CLUSTERED COLOURING IN MINOR-CLOSED CLASSES

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Abstract. The *clustered chromatic number* of a class of graphs is the minimum integer k such that for some integer c every graph in the class is k-colourable with monochromatic components of size at most c. We prove that for every graph H, the clustered chromatic number of the class of H-minor-free graphs is tied to the tree-depth of H. In particular, if H is connected with tree-depth t then every H-minor-free graph is $(2^{t+1}-4)$-colourable with monochromatic components of size at most $c(H)$. This provides evidence for a conjecture of Ossona de Mendez, Oum and Wood (2016). If $t=3$ then we prove that 4 colours suffice, which is best possible. We also determine those minor-closed graph classes with clustered chromatic number 2. Finally, we develop a conjecture for the clustered chromatic number of an arbitrary minor-closed class.

1 Introduction

In a vertex-coloured graph, a *monochromatic component* is a connected component of the subgraph induced by all the vertices of one colour. A graph G is *k-colourable with clustering c* if each vertex can be assigned one of k colours such that each monochromatic component has at most c vertices. We shall consider such colourings, where the first priority is to minimise the number of colours, with small clustering as a secondary goal. With this viewpoint the following definition arises. The *clustered chromatic number* of a graph class $\mathcal{G}$, denoted by $\chi_\star(\mathcal{G})$, is the minimum integer k such that, for some integer c, every graph in $\mathcal{G}$ has a k-colouring with clustering c.

This paper studies clustered colouring in minor-closed classes of graphs. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. A class of graphs $\mathcal{M}$ is *minor-closed* if for every graph $G \in \mathcal{M}$ every minor of G is in $\mathcal{M}$, and some graph is not in $\mathcal{M}$. For a graph H, let $\mathcal{M}_H$ be the class of H-minor-free graphs (that is, not containing H as a minor). Note that we only consider simple finite graphs.

We approach this topic via Hadwiger's Conjecture, which states that every graph with no K_t-minor is properly $(t-1)$-colourable. This conjecture is easy for $t \leqslant 4$, is equivalent to the 4-colour theorem for $t=5$, is true for $t=6$ [20], and is open for $t \geqslant 7$. The best known upper bound on the

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August 8, 2017

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Supported by the Australian Research Council.

chromatic number is $O(t\sqrt{\log t})$, independently due to Kostochka [11, 12] and Thomason [22, 23]. This conjecture is widely considered to be one of the most important open problems in graph theory; see [21] for a survey.

Clustered colourings of K_t-minor-free graphs provide an avenue for attacking Hadwiger's Conjecture. Kawarabayashi and Mohar [9] first proved a $O(t)$ upper bound on $\chi_\star(\mathcal{M}_{K_t})$. In particular, they proved that every K_t-minor-free graph is $\lceil\frac{31}{2}t\rceil$-colourable with clustering $f(t)$, for some function f. The number of colours in this result was improved to $\lceil\frac{7t-3}{2}\rceil$ by Wood [25], to $4t-4$ by Edwards, Kang, Kim, Oum, and Seymour [5], to $3t-3$ by Liu and Oum [14], and to $2t-2$ by Norin [16]. Thus $\chi_\star(\mathcal{M}_{K_t}) \leqslant 2t-2$. See [7, 8] for analogous results for graphs excluding odd minors. For all of these results, the function $f(t)$ is very large, often depending on constants from the Graph Minor Structure Theorem. Van den Heuvel and Wood [24] proved the first such result with $f(t)$ explicit. In particular, they proved that every K_t-minor-free graph is $(2t-2)$-colourable with clustering $\lceil\frac{t-2}{2}\rceil$. The result of Edwards et al. [5] mentioned below implies that $\chi_\star(\mathcal{M}_{K_t}) \geqslant t-1$. Dvořák and Norin [4] have announced a proof that $\chi_\star(\mathcal{M}_{K_t}) = t-1$.

Now consider the class $\mathcal{M}_H$ of H-minor-free graphs for an arbitrary graph H. Hadwiger's Conjecture would imply that the maximum chromatic number of a graph in $\mathcal{M}_H$ equals $|V(H)|-1$ (since $K_{|V(H)|-1}$ is H-minor-free). However, for clustered colourings, fewer colours often suffice. For example, Kawarabayashi and Thomassen [10] and Esperet and Ochem [6] proved that graphs embeddable on any fixed surface are 5-colourable with bounded clustering, whereas the chromatic number is $\Theta(\sqrt{g})$ for surfaces of Euler genus g. Van den Heuvel and Wood [24] proved that $K_{2,t}$-minor-free graphs are 3-colourable with clustering $t-1$, and that $K_{3,t}$-minor-free graphs are 6-colourable with clustering $2t$. These results show that $\chi_\star(\mathcal{M}_H)$ depends on the structure of H, unlike the usual chromatic number which only depends on $|V(H)|$.

At the heart of this paper is the following question: what property of H determines the $\chi_\star(\mathcal{M}_H)$? The following definitions help to answer this question. Let T be a rooted tree. The *depth* of T is the maximum number of vertices on a root–to–leaf path in T. The *closure* of T is obtained from T by adding an edge between every ancestor and descendent in T. The *connected tree-depth*[1] of a graph H, denoted by $\overline{\mathrm{td}}(H)$, is the minimum depth of a rooted tree T such that H is a subgraph of the closure of T.

