradiction that there is a function  $D'$  such that for every  $r \in \mathbb{N}$ , every  $G' \in \mathcal{G}_d^{k,p}$  can be covered by  $d$  sets whose  $r$ -components are  $D'(r)$ -bounded.

Consider a graph  $G \in \mathcal{G}_d$ , and a  $(k, p)$ -stretch  $G' \in \mathcal{G}_d^{k,p}$  of  $G$ . By assumption, there are  $d$  sets  $U'_1, \dots, U'_d$  that cover  $G'$  in such a way that for each  $1 \leq i \leq d$ , each  $(k + 16p \log_2 d)r$ -component of  $U'_i$  is  $D'((k + 16p \log_2 d)r)$ -bounded. For  $1 \leq i \leq d$ , let  $U_i$  be the projection of  $U'_i$  in  $G$ . Note that  $U_1, \dots, U_d$  forms a cover of  $G$ , and every  $r$ -component of some  $U_i$  is a subset of the projection of a  $(k + 16p \log_2 d)r$ -component of  $U'_i$  in  $G$ . Therefore, for every  $i$ , every  $r$ -component of  $U_i$  is  $D'((k + 16p \log_2 d)r)$ -bounded. By setting  $D(r) = D'((k + 16p \log_2 d)r)$ , this shows that  $D$  is a  $(d - 1)$ -dimensional control function for  $\mathcal{G}_d$ , a contradiction.  $\diamond$

**Claim.** For every  $d, k, p \in \mathbb{N}$  with  $p \geq d$  and  $k \geq 4p \log_2 d$ , the class  $\mathcal{G}_d^{k,p}$  has growth at most  $r \mapsto 3r + 1$  when  $r \leq p$ ,  $r \mapsto 4dr + 1$  when  $p \leq r \leq k$ , and  $r \mapsto 4dk \cdot (1 + 2 \lceil r/k \rceil)^d$  when  $r \geq k$ .

Given a graph  $G \in \mathcal{G}_d^{k,p}$  and a vertex  $u \in G$ , we bound the number  $|B_r(u)|$  of vertices at distance at most  $r$  of  $u$  in  $G$ , as a function of  $r$ . Observe that if  $B_r(u)$  induces a tree with at most  $t$  leaves, then  $|B_r(u)| \leq tr + 1$ .

If  $r \leq p$ , then  $B_r(u)$  is a tree with at most 3 leaves, and thus  $|B_r(u)| \leq 3r + 1$ . If  $p \leq r \leq k$ , then  $B_r(u)$  is a tree with at most  $4d$  leaves (corresponding to the leaves of two adjacent trees  $T_v$  and  $T_w$ , each having at most  $2d$  leaves), and thus  $|B_r(u)| \leq 4dr + 1$ . Assume now that  $r \geq k$ . A graph in  $\mathcal{G}_d$  has growth rate at most  $r \mapsto (2r + 1)^d$ . Moreover, a graph of  $\mathcal{G}_d$  can be obtained from  $G$  by contracting trees of radius at most  $\frac{k+1}{2} + 4p \log_2 d$  with  $2d$  leaves (corresponding to each tree  $T_v$  together with half of each of the  $2d$  incident subdivided edges) into single vertices. By the observation above, each such tree contains at most  $2d(\frac{k+1}{2} + 4p \log_2 d) + 1 \leq 4dk$  vertices. Therefore, the vertex  $u$  is at distance at most  $r$  from at most  $4dk \cdot (1 + 2 \lceil r/k \rceil)^d$  vertices in  $G$ .  $\diamond$

Consider now a superpolynomial function  $f$  with  $f(r) \geq 3r + 1$  for every  $r$ , and fix an integer  $d$ . Since  $f$  is superpolynomial, there exists an integer  $p \geq d$  such that  $f(r) \geq 4dr + 1$  for every  $r \geq p$ . Similarly, there exists an integer  $k \geq 4p \log_2 d$  such that  $f(r) \geq 4dk \cdot (1 + 2\lceil r/k \rceil)^d$  for every  $r \geq k$ . Therefore it follows from the two claims above that the class  $\mathcal{G}_d^{k,p}$  has growth at most  $f$  and asymptotic dimension at least  $d$ . Since  $d$  was arbitrary, it follows that the class of graphs of growth at most  $f$  has unbounded asymptotic dimension.  $\square$

We note that the intriguing assumption of Theorem 6.5, requiring that  $f(r) \geq 3r + 1$  for every  $r$ , turns out to be necessary. If  $f(r_0) \leq 3r_0$  for some  $r_0$ , then the class of graphs with growth at most  $f$  has asymptotic dimension 1. To see this, observe first that graphs with no cycle of length at least  $2r_0 + 1$  have no  $(r_0 + 1)$ -fat 2-banana, and thus it follows from Lemma 3.2 that they have a 1-dimensional control function. On the other hand, if a connected graph  $G$  of growth at most  $f$  contains a cycle  $C$  of length at least  $2r_0 + 1$ , then any vertex of  $G$  is at distance less than  $r_0$  from  $C$  (since otherwise some ball of radius  $r_0$  centered in a vertex of  $C$  would contain at least  $3r_0 + 1$  vertices). In this case it is easy to find a cover by two sets whose  $r$ -components are  $\max(2r + 2r_0, 4r_0)$ -bounded, by dividing  $C$  into an even number of intervals of length in  $(r, 2r]$ , coloring the intervals with alternating colors 1 and 2, and adding all the remaining vertices to their closest interval (breaking ties arbitrarily).

