**Bad News for Chordal Partitions**

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Abstract. Reed and Seymour [1998] asked whether every graph has a partition into induced connected non-empty bipartite subgraphs such that the quotient graph is chordal. If true, this would have significant ramifications for Hadwiger’s Conjecture. We prove that the answer is ‘no’. In fact, we show that the answer is still ‘no’ for several relaxations of the question.

1 Introduction

Hadwiger’s Conjecture [9] says that for all \( t \geq 0 \) every graph with no \( K_{t+1} \)-minor is \( t \)-colourable. This conjecture is easy for \( t \leq 3 \), is equivalent to the 4-colour theorem for \( t = 4 \), is true for \( t = 5 \) [18], and is open for \( t \geq 6 \). The best known upper bound on the chromatic number is \( O(t\sqrt{\log t}) \), independently due to Kostochka [14, 15] and Thomason [21, 22]. This conjecture is widely considered to be one of the most important open problems in graph theory; see [20] for a survey.

Throughout this paper, we employ standard graph-theoretic definitions (see [4]), with one important exception: we say that a graph \( G \) contains a graph \( H \) if \( H \) is isomorphic to an induced subgraph of \( G \).

Motivated by Hadwiger’s Conjecture, Reed and Seymour [17] introduced the following definitions. A *vertex-partition*, or simply *partition*, of a graph \( G \) is a set \( \mathcal{P} \) of non-empty induced subgraphs of \( G \) such that each vertex of \( G \) is in exactly one element of \( \mathcal{P} \). Each element of \( \mathcal{P} \) is called a *part*. The *quotient* of \( \mathcal{P} \) is the graph, denoted by \( G/\mathcal{P} \), with vertex set \( \mathcal{P} \) where distinct parts \( P, Q \in \mathcal{P} \) are adjacent in \( G/\mathcal{P} \) if and only if some vertex in \( P \) is adjacent in \( G \) to some vertex in \( Q \). A partition of \( G \) is *connected* if each part is connected. We (almost) only consider connected partitions. In this case, the quotient is the *minor* of \( G \) obtained by contracting each part into a single vertex. A partition is *chordal* if it is connected and the quotient is chordal (that is, contains no induced cycle of length at least four). Every graph

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Reed and Seymour [17] used different terminology: ‘chordal decomposition’ instead of chordal partition, and ‘touching pattern’ instead of quotient.
has a chordal partition (with a 1-vertex quotient). Chordal partitions are a useful tool when studying graphs $G$ with no $K_{t+1}$ minor. Then for every connected partition $\mathcal{P}$ of $G$, the quotient $G/\mathcal{P}$ contains no $K_{t+1}$, so if in addition $\mathcal{P}$ is chordal, then $G/\mathcal{P}$ is $t$-colourable (since chordal graphs are perfect). Reed and Seymour [17] asked the following question (repeated in [13, 20]).

**Question 1.** Does every graph have a chordal partition such that each part is bipartite?

If true, this would imply that every graph with no $K_{t+1}$-minor is $2t$-colourable, by taking the product of the $t$-colouring of the quotient with the 2-colouring of each part. This would be a major breakthrough for Hadwiger’s Conjecture. The purpose of this note is to answer Reed and Seymour’s question in the negative. In fact, we show the following stronger result.

**Theorem 2.** For every integer $k \geq 1$ there is a graph $G$, such that for every chordal partition $\mathcal{P}$ of $G$, some part of $\mathcal{P}$ contains $K_k$. Moreover, for every integer $t \geq 4$ there is a graph $G$ with tree-width at most $t - 1$ (and thus with no $K_{t+1}$-minor) such that for every chordal partition $\mathcal{P}$ of $G$, some part of $\mathcal{P}$ contains a complete graph on at least $\lfloor (3t - 11)/3 \rfloor$ vertices.