The following result is the primary contribution of this paper; it is proved in Section 2.

Theorem 1. *For every graph H, $\chi_\star(\mathcal{M}_H)$ is tied to the (connected) tree-depth of H. In particular,*

$$\overline{\mathrm{td}}(H) - 1 \leqslant \chi_\star(\mathcal{M}_H) \leqslant 2^{\overline{\mathrm{td}}(H)+1} - 4.$$

The upper bound in Theorem 1 gives evidence for, and was inspired by, a conjecture of Ossona de

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[1] This definition is a variant of the more commonly studied notion of the *tree-depth* of H, denoted by $\mathrm{td}(H)$, which equals the maximum connected tree-depth of the connected components of H. See [15] for background on tree-depth. If H is connected, then $\mathrm{td}(H) = \overline{\mathrm{td}}(H)$. In fact, $\mathrm{td}(H) = \overline{\mathrm{td}}(H)$ unless H has two connected components H_1 and H_2 with $\mathrm{td}(H_1) = \mathrm{td}(H_2) = \mathrm{td}(H)$, in which case $\overline{\mathrm{td}}(H) = \mathrm{td}(H) + 1$. We choose to work with connected tree-depth to avoid this distinction.

Mendez, Oum, and Wood [17], which we now introduce. A graph G is *k-colourable with defect d* if each vertex of G can be assigned one of k colours so that each vertex is adjacent to at most d neighbours of the same colour; that is, each monochromatic component has maximum degree at most d. The *defective chromatic number* of a graph class $\mathcal{G}$, denoted by $\chi_\Delta(\mathcal{G})$, is the minimum integer k such that, for some integer d, every graph in $\mathcal{G}$ is k-colourable with defect d. Every colouring of a graph with clustering c has defect $c-1$. Thus the defective chromatic number of a graph class is at most its clustered chromatic number. Ossona de Mendez et al. [17] conjectured the following behaviour for the defective chromatic number of $\mathcal{M}_H$.

Conjecture 2 (Ossona de Mendez et al. [17]). *For every graph H,*

$$\chi_\Delta(\mathcal{M}_H) = \overline{\mathrm{td}}(H) - 1.$$

Ossona de Mendez et al. [17] proved the lower bound, $\chi_\Delta(\mathcal{M}_H) \geqslant \overline{\mathrm{td}}(H) - 1$, in Conjecture 2. The lower bound in Theorem 1 follows since $\chi_\Delta \leqslant \chi_\star$ for every class. The upper bound in Conjecture 2 is known to hold in some special cases. Edwards et al. [5] proved it if $H = K_t$; that is, $\chi_\Delta(\mathcal{M}_{K_t}) = t - 1$, which can be thought of as a defective version of Hadwiger's Conjecture. Ossona de Mendez et al. [17] proved the upper bound in Conjecture 2 if $\overline{\mathrm{td}}(H) \leqslant 3$ or if H is a complete bipartite graph. In particular, $\chi_\Delta(\mathcal{M}_{K_{s,t}}) = \min\{s,t\}$.

Theorem 1 provides some evidence for Conjecture 2 by showing that $\chi_\Delta(\mathcal{M}_H)$ and $\chi_\star(\mathcal{M}_H)$ are bounded from above by some function of $\overline{\mathrm{td}}(H)$. This was previously not known to be true.

While it is conjectured that $\chi_\Delta(\mathcal{M}_H) = \overline{\mathrm{td}}(H) - 1$, the following lower bound shows that $\chi_\star(\mathcal{M}_H)$ might be larger, thus providing some distinction between defective and clustered colourings.

Theorem 3. *For each $k \geqslant 2$, there is a graph H_k with $\overline{\mathrm{td}}(H_k) = \mathrm{td}(H_k) = k$ such that*

$$\chi_\star(\mathcal{M}_{H_k}) \geqslant 2k - 2.$$

We conjecture an analogous upper bound:

Conjecture 4. *For every graph H,*

$$\chi_\star(\mathcal{M}_H) \leqslant 2\,\overline{\mathrm{td}}(H) - 2.$$

A further contribution of the paper is to precisely determine the minor-closed graph classes with clustered chromatic number 2. This result is introduced and proved in Section 3. Section 4 studies clustered colourings of graph classes excluding so-called fat stars as a minor. This leads to a proof of Conjecture 4 in the $\overline{\mathrm{td}}(H) = 3$ case. We conclude in Section 5 with a conjecture about the clustered chromatic number of an arbitrary minor-closed class that generalises Conjecture 4.

2 Tree-depth Bounds

The main goal of this section is to prove that $\chi_\star(\mathcal{M}_H)$ is bounded from above by some function of $\overline{\mathrm{td}}(H)$. We actually provide two proofs. The first proof, which uses less colours, depends on deep results from graph structure theory and gives no explicit bound on the clustering. The second proof is self-contained but uses more colours. Both proofs have their own merits, so we include both.

2.1 First Proof

The first proof depends on the following Erdős-Pósa Theorem by Robertson and Seymour [19]. For a graph H and integer $p \geqslant 1$, let $p\,H$ be the disjoint union of p copies of H.