It is worth mentioning that the class  $\mathcal{G}_d^{k,p}$  defined in the proof of Theorem 6.5 has interesting properties. The graphs in the class have maximum degree 3 and the vertices of degree 3 are arbitrarily far apart (by taking  $p$  arbitrarily large), so in particular it gives a simple example of a graph class of bounded degree and unbounded asymptotic dimension (without relying on expansion properties). It also allows for a precise forbidden (induced) subgraph characterisation of bounded asymptotic dimension. Given a graph  $H$ , let  $\text{Forb}(H)$  be the class of graphs excluding  $H$  as a subgraph.

Assume that every connected component of  $H$  is a path or a subdivided  $K_{1,3}$ , and let us argue by induction on the number of connected components that  $\text{Forb}(H)$  has asymptotic dimension at most 1. If  $H$  has a single component, then  $H$  is itself a path or a subdivided  $K_{1,3}$ , and an argument similar to the proof above that any class of growth  $f$  with  $f(r_0) \leq 3r_0$  for some  $r_0$  has asymptotic dimension 1 shows that  $\text{Forb}(H)$  has asymptotic dimension at most 1. To derive the general case, we note that for any two graph classes  $\mathcal{G}$  and  $\mathcal{G}'$ , if every graph in  $\mathcal{G}$  contains a bounded number of vertices whose removal yields a graph in  $\mathcal{G}'$ , then the asymptotic dimension of  $\mathcal{G}$  is at most that of  $\mathcal{G}'$ . It suffices then to observe that if  $H$  is not connected and  $H'$  is a connected component of  $H$ , then every graph in  $\text{Forb}(H)$  either belongs to  $\text{Forb}(H')$  or contains  $|V(H')|$  vertices whose removal yields a graph in  $\text{Forb}(H \setminus H')$ .

On the other hand, if  $H$  is a connected graph distinct from a path or a subdivided  $K_{1,3}$ , then the class  $\mathcal{G}_d^{k,p}$  (for increasing  $d$  and suitable parameters  $k, p$ ) shows that  $\text{Forb}(H)$  has unbounded asymptotic dimension. We thus obtain the following result.

**Observation 6.6.** *For any graph  $H$ , the class  $\text{Forb}(H)$  has asymptotic dimension at most 1 if  $H$  is a disjoint union of paths and subdivided  $K_{1,3}$ , and unbounded asymptotic dimension otherwise.*

Given a graph  $H$ , let  $\text{Forb}_i(H)$  denote the class of graphs excluding  $H$  as an induced subgraph. If  $H$  is a path (or a disjoint union of paths) then any graph in  $\text{Forb}_i(H)$  consists of connected components of bounded diameter, and thus  $\text{Forb}_i(H)$  has asymptotic dimension 0. On the other hand, if  $H$  has a connected component that is not a path then it is not difficult to check that it does not appear as an induced subgraph in a graph of  $\mathcal{G}_d^{k,p}$  for sufficiently large



$k$  and  $p$  (when  $H$  is a subdivided  $K_{1,3}$  we need to replace every vertex of degree 3 in graphs of  $\mathcal{G}_d^{k,p}$  with a triangle, which does not impact the asymptotic dimension). As a consequence, if  $H$  has a connected component other than a path, then  $\text{Forb}_i(H)$  has unbounded asymptotic dimension. We thus obtain the following.

**Observation 6.7.** *For any graph  $H$ , the class  $\text{Forb}_i(H)$  has asymptotic dimension 0 if  $H$  is a disjoint union of paths, and unbounded asymptotic dimension otherwise.*

## 7. PATHWIDTH

The *pathwidth* of a graph  $G$  is the minimum width of a tree-decomposition  $(T, \mathcal{B})$  of  $G$  of width at most  $k$  such that  $T$  is a path. Equivalently  $G$  has pathwidth at most  $k$  if and only if  $G$  is a spanning subgraph of some interval graph of clique number at most  $k + 1$ .

In this section it will be convenient to use the definition of asymptotic dimension based on covers of bounded  $r$ -multiplicity (see Section 1.7).

**Theorem 7.1.** *For every  $k \geq 1$ , and every integer  $r \geq 1$ , every graph of pathwidth at most  $k$  has a  $O((kr)^{2k+2})$ -bounded cover of  $r$ -multiplicity at most 2. In particular, for every  $k \geq 1$ , the class of graph of pathwidth at most  $k$  has asymptotic dimension at most 1.*

We start with some notation and a few preparatory lemmas. In all this section,  $G$  is a graph of pathwidth at most  $k$ . Let  $(P, \mathcal{B})$  be a path-decomposition of  $G$  of width  $k$  which minimizes  $\ell = |P|$ . Let  $P = v_1, \dots, v_\ell$ , and let the corresponding bags of  $\mathcal{B}$  be denoted by  $X_1, \dots, X_\ell$ . By minimality of  $\ell$  we have that for any  $1 \leq i \leq \ell - 1$ ,  $X_i \not\subseteq X_{i+1}$  and  $X_{i+1} \not\subseteq X_i$ . We write  $X_I = \cup_{i \in I} X_i$ , so  $|X_I| \leq (k + 1)|I|$ . The path-decomposition defines an interval representation  $(I_v)_{v \in V(G)}$  of a supergraph of  $G$  by letting  $I_v$  be the smallest real interval containing  $\{i \in [\ell] : v \in X_i\}$ . If  $u$  and  $v$  are adjacent in  $G$  then  $I_v$  and  $I_u$  intersect, and each point of the real line is contained in at most  $k + 1$  intervals  $I_v$ ,  $v \in V(G)$ . Note that by minimality of  $\ell$ , for any  $i \in [\ell]$  there is a vertex  $v$  such that  $I_v$  starts at  $i$ .