Theorem 2 says that it is not possible to find a chordal partition in which each part has bounded chromatic number. What if we work with a larger class of partitions? The following natural class arises. A partition of a graph is perfect if it is connected and the quotient graph is perfect. If $\mathcal{P}$ is a perfect partition of a $K_{t+1}$-minor free graph $G$, then $G/\mathcal{P}$ contains no $K_{t+1}$ and is therefore $t$-colourable. So if every part of $\mathcal{P}$ has small chromatic number, then we can control the chromatic number of $G$. We are led to the following relaxation of Question 1: does every graph have a perfect partition in which every part has bounded chromatic number? Unfortunately, this is not the case.

**Theorem 3.** For every integer $k \geq 1$ there is a graph $G$, such that for every perfect partition $\mathcal{P}$ of $G$, some part of $\mathcal{P}$ contains $K_k$. Moreover, for every integer $t \geq 6$ there is a graph $G$ with tree-width at most $t - 1$ (and thus with no $K_{t+1}$-minor), such that for every perfect partition $\mathcal{P}$ of $G$, some part of $\mathcal{P}$ contains a complete graph on at least $\lfloor (3t/2 - 8)/3 \rfloor$ vertices.

Theorems 2 and 3 say that it is hopeless to improve on the $O(t^{4/3})$ bound for the chromatic number of $K_t$-minor-free graphs using chordal or perfect partitions directly. Indeed, the best possible upper bound on the chromatic number using the above approach would be $O(t^{4/3})$ (since the quotient is $t$-colourable, and the best possible upper bound on the chromatic number of the parts would be $O(t^{1/3})$.)

What about using an even larger class of partitions? Chordal graphs contain no 4-cycle, and perfect graphs contain no 5-cycle. These are the only properties of chordal and perfect graphs used in the proofs of Theorems 2 and 3. Thus the following result is a qualitative generalisation
of both Theorems 2 and 3. It says that there is no hereditary class of graphs for which the above colouring strategy works.

**Theorem 4.** For every integer $k \geq 1$ and graph $H$, there is a graph $G$, such that for every connected partition $\mathcal{P}$ of $G$, either some part of $\mathcal{P}$ contains $K_k$ or the quotient $G/\mathcal{P}$ contains $H$.

Before presenting the proofs, we mention some applications of chordal partitions and related topics. Chordal partitions have proven to be a useful tool in the study of the following topics for $K_{t+1}$-minor-free graphs: cops and robbers pursuit games [1], fractional colouring [13, 17], generalised colouring numbers [11], and defective and clustered colouring [12]. These papers show that every graph with no $K_{t+1}$ minor has a chordal partition in which each part has desirable properties. For example, in [17], each part has a stable set on at least half the vertices, and in [12], each part has maximum degree $O(t)$ and is $2$-colourable with monochromatic components of size $O(t)$.

Several papers [7, 16, 23] have shown that graphs with tree-width $k$ have chordal partitions in which the quotient is a tree, and each part induces a subgraph with tree-width $k - 1$, amongst other properties. Such partitions have been used for queue and track layouts [7] and non-repetitive graph colouring [16]. A tree partition is a (not necessarily connected) partition of a graph whose quotient is a tree; these have also been widely studied [2, 3, 5, 6, 8, 10, 19, 24]. Here the goal is to have few vertices in each part of the partition. For example, a referee of [5] proved that every graph with tree-width $k$ and maximum degree $\Delta$ has a tree partition with $O(k\Delta)$ vertices in each part.

### 2 Chordal Partitions: Proof of Theorem 2

Let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a partition of a graph $G$, and let $X$ be an induced subgraph of $G$. Then the restriction of $\mathcal{P}$ to $X$ is the partition of $X$ defined by $$\mathcal{P}(X) := \{G[V(P_i) \cap V(X)] : i \in \{1, \ldots, m\}, V(P_i) \cap V(X) \neq \emptyset\}.$$ Note that the restriction of a connected partition to a subgraph need not be connected. The following lemma gives a scenario where the restriction is connected.