Theorem 5 ([19]; see [18, Lemma 3.10])**.** *For every graph H with c connected components and for all integers $p, w \geqslant 1$, for every graph G with treewidth at most w and with no $p\,H$ minor, there is a set $X \subseteq V(G)$ of size at most $(p-1)(wc-1)$ such that $G - X$ has no H minor.*

The next lemma is the heart of our proof. Let $C\langle h, k\rangle$ be the closure of the complete k-ary tree of depth h. If r is a vertex in a connected graph G and $V_i := \{v \in V(G) : \operatorname{dist}_G(v, r) = i\}$ for $i \geqslant 0$, then $V_0, V_1, \dots$ is called the *BFS layering* of G starting at r.

Lemma 6. *For all integers $h, k, w \geqslant 1$, every $C\langle h, k\rangle$-minor-free graph G of treewidth at most w is $(2^h - 2)$-colourable with clustering at most $(k-1)(w-1)$.*

Proof. We proceed by induction on $h \geqslant 1$, with w and k fixed. The case $h = 1$ is trivial since $C\langle 1, k\rangle$ is the 1-vertex graph. Now assume that $h \geqslant 2$, the claim holds for $h - 1$, and G is a $C\langle h, k\rangle$-minor-free graph with treewidth at most w. Let $V_0, V_1, \dots$ be the BFS layering of G starting at some vertex r.

Fix $i \geqslant 1$. Then $G[V_i]$ contains no $k\,C\langle h-1, k\rangle$ as a minor, as otherwise contracting $V_0 \cup \dots \cup V_{i-1}$ to a single vertex gives a $C\langle h, k\rangle$ minor (since every vertex in V_i has a neighbour in V_{i-1}). Since G has treewidth at most w, so does $G[V_i]$. By Theorem 5 with $H = C\langle h-1, k\rangle$ and $c = 1$, there is a set $X_i \subseteq V_i$ of size at most $(k-1)(w-1)$ such that $G[V_i \setminus X_i]$ has no $C\langle h-1, k\rangle$ minor. By induction, $G[V_i \setminus X_i]$ is $(2^{h-1} - 2)$-colourable with clustering $(k-1)(w-1)$. Use one new colour for X_i. Thus $G[V_i]$ is $(2^{h-1} - 1)$-colourable with clustering $(k-1)(w-1)$.

Use disjoint sets of colours for even and odd i, and colour r by one of the colours used for even i. No edge joins V_i with V_j for $j \geqslant i + 2$. Thus G is $(2^h - 2)$-coloured with clustering $(k-1)(w-1)$. □

To drop the assumption of bounded treewidth, we use the following result of DeVos, Ding, Oporowski, Sanders, Reed, Seymour, and Vertigan [3], the proof of which depends on the graph minor structure theorem.

Theorem 7 ([3])**.** *For every graph H there is an integer w such that for every graph G with no H-minor, there is a partition V_1, V_2 of $V(G)$ such that $G[V_i]$ has treewidth at most w, for $i \in \{1, 2\}$.*

Lemma 6 and Theorem 7 imply:

Lemma 8. *For all integers $h, k \geqslant 1$, there is an integer $g(h,k)$, such that every $C\langle h,k\rangle$-minor-free graph G is $(2^{h+1}-4)$-colourable with clustering at most $g(h,k)$.*

Fix a graph H. By definition, H is a subgraph of $C\langle \overline{\mathrm{td}}(H), |V(H)|\rangle$. Thus every H-minor-free graph contains no $C(\overline{\mathrm{td}}(H), |V(H)|)$-minor. Hence, Lemma 8 implies

$$\chi_\star(\mathcal{M}_H) \leqslant 2^{\overline{\mathrm{td}}(H)+1} - 4,$$

which is the upper bound in Theorem 1.

Note Theorem 26 below improves the $h = 3$ case in Lemma 6, which leads to a small constant-factor improvement in Theorem 1 for $h \geqslant 3$.

2.2 Second Proof

We now present our second proof that $\chi_\star(\mathcal{M}_H)$ is bounded from above by some function of $\overline{\mathrm{td}}(H)$. This proof is self-contained (not using Theorems 5 and 7).

Let T be a rooted tree. Recall that the *closure* of T is the graph G with vertex set $V(T)$, where two vertices are adjacent in G if one is an ancestor of the other in T. The *weak closure* of T is the graph G with vertex set $V(T)$, where two vertices are adjacent in G if one is a leaf and the other is one of its ancestors. For $h, k \geqslant 1$, let $T\langle h,k\rangle$ be the rooted complete k-ary tree of depth h. Let $W\langle h,k\rangle$ be the weak closure of $T\langle h,k\rangle$.