Given a subset  $U \subseteq V(G)$ , a vertex  $u$  is *maximal* in  $U$  if, for every  $v \neq u$  in  $U$ ,  $I_v$  does not strictly contain  $I_u$ . A subset  $U$  of vertices is *non-nested* if every vertex of  $U$  is maximal in  $U$ . Let  $V_1$  be the set of maximal vertices in  $V(G)$ , and for any  $j \geq 2$ , let  $V_j$  be the set of maximal vertices in  $V(G) - \cup_{i < j} V_i$ . Note that since  $G$  has pathwidth at most  $k$ , all the sets  $V_j$  with  $j > k + 1$  are empty (since otherwise we would find a vertex  $v$  such that  $I_v$  is included in at least  $k + 1$  other intervals  $I_u$ , contradicting the assumption that any point is contained in at most  $k + 1$  intervals). This way we have partitioned  $V(G)$  into at most  $k + 1$  sets  $V_1, \dots, V_p$ , such that  $V_j$  is non-nested for all  $j \in [p]$  (see Figure 4 for an example).

For each integer  $j \in [p]$ , let  $S_j$  be the set of elements  $i \in [\ell]$  such that an interval  $I_v$  with  $v \in V_j$  starts at  $i$ . Recall that for any  $i \in [\ell]$ , some interval starts at  $i$ , so it follows that the sets  $S_j$  cover  $[\ell]$ . For  $j \in [p]$ , we define a measure  $\mu_j$  on  $[\ell]$  by

$$\mu_j(S) = |S_j \cap S|,$$

for any subset  $S \subseteq [\ell]$ . In other words,  $\mu_j(S)$  is the number of elements  $i \in S$  such that at least one vertex of  $V_j$  starts at  $i$  (see Figure 4 for an illustration).

**Lemma 7.2.** *Let  $r, a, b$  be integers with  $[a, b] \subseteq [\ell]$ . If  $\mu_j([a, b]) \leq C$ , then any  $r$ -component of  $V_j \cap X_{[a, b]}$  (with respect to the distances in  $G$ ) has diameter at most  $(C + 1)(k + 1)r$  in  $G$ .*

*Proof.* We claim that  $|X_{[a, b]} \cap V_j| \leq (C + 1)(k + 1)$ . Indeed,  $X_a$  contains at most  $k + 1$  vertices, and intervals corresponding to the vertices of  $V_j$  intersecting  $[a, b]$  and not containing  $a$  start in

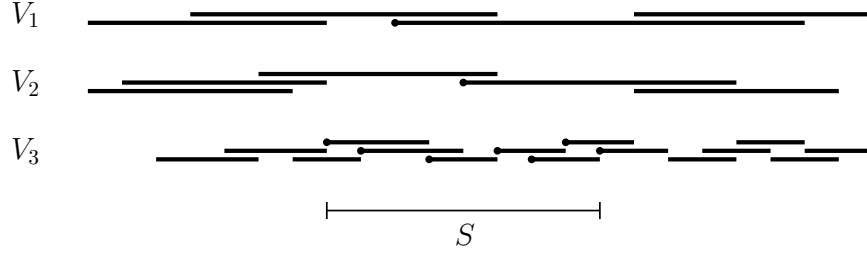


FIGURE 4. An example of a decomposition into 3 non-nested sets. The interval  $S$  has measures  $\mu_1(S) = \mu_2(S) = 1$  and  $\mu_3(S) = 7$  (each starting endpoint lying in  $S$  is pictured with a dot).

at most  $C$  elements of  $[a, b]$  (and at most  $k + 1$  intervals start at each point). The lemma now follows since any  $r$ -component of a set of  $M$  elements has diameter at most  $M \cdot r$  in  $G$ .  $\square$

We now partition the vertices of each set  $V_j$  into *sections*. We start by defining the following constants:

$$r_p := 100r, R_p := 3(k+1)kr_p^2, \text{ and for any } 1 \leq j < p, r_j := 10R_{j+1} \text{ and } R_j := 3(k+1)kr_j^2.$$

For each  $j \in [p]$ , we define a partition of  $[\ell]$  into consecutive intervals using  $\mu_j$ . Each interval  $I$  in the partition satisfies  $\mu_i(I) = Q_i := 2kr_i$ , apart from the last one that is allowed to have a smaller weight (these intervals are called *sections of label  $j$* , or  *$j$ -sections*). The set of all  $j$ -sections is denoted by  $P_j$ . To each  $j$ -section, we associate the set of vertices of  $V_j$  that *start* in the section. Abusing notation, we will often consider a  $j$ -section as the set of vertices we associated to it.

We also number the corresponding sections and hence might refer to the  $i$ th section of label  $j$ . With this we then mean those vertices in  $V_j$  that started in a bag  $a$  with  $\mu_j([a]) \in [(i-1) \cdot Q_j + 1, i \cdot Q_j]$ . For any  $1 \leq j \leq p$ , we denote by  $G_{\geq j}$  the subgraph of  $G$  induced by the vertices  $\bigcup_{i \geq j} V_i$ .

**Lemma 7.3.** *Let  $A$  and  $B$  be two non-consecutive  $j$ -sections. Then  $d_{G_{\geq j}}(A, B) > Q_j / (2k) = r_j$ .*

*Proof.* Let  $P = (v_0 := a, \dots, v_r := b)$  be a path from a vertex  $a \in A$  to a vertex  $b \in B$ , that only contains vertices of  $G_{\geq i}$ . Let  $S$  be a  $j$ -section lying between the  $j$ -sections  $A$  and  $B$ . Let us label the elements of  $S$  contributing to  $\mu_j$  as  $x_0, \dots, x_{Q_j-1}$ . We prove by induction on  $i$  that  $I_{v_i}$  finishes before  $x_{(k+1)(i+1)}$ . In particular, we require  $(k+1)(r+1) \geq Q_j - 1$  or  $r+1 \geq \frac{Q_j-1}{k+1}$  and hence  $r > \frac{Q_j}{2k} = r_j$ . Observe first that  $I_{v_1}$  starts before  $x_1$ . For  $i > 1$ , we find using the induction hypothesis that  $I_{v_i}$  starts before  $x_{(k+1)i}$ , since  $v_i v_{i-1}$  is an edge. We now show how to conclude from this that  $I_{v_i}$  finishes before  $x_{(k+1)(i+1)}$ .