**Lemma 5.** Let $X$ be an induced subgraph of a graph $G$, such that the neighbourhood of each component of $G - V(X)$ is a clique (in $X$). Let $\mathcal{P}$ be a connected partition of $G$ with quotient $G/\mathcal{P}$. Then $\mathcal{P}(X)$ is a connected partition of $X$, and the quotient of $\mathcal{P}(X)$ is the subgraph of $G/\mathcal{P}$ induced by those parts that intersect $X$.

**Proof.** We first prove that for every connected subgraph $G'$ of $G$, if $V(G') \cap V(X) \neq \emptyset$, then $G'[V(G') \cap V(X)]$ is connected. Consider non-empty sets $A, B$ that partition $V(G') \cap V(X)$.
Let $P$ be a shortest path from $A$ to $B$ in $G'$. Then no internal vertex of $P$ is in $V(X)$. If $P$ has an internal vertex, then all its interior belongs to one component $C$ of $G - V(X)$, implying the endpoints of $P$ are in the neighbourhood of $C$ and are therefore adjacent, a contradiction. Thus $P$ has no interior, and hence $G'[V(G') \cap V(X)]$ is connected.

Apply this observation with each part of $\mathcal{P}$ as $G'$. It follows that $\mathcal{P}(X)$ is a connected partition of $X$. Moreover, if adjacent parts $P$ and $Q$ of $\mathcal{P}$ both intersect $X$, then by the above observation with $G' = G[V(P) \cup V(Q)]$, there is an edge between $V(P) \cap V(X)$ and $V(Q) \cap V(X)$. Conversely, if there is an edge between $V(P) \cap V(X)$ and $V(Q) \cap V(X)$ for some parts $P$ and $Q$ of $\mathcal{P}$, then $PQ$ is an edge of $G/\mathcal{P}$. Thus the quotient of $\mathcal{P}(X)$ is the subgraph of $G/\mathcal{P}$ induced by those parts that intersect $X$. 

The next lemma with $r = 1$ implies Theorem 2. To obtain the second part of Theorem 2 apply Lemma 6 with $k = [(3t - 11)^{1/3}]$, in which case $s(k, 1) \leq t$.

**Lemma 6.** For all integers $k \geq 1$ and $r \geq 1$, if

$$s(k, r) := \frac{1}{8}(k^3 - k) + (r - 1)k + 4,$$

then there is a graph $G(k, r)$ with tree-width at most $s(k, r) - 1$ (and thus with no $K_{s(k, r) + 1}$-minor), such that for every chordal partition $\mathcal{P}$ of $G$, either:

1. $G$ contains a $K_{kr}$ subgraph intersecting each of $r$ distinct parts of $\mathcal{P}$ in $k$ vertices, or
2. some part of $\mathcal{P}$ contains $K_{k+1}$.

**Proof.** Note that $s(k, r)$ is the upper bound on the size of the bags in the tree-decomposition of $G(k, r)$ that we construct. We proceed by induction on $k$ and then $r$. When $k = r = 1$, the graph with one vertex satisfies (1) for every chordal partition and has a tree-decomposition with one bag of size $1 < s(1, 1)$.

First we prove that the $(k, 1)$ and $(k, r)$ cases imply the $(k, r + 1)$ case. Let $A := G(k, 1)$ and $B := G(k, r)$. Let $G$ be obtained from $A$ as follows. For each $k$-clique $C$ in $A$, add a copy $B_C$ of $B$ (disjoint from the current graph), where $C$ is complete to $B_C$. We claim that $G$ has the claimed properties of $G(k, r + 1)$.