Lemma 9. *For $h, k \geqslant 2$, the graph $W\langle h,k\rangle$ contains $C\langle h,k-1\rangle$ as a minor.*

Proof. Let r be the root vertex. Colour r blue. For each non-leaf vertex v, colour $k-1$ children of v *blue* and colour the other child of v *red*. Let X be the set of blue vertices v in $T\langle h,k\rangle$, such that every ancestor of v is blue. Note that X induces a copy of $T\langle h,k-1\rangle$ in $T\langle h,k\rangle$. Let v be a non-leaf vertex in X. Let w be the red child of v, and let T_v be the subtree of $T\langle h,k\rangle$ rooted at w. Then every leaf of T_v is adjacent in $W\langle h,k\rangle$ to v and to every ancestor of v. Contract T_v and the edge vw into v. Now v is adjacent to every ancestor of v in X. Do this for each non-leaf vertex in X. Note that T_u and T_v are disjoint for distinct non-leaf vertices $u, v \in X$. Thus, we obtain $C\langle h,k-1\rangle$ as a minor of $W\langle h,k\rangle$. □

A *model* of a graph H in a graph G is a collection $\{J_x : x \in V(H)\}$ of pairwise disjoint subtrees of G such that for every $xy \in E(H)$ there is an edge of G with one end in $V(J_x)$ and another in $V(J_y)$. A graph contains H as a minor if and only if it contains a model of H.

Lemma 10. *For $h \geqslant 2$ and $k \geqslant 1$, if a graph G contains $W\langle h,6k\rangle$ as a minor, then G contains subgraphs G' and G'', both containing $W\langle h,k\rangle$ as a minor, such that $|V(G') \cap V(G'')| \leqslant 1$.*

Proof. Let $\{J_x : x \in V(W\langle h,6k\rangle)\}$ be a model of $W\langle h,6k\rangle$ in G. Let r be the root vertex of $W\langle h,6k\rangle$. We may assume that for each leaf vertex x of $T\langle h,6k\rangle$, there is exactly one edge between J_x and J_r.

Let Q be a tree obtained from J_r by splitting vertices, where:

- Q has maximum degree at most 3,
- J_r is a minor of Q; let $\{Q_v : v \in V(J_r)\}$ be the model of J_r in Q, so each edge vw of J_r corresponds to an edge of Q between Q_v and Q_w,
- there is a set L of leaf vertices in Q, and a bijection ϕ from L to the set of leaves of $T\langle h, 6k\rangle$, such that for each leaf x of $T\langle h, 6k\rangle$, if the edge between J_x and J_r in G is incident with vertex v in J_r, then $\phi^{-1}(x)$ is a vertex z in $L \cap Q_v$, in which case we say x and z are *associated*.

Let $L' \subseteq L$. Apply the following 'propagation' process in $T\langle h, 6k\rangle$. Initially, say that the vertices in $\phi(L')$ are *alive* with respect to L'. For each parent vertex y of leaves in $T\langle h, 6k\rangle$, if at least $2k$ of its $6k$ children are alive with respect to L', then y is also alive with respect to L'. Now propagate up $T\langle h, 6k\rangle$, so that a non-leaf vertex y of $T\langle h, 6k\rangle$ is *alive* if and only if at least $2k$ of its children are alive with respect to L'. Say L' is *good* if r is alive with respect to L'.

For an edge vw of Q let L_{vw} be the set of vertices in L in the subtree of $Q - vw$ containing v, and let L_{wv} be the set of vertices in L in the subtree of $Q - vw$ containing w. Since L is the disjoint union of L_{vw} and L_{wv}, every leaf vertex of $T\langle h, 6k\rangle$ is in exactly one of $\phi(L_{vw})$ or $\phi(L_{wv})$. By induction, every vertex in $T\langle h, 6k\rangle$ is alive with respect to L_{vw} or L_{wv} (possibly both). In particular, L_{vw} or L_{wv} is good (possibly both).

Suppose that both L_{vw} and L_{wv} are good. Then at least $2k$ children of r are alive with respect to L_{vw}, and at least $2k$ children of r are alive with respect to L_{wv}. Thus there are disjoint sets A and B, each consisting of k children of r, where every vertex in A is alive with respect to L_{vw}, and every vertex in B is alive with respect to L_{wv}. We now define a set of vertices, said to be *chosen* by v, all of which are alive with respect to L_{vw}. First, each vertex in A is *chosen* by v. Then for each non-leaf vertex z chosen by v, choose k children of z that are also alive with respect to L_{vw}, and say they are *chosen* by v. Continue this process down to the leaves of $T\langle h, 6k\rangle$. We now define the graph G', which is initially empty. For each vertex z chosen by v, add the subgraph J_z to G'. Furthermore, for each leaf vertex z of $T\langle h, 6k\rangle$ chosen by v and for each ancestor y of z chosen by v, add the edge in G between J_z and J_y to G'. Define G'' analogously with respect to B and L_{wv}. At this point, G' and G'' are disjoint.

The edge vw in Q either corresponds to an edge or a vertex of J_r. First suppose that vw corresponds to an edge ab of J_r, where v is in Q_a and w is in Q_b. Let J_r^1 be the subtree of $J_r - ab$ containing a. Add J_r^1 to G', plus the edge in G between J_r^1 and J_z for each leaf z of $T\langle h, 6k\rangle$ chosen by v. Similarly, let J_r^2 be the subtree of $J_r - ab$ containing b, and add J_r^2 to G'', plus the edge in G between J_r^2 and J_z for each leaf z of $T\langle h, 6k\rangle$ chosen by w. Observe that G' and G'' are disjoint, and they both contain $W\langle h, k\rangle$ as a minor, as desired.