Since  $x_{(k+1)i} \in I_{v_i}$ , if we also have  $x_{(k+1)(i+1)} \in I_{v_i}$  then  $I_{v_i}$  contains at least  $k+1$  elements contributing to  $\mu_j$ . Hence the intervals of  $k+1$  new vertices of  $V_j$  must start at each of these elements. But  $x_{(k+1)(i+1)}$  is contained in at most  $k+1$  intervals, hence the interval of at least one of the new vertices finishes before  $x_{(k+1)(j+1)}$ . In this case,  $I_{v_i}$  strictly contains an interval  $I_v$  with  $v \in V_j$ , contradicting the fact that  $v_i$  is in  $G_{\geq j}$  and  $v$  is maximal in  $V(G_{\geq j})$ .  $\square$

We are now ready to define the desired cover (actually a partition) of  $V(G)$  into bounded sets of  $r$ -multiplicity at most 2. For each  $j \in [p]$  and each  $j$ -section  $S$ , each  $r_j$ -component of the associated vertices of  $V_i$  (where the graph  $G$  is used to define the components) is called an *initial cluster* of label  $j$ , or an *initial  $j$ -cluster*. Our goal now is to merge the different initial

clusters into sets, such that the resulting family has  $r$ -multiplicity at most 2. We first make some observations about the initial clusters.

**Lemma 7.4.** *The following holds:*

- (1) *Every initial  $j$ -cluster is  $R_j$ -bounded.*
- (2) *Every  $r$ -ball with  $r \leq r_j/2$  intersects at most one initial  $j$ -cluster in each  $j$ -section.*
- (3) *In the graph  $G_{\geq j}$ , each  $r$ -ball, with  $r \leq r_j/2$ , intersects at most two initial  $j$ -clusters. If there are two such clusters, they belong to consecutive  $j$ -sections.*

*Proof.* Each  $j$ -section has weight at most  $Q_j$ , and thus by Lemma 7.2, the diameter of an initial  $j$ -cluster is at most  $(k+1)(Q_j+1)r_j = (k+1)(2kr_j+1)r_j \leq 3k(k+1)r_j^2 = R_j$ , thus proving (1). (2) follows from the definition of  $r_j$ -components. (3) follows by (2) and Lemma 7.3.  $\square$

We are now ready to prove Theorem 7.1.

*Proof of Theorem 7.1.* Let  $G$  be a graph of pathwidth at most  $k$  and  $r \geq 1$  be an integer. We use all the notation that has been introduced above (note in particular that the definition of the integers  $r_j$  and  $R_j$  depends on  $r$ ). For each  $j \in [p]$  (recall that  $p \leq k+1$ ), we divide  $V_j$  into  $j$ -sections, and then each  $j$ -section into initial  $j$ -clusters, as explained above. For each  $j = 1, \dots, p$ , we now define  $j$ -merged clusters inductively as follows. The 1-merged clusters are precisely the initial 1-clusters.

For  $j \geq 2$ , each  $j$ -merged cluster is either an initial  $j$ -cluster, or the union of a  $(j-1)$ -merged cluster and a (possibly empty) set of initial clusters of  $P_j$ . Each initial  $j$ -cluster will be included in a single  $j$ -merged cluster, and in particular the set of  $j$ -merged clusters form a partition of  $V_1 \cup \dots \cup V_j$ . We will maintain the following properties by induction on  $j$ .

- (i) Each  $j$ -merged cluster contains a unique initial cluster  $X$  of minimum label  $i$ . The cluster  $X$  is called the *oldest cluster* and  $i$  is called the *age* of the  $j$ -merged cluster.
- (ii) If a  $j$ -merged cluster contains initial clusters of labels in  $[j_1, j_2] \subseteq [j]$  then the diameter of the  $j$ -merged cluster is at most  $(1+j_2)R_{j_1}$ .
- (iii) Every  $r_j/2$ -ball in  $G$  intersects at most two  $j$ -merged clusters.

The merging works as follows. Let  $C$  be an initial  $j$ -cluster, and let  $\mathcal{C}$  the set of  $(j-1)$ -merged clusters at distance at most  $r_j$  from  $C$  in  $G$ . Note that  $\mathcal{C}$  might be empty in which case  $C$  will become a  $j$ -merged cluster on its own. Otherwise, arbitrarily choose a cluster  $C' \in \mathcal{C}$  of minimal age and merge  $C$  and  $C'$  (i.e. the  $j$ -merged cluster containing  $C$  will be the union of  $C'$  and all the initial  $j$ -clusters that have merged onto it).

The  $p$ -merged clusters are our desired partition of  $V(G)$ : since  $r_p > 2r$ , property (iii) ensures that every ball of radius at most  $r$  intersects at most two  $p$ -merged clusters, and thus the family of  $p$ -merged clusters has  $r$ -multiplicity at most 2. Moreover (ii) gives each  $p$ -merged cluster has diameter at most  $(k+2)R_1 = O((kr)^{2k+2})$ , as desired.