By assumption, in every chordal partition of $A$ some part contains $K_{ki}$. $A$ has a tree-decomposition with bags of size at most $s(k, 1)$, and for each $k$-clique $C$ in $A$, there is a tree-decomposition of $B_C$ with bags of size at most $s(k, r)$. For every tree-decomposition of a graph and for each clique $C$, there is a bag containing $C$. Add an edge between the node corresponding to a bag containing $C$ in the tree-decomposition of $A$ and any node of the tree in the tree-decomposition of $B_C$, and add $C$ to every bag of the tree-decomposition of $B_C$. We obtain a tree-decomposition of $G$ with bags of size at most $\max\{s(k, 1), s(k, r) + k\} = s(k, r) + k = s(k, r + 1)$, as desired.
Consider a chordal partition $\mathcal{P}$ of $G$. By Lemma 5, $\mathcal{P}(A)$ is a connected partition of $A$, and the quotient of $\mathcal{P}(A)$ equals the subgraph of $G/\mathcal{P}$ induced by those parts that intersect $A$. Since $G/\mathcal{P}$ is chordal, the quotient of $\mathcal{P}(A)$ is chordal. Since $A = G(k, 1)$, by induction, $\mathcal{P}(A)$ satisfies (1) with $r = 1$ or (2). If outcome (2) holds, then some part of $\mathcal{P}$ contains $K_{k+1}$ and outcome (2) holds for $G$.

Now assume that $\mathcal{P}(A)$ satisfies outcome (1) with $r = 1$; that is, some part $P$ of $\mathcal{P}$ contains some $k$-clique $C$ of $A$. If some vertex of $B_C$ is in $P$, then $P$ contains $K_{k+1}$ and outcome (2) holds for $G$. Now assume that no vertex of $B_C$ is in $P$. Since each part of $\mathcal{P}$ is connected, the parts of $\mathcal{P}$ that intersect $B_C$ do not intersect $G - V(B_C)$. Thus, $\mathcal{P}(B_C)$ is a connected partition of $B_C$, and the quotient of $\mathcal{P}(B_C)$ equals the subgraph of $G/\mathcal{P}$ induced by those parts that intersect $B_C$, and is therefore chordal. Since $B = G(k, r)$, by induction, $\mathcal{P}(B_C)$ satisfies (1) or (2). If outcome (2) holds, then the same outcome holds for $G$. Now assume that outcome (1) holds for $B_C$. Thus $B_C$ contains a $K_{kr}$ subgraph intersecting each of $r$ distinct parts of $\mathcal{P}$ in $k$ vertices. None of these parts are $P$. Since $C$ is complete to $B_C$, $G$ contains a $K_{k(r+1)}$ subgraph intersecting each of $r+1$ distinct parts of $\mathcal{P}$ in $k$ vertices, and outcome (1) holds for $G$. Hence $G$ has the claimed properties of $G(k, r+1)$.

It remains to prove the $(k, 1)$ case for $k \geq 2$. By induction, we may assume the $(k-1, k+1)$ case. Let $A := G(k-1, k+1)$. As illustrated in Figure 1, let $G$ be obtained from $A$ as follows: for each set $\mathcal{C} = \{C_1, \ldots, C_{k+1}\}$ of pairwise-disjoint $(k-1)$-cliques in $A$, whose union induces $K_{(k-1)(k+1)}$, add a $K_{k+1}$ subgraph $B_\mathcal{C}$ (disjoint from the current graph), whose $i$-th vertex is adjacent to every vertex in $C_i$. We claim that $G$ has the claimed properties of $G(k, 1)$.

By assumption, $A$ has a tree-decomposition with bags of size at most $s(k-1, k+1)$. For each set $\mathcal{C} = \{C_1, \ldots, C_{k+1}\}$ of pairwise-disjoint $(k-1)$-cliques in $A$, whose union induces $K_{(k-1)(k+1)}$, choose a node $x$ corresponding to a bag of the tree-decomposition of $A$ containing $C_1 \cup \cdots \cup C_{k+1}$, and add a new node adjacent to $x$ with corresponding bag $V(B_\mathcal{C}) \cup C_1 \cup \cdots \cup C_{k+1}$. We
obtain a tree-decomposition of \( G \) with bags of size at most \( \max\{s(k-1, k+1), (k+1)k\} = s(k-1, k+1) = s(k, 1) \), as desired.