Now consider the case in which vw corresponds to a vertex z in J_r; that is, v and w are both in Q_z. Let J_r^1 be the subtree of J_r corresponding to the subtree of $Q - vw$ containing v (which includes z). Add J_r^1 to G', plus the edge in G between J_r^1 and J_z for each leaf z of $T\langle h, 6k\rangle$ chosen by v. Similarly, let J_r^2 be the subtree of J_r corresponding to the subtree of $Q - vw$ containing w (which includes z). Add J_r^2 to G'', plus the edge in G between J_r^2 and J_z for each leaf z of $T\langle h, 6k\rangle$ chosen by w. Observe that both G' and G'' contain $W\langle h, k\rangle$ as a minor, and $V(G_1) \cap V(G_2) = \{z\}$, as desired.

We may therefore assume that for each edge vw of Q, exactly one of L_{vw} and L_{wv} is good. Orient vw towards v if L_{vw} is good, and towards w if L_{wv} is good. Since at most one leaf of $T\langle h, 6k\rangle$ is associated with each leaf of Q, each edge incident with a leaf of Q is oriented away from the leaf. Since Q is a tree, Q contains a sink vertex v, which is therefore not a leaf. Let w_1, w_2 and possibly w_3 be the neighbours of v in Q. Let L_i be the set of vertices in L in the subtree of $Q - vw_i$ containing w_i. Since vw_i is oriented towards v, with respect to vw_i, the set L_i is not good. Since no leaf of $T\langle h, 6k\rangle$ is associated with v, the sets $\phi(L_1)$, $\phi(L_2)$ and $\phi(L_3)$ partition the leaves of $T\langle h, 6k\rangle$. Since each non-leaf vertex y in $T\langle h, 6k\rangle$ has $6k$ children, y is alive with respect to at least one of L_1, L_2 or L_3. In particular, at least one of L_1, L_2 or L_3 is good. This is a contradiction. □

Theorem 11. *Let* $f(h) := \frac{1}{6}(4^h - 4)$ *for every* $h \geqslant 1$. *Then there is a function* $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *such that for every* $k \geqslant 1$, *every graph either contains* $W\langle h, k\rangle$ *as a minor or is* $f(h)$*-colourable with clustering* $g(h, k)$.

Proof. We proceed by induction on $h \geqslant 1$. In the base case, $h = 1$, since $W\langle 1, k\rangle$ is the 1-vertex graph, the result holds with $f(1) = g(1, k) = 0$. Now assume that $h \geqslant 2$ and the result holds for $h - 1$ and all k.

Let G be a graph, which we may assume is connected. Let $V_0, V_1, \dots$ be a BFS layering of G.

Fix $i \geqslant 1$. Let s be the maximum integer such that $G[V_i]$ contains s disjoint subgraphs $G_1, \dots, G_s$ each containing a $W\langle h - 1, \max\{1, 6^{k-s}\}k\rangle$ minor. First suppose that $s \geqslant k$. Then $G[V_i]$ contains k disjoint subgraphs each containing a $W\langle h - 1, k\rangle$ minor. Contracting $V_0 \cup \dots \cup V_{i-1}$ to a single vertex gives a $W\langle h, k\rangle$ minor (since every vertex in V_i has a neighbour in V_{i-1}), and we are done. Now assume that $s \leqslant k - 1$.

If $s = 0$, then $G[V_i]$ contains no $W\langle h - 1, 6^{k-1}k\rangle$ minor. By induction, $G[V_i]$ is $f(h-1)$-colourable with clustering $g(h - 1, 6^{k-1}k)$.

Now consider the case that $s \in [1, k - 1]$. Apply Lemma 10 to G_j for each $j \in [1, r]$. Thus G_j contains subgraphs G'_j and G''_j, both containing $W\langle h - 1, 6^{k-s-1}k\rangle$ as a minor, such that $|V(G'_j) \cap V(G''_j)| \leqslant 1$. Let $X := \bigcup_{j=1}^{s} V(G'_j) \cap V(G''_j)$. Thus $|X| \leqslant s \leqslant k - 1$. Let $A := G[V_i] - \bigcup_{j=1}^{s} V(G'_j)$ and $B := G[V_i] - \bigcup_{j=1}^{s} V(G''_j)$. By the maximality of s, the subgraph A contains no $W\langle h - 1, 6^{k-s-1}k\rangle$ minor (as otherwise $A, G'_1, \dots, G'_s$ would give $s + 1$ pairwise disjoint subgraphs satisfying the requirements). By induction, A is $f(h - 1)$-colourable with clustering $g(h - 1, 6^k k)$ since $6^{k-s-1}k \leqslant 6^k k$. Similarly, B is $f(h - 1)$-colourable with clustering

$g(h-1,6^k k)$. By construction, each vertex in $G[V_i]$ is in at least one of X, A or B. Use one new colour for X, which has size at most $s \leqslant k-1$.