We now prove that properties (i)–(iii) hold by induction on  $j \geq 1$ . For  $i = 1$ , the 1-merged clusters are the initial 1-clusters, which are  $R_1$ -bounded by Lemma 7.4(1). This proves (i) and (ii). Since  $G = G_{\geq 1}$ , (iii) is a direct consequence of Lemma 7.4(3). We now assume we have proved the properties for  $j-1 \geq 1$  and prove the assertions for  $j$ .

(i) *Each  $j$ -merged cluster contains a unique initial cluster  $X$  of minimum age.*

This is true just before the merging for  $j-1$  by induction. Note that we merge each initial  $j$ -clusters with a single  $(j-1)$ -merged cluster or leave the initial  $j$ -cluster alone. So the conclusion indeed follows.

(ii) *If a  $j$ -merged cluster contains initial clusters of labels in  $[j_1, j_2]$  then the diameter of the merged cluster is at most  $(1+j_2)R_{j_1}$ .*

For any  $j$ -merged cluster that just consists of an initial  $j$ -cluster, the claim follows immediately from Lemma 7.4(1). If a  $j$ -merged cluster is also a  $(j - 1)$ -merged cluster (i.e. no initial  $j$ -cluster was added to it), then the property follows by induction. Finally, consider a  $j$ -merged cluster  $C$  obtained from a  $(j - 1)$ -merged cluster  $C'$  of age  $j_1$  by adding some initial  $j$ -clusters to it. Then  $C$  also has age  $j_1$  and it only contains initial clusters whose labels lie in  $[j_1, j]$ . All the initial  $j$ -clusters contained in  $C$  have diameter at most  $R_j$  and are at distance at most  $r_j$  from  $C'$  (whose diameter is at most  $(1 + j - 1)R_{j_1}$  by induction). It follows that the diameter of  $C$  is at most  $(1 + j - 1)R_{j_1} + 2r_j + 2R_j \leq (1 + j)R_{j_1}$ , since  $2r_j + 2R_j \leq R_{j_1}$ , as desired.

(iii) *Every  $r_j/2$ -ball in  $G$  is incident to at most two  $j$ -merged clusters.*

Fix some arbitrary vertex  $v \in G$ , and consider the ball  $B(v, r_j/2)$ . Note that it follows from Lemma 7.4(3) that it intersects at most two  $j$ -merged clusters if the ball is included in  $V(G_{\geq j})$ , so we can assume that  $B(v, r_j/2)$  intersects some  $(j - 1)$ -merged cluster, and thus some  $j$ -merged cluster  $C$  of age  $i < j$ .

We now claim that the age of all the  $j$ -merged clusters intersecting  $B(v, r_j/2)$  has to be smaller than  $j$ . Indeed, all initial  $j$ -clusters at distance at most  $r_j/2$  of  $v$  are at distance at most  $r_j$  from  $C$ . Hence any such initial cluster will be merged onto a cluster of age at most the age of  $C$ .

Let  $A, B, C$  be three  $j$ -merged clusters intersecting  $B(v, r_j/2)$ . Be the previous paragraph, the age of each of  $A, B, C$  is less than  $j$ . Hence there exist  $(j - 1)$ -merged clusters  $A', B', C'$  that have been transformed into the  $j$ -merged clusters  $A, B, C$  (possibly by adding an empty set of initial  $j$ -clusters). Since the initial  $j$ -clusters have diameter at most  $R_j$ , and are within  $r_j$  of the  $(j - 1)$ -merged clusters they merge onto (if any), the vertex  $v$  is at distance at most  $r_j/2 + R_j + r_j < r_{j-1}/2$  from  $A', B'$  and  $C'$ . This contradicts the induction hypothesis, and concludes the proof of Theorem 7.1.  $\square$

The *layered pathwidth* of a graph can be defined analogously as the layered treewidth (see Section 5), by restricting the tree-decomposition to be a path. Using the same proof as that of Theorem 5.2, and replacing (layered) treewidth by (layered) patwidth, we obtain the following immediate consequence of Theorem 7.1.

**Corollary 7.5.** *For any  $k$ , the class of graphs of layered pathwidth at most  $k$  has asymptotic dimension at most 2.*

A graph  $H$  is an *apex-forest* if  $H - v$  is a forest for some  $v \in V(H)$ . It was proved in [10] that for any apex-forest  $H$ , the class of  $H$ -minor free graphs has bounded layered pathwidth. We obtain the following as a direct consequence.

**Corollary 7.6.** *For any apex-forest  $H$ , the class of  $H$ -minor free graphs has asymptotic dimension at most 2.*

## 8. CONCLUSION AND OPEN PROBLEMS

Recall Question 2, by Fujiwara and Papasoglu [16], asking whether there is a constant  $k$  such that any proper minor-closed class has asymptotic dimension at most  $k$  (and speculating that this might even be true with  $k = 2$ ).

While we have proved that this is true for classes of graphs excluding a  $K_{3,p}$  minor (Theorem 3.5), or an apex-forest minor (Corollary 7.6) and for classes of graphs of bounded degree excluding a fixed minor (Theorem 1.7), we do not know how to answer this question in general. In fact we do not know the answer to the following more specific question.

**Question 3.** Is there a constant  $k$  such that for any  $t \geq 0$ , the class of graphs of treewidth at most  $t$  has asymptotic dimension at most  $k$ ?

As we have proved in Theorem 5.2, if this question had a positive answer (for some integer  $k$ ), then any class of graphs of bounded layered treewidth would have asymptotic dimension at most  $k + 1$ , and in particular for any apex graph  $H$ , the class of  $H$ -minor free graphs would have asymptotic dimension at most  $k + 1$ .

**Question 4.** Do 1-planar graphs have bounded asymptotic dimension?

Again, by Corollary 5.3, a negative answer to Question 4 would imply a negative answer to Question 3.