Consider a chordal partition \( \mathcal{P} \) of \( G \). By Lemma 5, \( \mathcal{P}(A) \) is a connected partition of \( A \) and the quotient of \( \mathcal{P}(A) \) equals the subgraph of \( G/\mathcal{P} \) induced by those parts that intersect \( A \), and is therefore chordal. Since \( A = G(k-1, k+1) \), by induction, \( \mathcal{P}(A) \) satisfies (1) or (2). If outcome (2) holds for \( \mathcal{P}(A) \), then some part of \( \mathcal{P} \) contains \( K_k \) and outcome (1) holds for \( G \) (with \( r = 1 \)). Now assume that outcome (1) holds for \( \mathcal{P}(A) \). Thus \( A \) contains a \( K_{(k-1)(k+1)} \) subgraph intersecting each of \( k+1 \) distinct parts \( P_1, \ldots, P_{k+1} \) of \( \mathcal{P} \) in \( k-1 \) vertices. Let \( C_i \) be the corresponding \((k-1)\)-clique in \( P_i \). Let \( \mathcal{E} := \{C_1, \ldots, C_{k+1}\} \) and \( \mathcal{E} : = C_1 \cup \cdots \cup C_{k+1} \).

If for some \( i \in \{1, \ldots, k+1\} \), the neighbour of \( C_i \) in \( B_\mathcal{E} \) is in \( P_i \), then \( P_i \) contains \( K_k \) and outcome (1) holds for \( G \). Now assume that for each \( i \in \{1, \ldots, k+1\} \), the neighbour of \( C_i \) in \( B_\mathcal{E} \) is not in \( P_i \). Suppose that some vertex \( x \) in \( B_\mathcal{E} \) is in \( P_i \) for some \( i \in \{1, \ldots, k+1\} \). Then since \( P_i \) is connected, there is a path in \( G \) between \( C_i \) and \( x \) avoiding the neighbourhood of \( C_i \) in \( B_\mathcal{E} \). Every such path intersects \( \mathcal{E} \setminus C_i \), but none of these vertices are in \( P_i \). Thus, no vertex in \( B_\mathcal{E} \) is in \( P_1 \cup \cdots \cup P_{k+1} \). If \( B_\mathcal{E} \) is contained in one part, then outcome (2) holds. Now assume that there are vertices \( x \) and \( y \) of \( B_\mathcal{E} \) in distinct parts \( Q \) and \( R \) of \( \mathcal{P} \). Then \( x \) is adjacent to every vertex in \( C_i \) and \( y \) is adjacent to every vertex in \( C_j \), for some distinct \( i, j \in \{1, \ldots, k+1\} \). Observe that \((Q, R, P_i, P_j)\) is a 4-cycle in \( G/\mathcal{P} \). Moreover, there is no \( QP_j \) edge in \( G/\mathcal{P} \) because \( (\mathcal{E} \setminus C_j) \cup \{y\} \) separates \( x \in Q \) from \( C_j \subseteq P_j \), and none of these vertices are in \( Q \cup P_j \). Similarly, there is no \( RP_i \) edge in \( G/\mathcal{P} \). Hence \((Q, R, P_j, P_i)\) is an induced 4-cycle in \( G/\mathcal{P} \), which contradicts the assumption that \( \mathcal{P} \) is a chordal partition. Therefore \( G \) has the claimed properties of \( G(k, 1) \).

\[ \square \]

3 Perfect Partitions: Proof of Theorem 3

The following lemma with \( r = 1 \) implies Theorem 3. To obtain the second part of Theorem 3 apply Lemma 6 with \( k = \lfloor (\frac{3}{2}t - 8)^{1/3} \rfloor \), in which case \( t(k, 1) \leq t \). The proof is very similar to Lemma 6 except that we force \( C_5 \) in the quotient instead of \( C_4 \).