In both cases, $G[V_i]$ is $(2f(h-1)+1)$-colourable with clustering $\max\{g(h-1,6^k k),k-1\}$. Use a different set of $2f(h-1)+1$ colours for even i and for odd i, and colour r by one of the colours used for even i. No edge joins V_i with V_j for $j \geqslant i+2$. Since $f(h) = 4f(h-1)+2$, G is $f(h)$-colourable with clustering $g(h,k) := \max\{g(h-1,6^k k),k-1\}$. □

Note that the clustering function $g(h,k)$ in Theorem 11 satisfies

$$g(h,k) \leqslant k6^{k6^{k6^{\cdot^{\cdot^{\cdot^{k6^k}}}}}},$$

where the number of ks is h.

Theorem 12. *For every graph H,*

$$\chi_\star(\mathcal{M}_H) \leqslant \tfrac{1}{6}(4^{\overline{\mathrm{td}}(H)}-4).$$

Proof. Let G be a graph not containing H as a minor. By definition, H is a subgraph of $C\langle \overline{\mathrm{td}}(H), |V(H)|\rangle$. Thus G does not contain $C\langle \overline{\mathrm{td}}(H), |V(H)|\rangle$ as a minor. By Lemma 9, G does not contain $W\langle \overline{\mathrm{td}}(H), |V(H)|+1\rangle$ as a minor. By Theorem 11, there is a constant $c = c(H)$, such that G is $\frac{1}{6}(4^{\overline{\mathrm{td}}(H)}-4)$-colourable with clustering at most c. □

2.3 Lower Bound

We now prove Theorem 3, where $H_k := C\langle k,3\rangle$, the closure of the complete ternary tree of depth k (which has tree-depth and connected tree-depth k).

Lemma 13. $\chi_\star(\mathcal{M}_{C\langle k,3\rangle}) \geqslant 2k-2$ *for* $k \geqslant 2$.

Proof. Fix an integer c. We now recursively define graphs G_k (depending on c), and show by induction on k that G_k has no $(2k-3)$-colouring with clustering c, and $C\langle k,3\rangle$ is not a minor of G_k.

For the base case $k=2$, let G_2 be the path on $c+1$ vertices. Then G_2 has no $C\langle 2,3\rangle = K_{1,3}$ minor, and G_2 has no 1-colouring with clustering c.

Assume G_{k-1} is defined for some $k \geqslant 3$, that G_{k-1} has no $(2k-5)$-colouring with clustering c, and $C\langle k-1,3\rangle$ is not a minor of G_{k-1}. As illustrated in Figure 1, let G_k be obtained from a path $(v_1,\dots,v_{c+1})$ as follows: for $i \in \{1,\dots,c\}$ add $2c-1$ pairwise disjoint copies of G_{k-1} complete to $\{v_i,v_{i+1}\}$.

Suppose that G_k has a $(2k-3)$-colouring with clustering c. Then v_i and v_{i+1} receive distinct colours for some $i \in \{1,\dots,c\}$. Consider the $2c-1$ copies of G_{k-1} complete to $\{v_i,v_{i+1}\}$. At

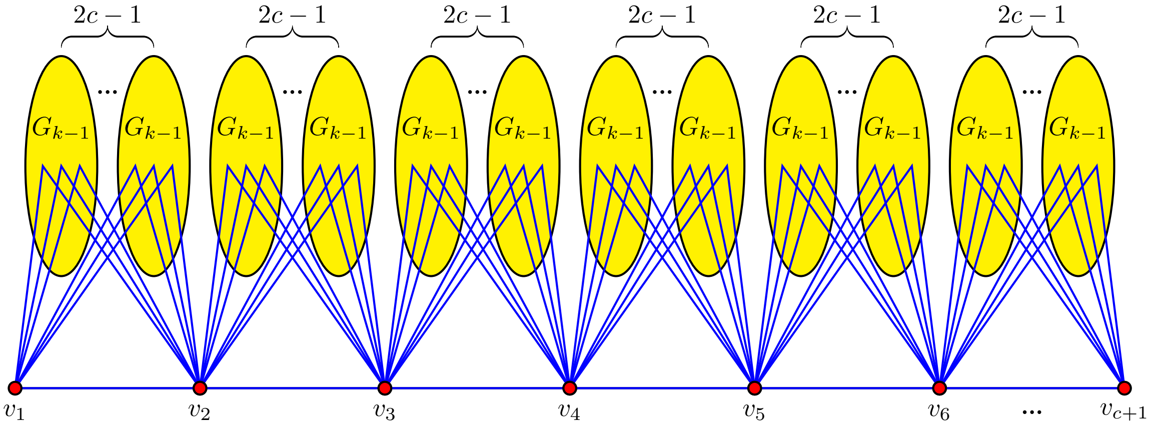

Figure 1: Construction of G_k.

most $c-1$ such copies contain a vertex assigned the same colour as v_i, and at most $c-1$ such copies contain a vertex assigned the same colour as v_{i+1}. Thus some copy avoids both colours. Hence G_{k-1} is $(2k-5)$-coloured with clustering c, which is a contradiction. Therefore G_k has no $(2k-3)$-colouring with clustering c.