Recall that Ostrovskii and Rosenthal [28] proved that the class of  $K_t$ -minor free graphs has asymptotic dimension at most  $4^t$ . Independently of the answer of Question 2, improving the bound (as a function of  $t$ ) is an interesting open problem. The following possible extension of Theorem 3.5 is fairly natural.

**Question 5.** Is it true that for every integers  $s$  and  $t$ , the class of graphs with no  $K_{s,t}$  minor has asymptotic dimension at most  $s - 1$ ? Or more generally at most  $f(s)$ , for some function  $f$ ?

We have seen that Question 3 has a negative answer if we replace asymptotic dimension by Assouad-Nagata dimension (Theorem 5.9). In this case finding the best function  $f(t)$  such that graphs of treewidth at most  $t$  have Assouad-Nagata dimension at most  $f(t)$  is an interesting open problem. The best known bound [28] is exponential in  $t$ . A related question of Filtser [15] asks whether graphs of treewidth at most  $t$  have a  $O(t^a)$ -dimensional control function  $D_t(r) = O(t^b) \cdot r$ , where  $a, b > 0$  are some constants.

It should be noted that Questions 2 and 3 are already open if we fix the radius  $r = 1$ . As explained in Section 1.4, in this case the questions become:

**Question 6.** Is there a constant  $k$  such that for any  $t \geq 0$ , the class of graphs with no  $K_t$  minor has weak diameter chromatic number at most  $k$ ? Can we take  $k = 3$ ?

**Question 7.** Is there a constant  $k$  such that for any  $t \geq 0$ , the class of graphs of treewidth at most  $t$  has weak diameter chromatic number at most  $k$ ? Can we take  $k = 2$ ?

As observed in [24, Theorem 4.1], Question 7 has a negative answer if we replace the weak diameter by the strong diameter (i.e. if we ask that the subgraph induced by each monochromatic component has bounded diameter, at least  $t + 1$  colors are necessary in some graphs of treewidth at most  $t$ ).

For some function  $f$ , we say that a class of graphs  $\mathcal{G}$  has *expansion* at most  $f$  if any minor obtained from contracting connected subgraphs of radius at most  $r$  in a graph of  $\mathcal{G}$  has average degree at most  $f(r)$  (see [26] for more details on this notion). In Section 6 we have proved that classes of graphs of polynomial growth have bounded asymptotic dimension. Note that if a class has bounded (resp. polynomial) growth, then it has bounded (resp. polynomial) expansion.

**Question 8.** Is it true that every class of graphs of polynomial expansion has bounded asymptotic dimension?

Observe that polynomial expansion would again be best possible here, as we have constructed classes of graphs of (barely) superpolynomial growth (and therefore expansion) with unbounded asymptotic dimension. It should be noted that there are important connections between

polynomial expansion and the existence of strictly sublinear separators [14]. On the other hand, Hume [17] proved that classes of graphs of bounded growth and bounded asymptotic dimension have sublinear separators.

We conclude with a remark and a question on  $q$ -fat minors, which were introduced in Section 3.3. We note that the proof of Ostrovskii and Rosenthal [28] can be adapted to prove that for any integers  $q$  and  $t \geq 3$ , the class of graphs with no  $q$ -fat  $K_t$ -minor has asymptotic dimension at most  $4^t$ . We believe that excluding a  $K_t$ -minor or a  $q$ -fat  $K_t$ -minor should not make a difference for the asymptotic dimension, and that  $q$ -fat minors are the key notion to answer Question 2 and 5 (in the same way  $q$ -fat bananas were the key to the proof of Theorem 3.5).

**Question 9.** Is it true that for any integers  $q$  and  $t \geq 3$ , the class of  $K_t$ -minor free graphs and the class of  $q$ -fat  $K_t$ -minor free graphs have the same asymptotic dimension?

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## APPENDIX A. PROOF OF THEOREM 2.1

In this appendix we prove Theorem 2.1. We closely follow [7].

**A.1. Control function.** For two integers  $n \geq 0$  and  $k \geq n + 1$  and a metric space  $X$ , we say that a function  $D_X$  is an  $(n, k)$ -dimensional control function for  $X$  if for all  $r > 0$ , there are subsets  $U_1, \dots, U_k$  of  $X$  such that

- (1) Each  $r$ -component of  $U_i$  is  $D_X(r)$ -bounded.
- (2) Each  $x \in X$  belongs to at least  $k - n$  sets  $U_i$ .

Note that an  $(n, n + 1)$ -dimensional control function is the same as an  $n$ -dimensional control function. We will need the following result.

**Lemma A.1** (Theorem 2.4 in [7]). *If  $X$  admits an  $(n, k)$ -dimensional control function  $D$ , then  $D'(r) := D(3r) + 2r$  is an  $(n, k + 1)$ -dimensional control function for  $G$ .*

Note that  $\mathbb{R}$  has Assouad-Nagata dimension 1, so in particular Lemma A.1 implies that it has a linear  $(1, n)$ -control function for any  $n \geq 2$ . Observe that such a function (and the associated sets  $U_1, \dots, U_n$ ) can be constructed explicitly in this case (and with better bounds than those implied by repeated applications of Lemma A.1). We first divide  $\mathbb{R}$  into consecutive intervals  $V_k := [rk, r(k + 1))$ ,  $k \in \mathbb{Z}$  of length  $r$ . For any  $1 \leq i \leq n$ , we then define

$$U_i = \mathbb{R} \setminus \bigcup \{V_k \mid k \in \mathbb{Z}, k \equiv i \pmod{n}\}.$$

Note each  $r$ -component of  $U_i$  is a single interval of length  $(n - 1)r$ , and is thus  $(n - 1)r$ -bounded. We write this as an observation for future reference.