Lemma 7. For all integers \( k \geq 1 \) and \( r \geq 1 \), if

\[ t(k, r) := \frac{3}{2}(k^3 - k) + (r - 1)k + 6, \]

then there is a graph \( G(k, r) \) with tree-width at most \( t(k, r) \) (and thus with no \( K_{t(k, r) + 1} \)-minor), such that for every perfect partition \( \mathcal{P} \) of \( G \), either:

(1) \( G \) contains a \( K_{kr} \) subgraph intersecting each of \( r \) distinct parts of \( \mathcal{P} \) in \( k \) vertices, or
(2) some part of \( \mathcal{P} \) contains \( K_{k+1} \).

Proof. Note that \( t(k, r) \) is the upper bound on the size of the bags in the tree-decomposition of \( G(k, r) \) that we construct. We proceed by induction on \( k \) and then \( r \). For the base case,
the graph with one vertex satisfies (1) for \( k = r = 1 \) and has a tree-decomposition with one bag of size \( 1 < t(1, 1) \). The proof that the \((k, 1)\) and \((k, r)\) cases imply the \((k, r + 1)\) case is identical to the analogous step in the proof in Lemma 6, so we omit it.

It remains to prove the \((k, 1)\) case for \( k \geq 2 \). By induction, we may assume the \((k - 1, 2k + 1)\) case. Let \( A := G(k - 1, 2k + 1) \). Let \( B \) be the graph consisting of two copies of \( K_{k+1} \) with one vertex in common. Note that \( A \) is isomorphic to the analogous step in the proof in Lemma 6, so we omit it.

By assumption, \( A \) has a tree-decomposition with bags of size at most \( t(k - 1, 2k + 1) \). For each set \( \mathcal{C} = \{C_1, \ldots, C_{2k+1}\} \) of pairwise-disjoint \((k - 1)\)-cliques in \( A \), whose union induces \( K_{(k-1)(2k+1)} \), choose a node \( x \) corresponding to a bag containing \( C_1 \cup \cdots \cup C_{2k+1} \) in the tree-decomposition of \( A \), and add a new node adjacent to \( x \) with corresponding bag \( V(B_\mathcal{C}) \cup C_1 \cup \cdots \cup C_{2k+1} \). We obtain a tree-decomposition of \( G \) with bags of size at most \( \max\{t(k - 1, 2k + 1), (2k + 1)k\} = t(k - 1, 2k + 1) + t(k, 1) \), as desired.

Consider a perfect partition \( \mathcal{P} \) of \( G \). By Lemma 5, \( \mathcal{P}(A) \) is a connected partition of \( A \) and the quotient of \( \mathcal{P}(A) \) equals the subgraph of \( G/\mathcal{P} \) induced by those parts that intersect \( A \), and is therefore perfect. Recall that \( A = G(k - 1, 2k + 1) \). If outcome (2) holds for \( \mathcal{P}(A) \), then some part of \( \mathcal{P} \) contains \( K_k \) and outcome (1) holds for \( G \) (with \( r = 1 \)). Now assume that outcome (1) holds for \( \mathcal{P}(A) \). Thus \( A \) contains a \( K_{(k-1)(2k+1)} \) subgraph intersecting each of \( 2k + 1 \) distinct parts \( P_1, \ldots, P_{2k+1} \) of \( \mathcal{P} \) in \( k - 1 \) vertices. Let \( C_i \) be the corresponding \((k - 1)\)-clique in \( P_i \). Let \( \mathcal{C} := \{C_1, \ldots, C_{2k+1}\} \) and \( \hat{\mathcal{C}} := C_1 \cup \cdots \cup C_{2k+1} \).

If for some \( i \in \{1, \ldots, 2k + 1\} \), the neighbour of \( C_i \) in \( B_\mathcal{C} \) is in \( P_i \), then \( P_i \) contains a \( K_k \).
subgraph and outcome (1) holds for \( G \). Now assume that for each \( i \in \{1, \ldots, 2k+1\} \), the neighbour of \( C_i \) in \( B_G \) is not in \( P_i \). Suppose that some vertex \( x \) in \( B_G \) is in \( P_i \) for some \( i \in \{1, \ldots, k+1\} \). Then since \( P_i \) is connected, there is a path in \( G \) between \( C_i \) and \( x \) avoiding the neighbourhood of \( C_i \) in \( B_G \). Every such path intersects \( C \setminus C_i \), but none of these vertices are in \( P_i \). Thus, no vertex in \( B_G \) is in \( P_1 \cup \cdots \cup P_{2k+1} \).