It remains to show that $C\langle k,3\rangle$ is not a minor of G_k. Suppose that G_k contains a model $\{J_x : x \in V(C\langle k,3\rangle)\}$ of $C\langle k,3\rangle$. Let r be the root vertex in $C\langle k,3\rangle$. Choose the $C\langle k,3\rangle$-model to minimise $\sum_{x\in V(H)} |V(J_x)|$. Thus J_r is a connected subgraph of $(v_1,\dots,v_{c+1})$. Say $J_r=(v_i,\dots,v_j)$. Note that $C\langle k,3\rangle - r$ consists of three pairwise disjoint copies of $C\langle k-1,3\rangle$. The model X of one such copy avoids v_{i-1} and v_{j+1} (if these vertices are defined). Since $C\langle k-1,3\rangle$ is connected, X is contained in a component of $G_k - \{v_{i-1},\dots,v_{j+1}\}$ and is adjacent to $(v_i,\dots,v_j)$. Each such component is a copy of G_{k-1}. Thus $C\langle k-1,3\rangle$ is a minor of G_{k-1}, which is a contradiction. Thus $C\langle k,3\rangle$ is not a minor of G_k. □

3 2-Colouring with Bounded Clustering

This section considers the following question: which minor-closed graph classes have clustered chromatic number 2? To answer this question we introduce three classes of graphs that are not 2-colourable with bounded clustering, as illustrated in Figure 2.

The first example is the n-fan, which is the graph obtained from the n-vertex path by adding one dominant vertex. If the n-fan is 2-colourable with clustering c, then the underlying path contains at most $c-1$ vertices of the same colour as the dominant vertex, implying that the other colour has at most c monochromatic components each with at most c vertices, and $n \leqslant c^2+c-1$. That is, if $n \geqslant c^2+c$ then the n-fan is not 2-colourable with clustering c.

The second example is the n-fat star, which is the graph obtained from the n-star (the star with n leaves) as follows: for each edge vw in the n-star, add n degree-2 vertices adjacent to v and w. Note that the n-fat star is $C\langle 3,n\rangle$. Suppose that the n-fat star has a 2-colouring with clustering

$c \leqslant n$. Deleting the dominant vertex in the n-fat star gives n disjoint n-stars. Since $n \geqslant c$, in at least one of these n-stars, no vertex receives the same colour as the dominant vertex, implying there is a monochromatic component on $n+1 \geqslant c+1$ vertices. Thus, for $n \geqslant c$ there is no 2-colouring of the n-fat star with clustering c.

The third example is the n-fat path, which is the graph obtained from the n-vertex path as follows: for each edge vw of the n-vertex path, add n degree-2 vertices adjacent to v and w. If $n \geqslant 2c$ then in every 2-colouring of the n-fat path with clustering c, adjacent vertices in the underlying path receive the same colour, implying that the underlying path is contained in a monochromatic component with more than c vertices. Thus, for $n \geqslant 2c$ there is no 2-colouring of the n-fat path with clustering c.

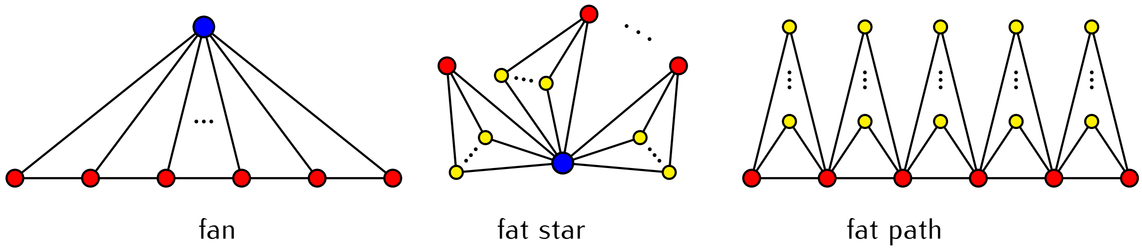

Figure 2: Graph classes that are not 2-colourable with bounded clustering.

These three examples all need three colours in a colouring with bounded clustering. The main result of this section is the following converse result.

Theorem 14. *Let $\mathcal{G}$ be a minor-closed graph class. Then $\chi_\star(\mathcal{G}) \leqslant 2$ if and only if for some integer $k \geqslant 1$, the k-fan, the k-fat path, and the k-fat star are not in $\mathcal{G}$.*

The following definition is a key to the proof of Theorem 14. For an integer $k \geqslant 1$ and a graph H with vertex set $\{v_1, \dots, v_h\}$, a *k-strong H-model* in a graph G consists of h pairwise disjoint connected subgraphs $X_1, \dots, X_h$ in G, such that for each edge v_iv_j of H there are k vertices in $V(G) \setminus \bigcup_{i=1}^h V(X_i)$ adjacent to both X_i and X_j. This definition leads to the following sufficient condition for a graph to contain a k-fat star or k-fat path

Lemma 15. *If a graph G contains a $k(k+1)$-strong H-model for some connected graph H with k^k edges, then G contains a k-fat star or a k-fat path as a minor.*