**Observation A.2.** *For any  $n \geq 1$ , and  $r > 0$ , there are subsets  $(U_i)_{1 \leq i \leq n}$  of  $\mathbb{R}$ , such that for each  $1 \leq i \leq n$ , each  $r$ -component of  $U_i$  is  $(n - 1)r$ -bounded, and each element of  $\mathbb{R}$  is contained in at least  $n - 1$  sets.*

**A.2. Control and dimension of real projections.** Let  $X$  be a metric space and  $f : X \rightarrow \mathbb{R}$  be a real projection of  $X$ . For  $n \geq 0$  and  $k \geq n + 1$ , a function  $D_f$  is an  $(n, k)$ -dimensional control function for  $f$  if for all  $r, S > 0$ , and any  $(\infty, S)$ -bounded set  $A$  (with respect to  $f$ ), there are subsets  $A_1, \dots, A_k$  of  $A$  such that

- (1) all  $r$ -components of  $A_i$  are  $D_f(r, S)$ -bounded
- (2) any  $x \in A$  belongs to at least  $k - n$  elements of  $\{A_1, \dots, A_k\}$ .

As before, an  $(n, n + 1)$ -dimensional control function for  $f$  is the same as an  $n$ -dimensional control function for  $f$ , defined in Section 2.2. Recall that a control function  $D_f$  for a real projection  $f$  is *linear* if  $D_f(r, S) = ar + bS + c$ , for some constants  $a, b, c > 0$ , and a *dilation* if  $D_f(r, S) = ar + bS$ , for some constants  $a, b > 0$ .

Control functions for real projections satisfy the following counterpart of Lemma A.1.

**Lemma A.3** (Proposition 4.7 in [7]). *Let  $X$  be a metric space and  $f : X \rightarrow \mathbb{R}$  be a real projection of  $X$ . If  $f$  admits an  $(n, k)$ -dimensional control function  $D_f$ , then  $D'_f(r, S) := D_f(3r, S) + 2r$  is an  $(n, k + 1)$ -dimensional control function for  $f$ . In particular, if  $D_f$  is linear, then  $D'_f$  is also linear, and  $D_f$  is dilation, then  $D'_f$  is also a dilation.*

The definition of control functions for a real projection  $f : X \rightarrow \mathbb{R}$  of  $X$  involves  $(\infty, S)$ -bounded sets, which are the preimage of  $S$ -bounded sets of  $\mathbb{R}$  under  $f$ . The following stronger version can be easily derived from the definition.

**Lemma A.4** (Proposition 4.8 in [7]). *Let  $X$  be a metric space and  $f : X \rightarrow \mathbb{R}$  be a real projection of  $X$ . If  $f$  admits an  $(n, k)$ -dimensional control function  $D_f$ , then for any  $r, s, S > 0$ , and any subset  $B \subseteq \mathbb{R}$  whose  $s$ -components are  $S$ -bounded, the preimage  $f^{-1}(B)$  of  $B$  in  $X$  can be covered by  $k$  sets with  $(r, s)$ -components that are  $D_f(r, S)$ -bounded, and such that any element of  $f^{-1}(B)$  is covered by at least  $k - n$  sets.*

**A.3. Union of sets.** We will need the following result on the union of sets.

**Lemma A.5** (Lemma 3.7 in [7]). *Let  $X$  be a metric space and  $f : X \rightarrow \mathbb{R}$  be a real projection of  $X$ . Let  $A_1$  and  $A_2$  be subsets of  $X$  such that all  $(r_i, s_i)$ -components of  $A_i$  are  $(R_i, S_i)$ -bounded, for  $i \in \{1, 2\}$ . If  $R_1 + 2r_1 < r_2$  and  $S_1 + 2s_1 < s_2$  then all  $(r_1, s_1)$ -components of  $A_1 \cup A_2$  are  $(R_2 + 2r_2, S_2 + 2s_2)$ -bounded.*

**A.4. Kolmogorov trick for real projections.** We are now ready to prove Theorem 2.1. We divide it into two separate parts, for convenience. The first part is proved explicitly in [7] (so we do not reprove it here).

**Theorem A.6** (Theorem 4.9 in [7]). *Let  $X$  be a metric space and  $f : X \rightarrow \mathbb{R}$  be a real projection of  $X$ . If  $f$  admits an  $n$ -dimensional control function  $D_f$ , then  $X$  admits an  $(n + 1)$ -dimensional control function  $D_X$  such that  $D_X(r)$  only depends on  $r$ ,  $D_f$  and  $n$ . Moreover, if  $D_f$  is linear then  $D_X$  is also linear and if  $D_f$  is a dilation then  $D_X$  is also a dilation.*

The second part is not stated explicitly in [7] but follows from an optimized version of the proof of their Theorem 4.9. We prove it here for completeness.

**Theorem A.7** (Theorem 4.9 in [7]). *Let  $X$  be a metric space and  $f : X \rightarrow \mathbb{R}$  be a real projection of  $X$ . Let  $a, b \geq 1$  be real numbers. If  $f$  admits an  $n$ -dimensional control function  $D_f$  with  $D_f(r, S) \leq ar + bS$ , for any  $r, S > 0$ , then  $X$  admits an  $(n + 1)$ -dimensional control function  $D_X$  such that  $D_X(r) \leq 20a(6a + b(n + 4)) \cdot r$ .*



*Proof.* Given  $r > 0$ , we set  $s_1 := r$ ;  $S_1 := (n+1)s_1$ ; and  $s_2 := S_1 + 2s_1 + r = (n+4)r$  (note that by definition,  $S_1 + 2s_1 < s_2$ ).