By construction, \( B_G \) consists of two \((k+1)\)-cliques \( B^1 \) and \( B^2 \), intersecting in one vertex \( v \). Say \( v \) is in part \( P \) of \( \mathcal{P} \). If \( B^1 \subseteq V(P) \), then outcome (2) holds. Now assume that there is a vertex \( x \) of \( B^1 \) in some part \( Q \) distinct from \( P \). Similarly, assume that there is a vertex \( y \) of \( B^2 \) in some part \( R \) distinct from \( P \). Now, \( Q \neq R \), since \( C \setminus \{v\} \) separates \( x \) and \( y \), and none of these vertices are in \( Q \cup R \). By construction, \( x \) is adjacent to every vertex in \( C_i \) and \( y \) is adjacent to every vertex in \( C_j \), for some \( i, j \in \{1, \ldots, 2k+1\} \). Observe that \((Q,P,R,P_j,P_i)\) is a 5-cycle in \( G/\mathcal{P} \). Moreover, there is no \( PQ \) edge in \( G/\mathcal{P} \) because \((C \setminus C_j) \cup \{y\} \) separates \( x \in Q \) from \( C_j \subseteq P_j \), and none of these vertices are in \( Q \cup P_j \). Similarly, there is no \( RP \) edge in \( G/\mathcal{P} \). There is no \( PP_j \) edge in \( G/\mathcal{P} \) because \((C \setminus C_j) \cup \{y\} \) separates \( v \in P \) from \( C_j \subseteq P_j \), and none of these vertices are in \( P \cup P_j \). Similarly, there is no \( PP_i \) edge in \( G/\mathcal{P} \). Hence \((Q,P,R,P_j,P_i)\) is an induced 5-cycle in \( G/\mathcal{P} \), which contradicts the assumption that \( \mathcal{P} \) is a perfect partition. Therefore \( G \) has the claimed properties of \( G(k,1) \).

\[ \square \]

4 General Partitions: Proof of Theorem 4

To prove Theorem 4 we show the following stronger result, in which \( G \) only depends on \( |V(H)| \).

**Lemma 8.** For all integers \( k, t, r \geq 1 \), there is a graph \( G = G(k,t,r) \), such that for every connected partition \( \mathcal{P} \) of \( G \) either:

1. \( G \) contains a \( K_{kr} \) subgraph intersecting each of \( r \) distinct parts of \( \mathcal{P} \) in \( k \) vertices, or
2. \( G/\mathcal{P} \) contains every \( t \)-vertex graph, or
3. some part of \( \mathcal{P} \) contains \( K_{k+1} \).

**Proof.** We proceed by induction on \( k+t \) and then \( r \). We first deal with two base cases. First suppose that \( t = 1 \). Let \( G := G(k,1,r) := K_1 \). Then for every partition \( \mathcal{P} \) of \( G \), the quotient \( G/\mathcal{P} \) has at least one vertex, and (2) holds. Now assume that \( t \geq 2 \). Now suppose that \( k = 1 \). Let \( G := G(1,t,r) := K_r \). Then for every connected partition \( \mathcal{P} \) of \( G \), if some part of \( \mathcal{P} \) contains an edge, then (3) holds; otherwise each part is a single vertex, and (1) holds. Now assume that \( k \geq 2 \).

The proof that the \((k,t,1)\) and \((k,t,r)\) cases imply the \((k,t,r+1)\) case is identical to the analogous step in the proof in Lemma 6, so we omit it.