For  $i \in \{1, 2\}$ , we consider the set  $B_i = \bigcup_{k \in \mathbb{Z}} [(2k+i-1)s_2, (2k+i)s_2]$ . Note that the sets  $B_1$  and  $B_2$  partition  $\mathbb{R}$  and any  $s_2$ -component of  $B_i$  ( $i \in \{1, 2\}$ ) is  $s_2$ -bounded.

By Observation A.2, there are subsets  $(B_1^j)_{1 \leq j \leq n+2}$  of  $B_1$ , such that for each  $1 \leq j \leq n+2$ , each  $s_1$ -component of  $B_1^j$  is  $(n+1)s_1$ -bounded (and thus  $S_1$ -bounded), and each element of  $B_1$  is contained in at least  $n+1$  sets  $B_1^j$ .

Recall that an  $n$ -dimensional control function for  $f$  is the same as an  $(n, n+1)$ -control function for  $f$ . By Lemma A.3, the function  $D'_f(r, S) := D_f(3r, S) + 2r$  is an  $(n, n+2)$ -dimensional control function for  $f$ . We now define  $r_1 := r$ ;  $R_1 := D'_f(r_1, s_2)$ ;  $r_2 := R_1 + 2r_1 + r$  (observe that  $R_1 + 2r_1 < r_2$ ); and  $R_2 := D'_f(r_2, s_2)$ . By Lemma A.4, for each  $i \in \{1, 2\}$ , since the  $s_2$ -components of  $B_i$  are  $s_2$ -bounded, the subset  $f^{-1}(B_i)$  of elements of  $X$  can be covered by  $n+2$  sets  $(A_i^j)_{1 \leq j \leq n+2}$  with  $(r_i, s_2)$ -components that are  $D'_f(r_i, s_2)$ -bounded (and thus  $R_i$ -bounded), and such that any element in  $f^{-1}(B_i)$  is covered by at least  $n+2 - n = 2$  sets  $A_i^j$ . (In what follows the property that the elements are covered at least twice instead of once will only be used for  $i = 1$ .)

For any  $1 \leq j \leq n+2$ , set  $D^j := (A_1^j \cap f^{-1}(B_1^j)) \cup A_2^j$ . We first observe that the sets  $D^j$  form a cover of  $X$ . To see this, consider an element  $x \in X$ . Note that  $f(x) \in B_i$  for some  $i \in \{1, 2\}$  since  $B_1$  and  $B_2$  partition  $\mathbb{R}$ . If  $f(x) \in B_2$ , then  $x$  is covered by a set  $A_2^j \subseteq D^j$ , for some  $1 \leq j \leq n+2$ . Assume now that  $x \in B_1$ . Then  $f(x)$  is covered by at least  $n+1$  sets  $B_1^j$ , while  $x$  (as an element of  $f^{-1}(B_1)$ ) is also covered by at least two sets  $A_1^j$ . Since  $1 \leq j \leq n+2$ , there is an index  $j$  such that  $v \in A_1^j \cap f^{-1}(B_1^j) \subseteq D^j$ , as desired.

Fix some  $1 \leq j \leq n+2$  and some  $i \in \{1, 2\}$ . Recall that all  $(r_i, s_2)$ -components of  $A_i^j$  are  $(R_i, s_2)$ -bounded. Consider two elements  $x, y$  lying in the same  $(r_1, s_1)$ -component of  $A_1^j \cap f^{-1}(B_1^j)$ . Since  $s_1 \leq s_2$ ,  $x$  and  $y$  are included in a  $(r_1, s_2)$ -component of  $A_1^j$  and thus  $\{x, y\}$  is  $(R_1, s_2)$ -bounded. On the other hand, since all  $s_1$ -components of  $B_1^j$  are  $S_1$ -bounded,  $|f(x) - f(y)| \leq S_1$ , and thus  $\{x, y\}$  is  $(R_1, S_1)$ -bounded. This proves that all  $(r_1, s_1)$ -components of  $A_1^j \cap f^{-1}(B_1^j)$  are  $(R_1, S_1)$ -bounded.

Recall that all  $(r_2, s_2)$ -components of  $A_2^j$  are  $(R_2, s_2)$ -bounded. By Lemma A.5, since  $R_1 + 2r_1 < r_2$  and  $S_1 + 2s_1 < s_2$ , this implies that all  $(r_1, s_1)$ -components of  $D^j = (A_1^j \cap f^{-1}(B_1^j)) \cup A_2^j$  are  $(R_2 + 2r_2, s_2 + 2s_2)$ -bounded. Since  $(r_1, s_1) = (r, r)$ ,  $(r_1, s_1)$ -components of  $D^j$  are the same as  $r$ -components of  $D^j$ , so it follows that  $r$ -components of  $D^j$  are  $(R_2 + 2r_2)$ -bounded. We obtain that  $D_X(r) := R_2 + 2r_2$  is an  $(n+1)$ -dimensional control function for  $X$ .

Recall that  $D_f(r, S) \leq ar + bS$ , for some constants  $a \geq 1, b \geq 1$ . It follows that  $D'_f(r, S) = D_f(3r, S) + 2r \leq (3a+2)r + bS$ . As a consequence,  $R_1 \leq (3a+2)r + bs_2 \leq (3a+2+b(n+4)) \cdot r$  and  $r_2 = R_1 + 3r \leq (3a+5+b(n+4)) \cdot r$ . This implies  $R_2 \leq ((3a+2)(3a+5+b(n+4)) + b(n+4)) \cdot r \leq (40a^2 + 6ab(n+4)) \cdot r$ , using that  $a, b \geq 1$ . We obtain

$$D_X(r) = R_2 + 2r_2 \leq 20a(6a + b(n+4)) \cdot r,$$

as desired.  $\square$

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