It remains to prove the \((k,t,1)\) case for \( k \geq 2 \) and \( t \geq 2 \). By induction, we may assume the \((k,t-1,1)\) case and the \((k-1,t,r)\) case for all \( r \). Let \( B := G(k,t-1,1) \) and \( n := |V(B)| \).
Consider a connected partition $\mathcal{P}$ of $G$. By Lemma 5, $\mathcal{P}(A)$ is a connected partition of $A$, and the quotient of $\mathcal{P}(A)$ equals the subgraph of $G/\mathcal{P}$ induced by those parts that intersect $A$. Recall that $A = G(k-1,t,2^n)$. If $\mathcal{P}(A)$ satisfies outcome (2), then the quotient of $\mathcal{P}(A)$ contains every $t$-vertex graph and outcome (2) is satisfied for $G$. If outcome (3) holds for $\mathcal{P}(A)$, then some part of $\mathcal{P}$ contains $K_k$ and outcome (1) holds for $G$ (with $r = 1$). Now assume that outcome (1) holds for $\mathcal{P}(A)$. Thus $A$ contains a $K_{(k-1)2^n}$ subgraph intersecting each of $2^n$ distinct parts $P_1, \ldots, P_{2^n}$ of $\mathcal{P}$ in $k-1$ vertices. Let $C_i$ be the corresponding $(k-1)$-clique in $P_i$. Let $\mathcal{C} := \{C_1, \ldots, C_{2^n}\}$.

If for some $i \in \{1, \ldots, 2^n\}$, some neighbour of $C_i$ in $B_\mathcal{C}$ is in $P_i$, then $P_i$ contains $K_k$ and outcome (1) holds for $G$. Now assume that for each $i \in \{1, \ldots, 2^n\}$, no neighbour of $C_i$ in $B_\mathcal{C}$ is in $P_i$. Suppose that some vertex $x$ in $B_\mathcal{C}$ is in $P_i$, for some $i \in \{1, \ldots, 2^n\}$. Then since $P_i$ is connected, $G$ contains a path between $C_i$ and $x$ avoiding the neighbourhood of $C_i$ in $B_\mathcal{C}$. Every such path intersects $C_1 \cup \cdots \cup C_{i-1} \cup C_{i+1} \cup \cdots \cup C_{2^n}$, but none of these vertices are in $P_i$. Thus, no vertex in $B_\mathcal{C}$ is in $P_1 \cup \cdots \cup P_{2^n}$. Hence, no part of $\mathcal{P}$ contains vertices in both $B_\mathcal{C}$ and in the remainder of $G$. Therefore, $\mathcal{P}(B_\mathcal{C})$ is a connected partition of $B_\mathcal{C}$, and the quotient of $\mathcal{P}(B_\mathcal{C})$ equals the subgraph of $G/\mathcal{P}$ induced by those parts that intersect $B_\mathcal{C}$. Since $B = G(k,t-1,1)$, by induction, $\mathcal{P}(B_\mathcal{C})$ satisfies (1), (2) or (3). If outcome (1) or (3) holds for $\mathcal{P}(B_\mathcal{C})$, then the same outcome holds for $G$. Now assume that outcome (2) holds for $\mathcal{P}(B_\mathcal{C})$.

We now show that outcome (2) holds for $G$. Let $H$ be a $t$-vertex graph, let $v$ be a vertex of $H$, and let $N_H(v) = \{w_1, \ldots, w_d\}$. Since outcome (2) holds for $\mathcal{P}(B_\mathcal{C})$, the quotient of $\mathcal{P}(B_\mathcal{C})$ contains $H - v$. Let $Q_1, \ldots, Q_d$ be the parts corresponding to $w_1, \ldots, w_d$. Then $S_i = V(Q_1 \cup \cdots \cup Q_d)$ for some $i \in \{1, \ldots, 2^n\}$. In $G/\mathcal{P}$, the vertex corresponding to $P_i$ is adjacent to $Q_1, \ldots, Q_d$ and to no other vertices corresponding to parts contained in $B_\mathcal{C}$. Thus, including $P_i$, $G/\mathcal{P}$ contains $H$ and outcome (2) holds for $\mathcal{P}$. Hence $G$ has the claimed properties of $G(k,t,1)$.

References