Proof. Use the notation introduced in the definition of k-strong H-model. Since H is connected with k^k edges, H contains a k-vertex path or a k-leaf star as a subgraph. Suppose that $(v_1, \dots, v_k)$ is a k-vertex path in H. For $i = 1, 2, \dots, k-1$, let N_i be a set of $k+1$ vertices in

$$(V(G) \setminus \bigcup_{i=1}^{h} V(X_i)) \setminus \bigcup_{j=1}^{i-1} N_j,$$

each of which is adjacent to both X_i and X_{i+1}. Such a set exists since X_i and X_{i+1} have at least $k(k+1)$ common neighbours in $V(G) \setminus \bigcup_{i=1}^h V(X_i)$. For $i \in [1, k-1]$, contract one vertex of N_i

into X_i. Then contract each of $X_1,\dots,X_h$ into a single vertex. We obtain the k-fat path as a minor in G. The case of a k-leaf star is analogous. $\square$

Lemma 16. *If a connected graph G contains a $(k+2c-2)$-strong H-model, for some graph H with c connected components, then G contains a k-strong H'-model for some connected graph H' with $|E(H')| = |E(H)|$.*

Proof. We proceed by induction on $c \geqslant 1$. The case $c = 1$ is vacuous. Assume $c \geqslant 2$, and the result holds for $c-1$. Let $H_1,\dots,H_c$ be the components of H. We may assume that H has no isolated vertices. Say $X_1,\dots,X_h$ is a $(k+2c-2)$-strong H-model in G. For each edge v_iv_j in H, let N_{ij} be a set of $k+2c-2$ common neighbours of X_i and X_j. For each component H_a of H, note that $(\bigcup_{v_i\in V(H_a)} V(X_i)) \cup (\bigcup_{v_iv_j\in E(H_a)} N_{ij})$ induces a connected subgraph in G, which we denote by G_a. Since G is connected, there is a path P between G_a and G_b, for some distinct $a,b\in[1,c]$, such that no internal vertex of P is in $G_1\cup\dots\cup G_c$. Note that P might be a single vertex. For some edge $v_iv_{i'}$ in H_a and some edge $v_jv_{j'}$ in H_b, without loss of generality, P joins some vertex x in $V(X_i)\cup N_{ii'}$ and some vertex y in $V(X_j)\cup N_{jj'}$. Let H' be the graph obtained from H by identifying v_i and v_j into a new vertex v_0. Now H' has $c-1$ components and $|E(H')| = |E(H)|$. Define $X_0 := X_i \cup X_j \cup P$. If $x \notin V(X_i)$ then add the edge between x and X_i to X_0. Similarly, if $y \notin V(X_j)$ then add the edge between y and X_j to X_0. Remove x and/or y from $N_{\alpha\beta}$ for each edge $v_\alpha v_\beta$ of H'. Now $|N_{\alpha\beta}| \geqslant k+2(c-1)-2$. We obtain a $(k+2(c-1)-2)$-strong H'-model in G. By induction, G contains a k-strong H''-model for some connected graph H'' with $|E(H'')| = |E(H)|$. $\square$

Lemma 17. *If a connected graph G contains a $3k^k$-strong H-model for some graph H with at least k^k edges, then G contains a k-fat star or a k-fat path as a minor.*

Proof. We may assume that H has exactly k^k edges and has no isolated vertices. Say H has c connected components. Then $c \leqslant k^k$ and $3k^k \geqslant k^2+k+2c-2$. Hence G contains a $(k^2+k+2c-2)$-strong H-model. The result then follows from Lemmas 15 and 16. $\square$

Lemma 18. *Let G be a connected graph such that $\deg_G(v) \geqslant \ell k$ for some non-cut-vertex v and integers $k,\ell \geqslant 1$. Then G contains a k-fan as a minor, or G contains a connected subgraph X and v has ℓ neighbours not in X and all adjacent to X (thus contracting X gives a $K_{2,\ell}$ minor).*

Proof. Let r be a vertex of $G-v$. For each $w\in N_G(v)$, let P_w be a wr-path in $G-v$. If $|P_w\cap N_G(v)| \geqslant k$ then G contains a k-fan minor. Now assume that $|P_w \cap N_G(v)| \leqslant k-1$ for each $w\in N_G(v)$. Let H be the digraph with vertex set $N_G(v)$, where $N^+_H(w) := V(P_w)\cap N_G(v)$ for each vertex w. Thus H has maximum outdegree at most $k-1$. Since $|V(H)| \geqslant \ell k$, H contains a stable set S of size ℓ. Let $X := \bigcup\{P_w : w\in S\} - S$, which is connected since S is stable. Each vertex in S is adjacent to v and to X, as desired. $\square$

Lemma 19. *Let G be a graph with distinct vertices $v_1,\dots,v_k$, such that $C := G-\{v_1,\dots,v_k\}$ is connected and $\deg_C(v_i) \geqslant k^3$ for each $i\in[1,k]$. Then G contains a k-fan or k-fat star as a minor.*

Proof. The idea of the proof is to attempt to build a k-fan model by constructing a subtree X such that each v_i is adjacent to a subset S_i of k leaves of X (where the S_i are disjoint). We construct X and the S_i by adding, one at a time, paths to some neighbour w of some v_i to increase the size of S_i. We always choose a neighbour at maximal distance from r among all neighbours of all v_i for which S_i is not yet large enough: this ensures that later paths will not pass through the sets S_i that have been previously constructed.

We now formalise this idea. Let r be a vertex in C. Let $V_0, V_1, \dots, V_n$ be a BFS layering of C starting at r. Initialise $t := n$ and $X := \{r\}$ and $S_i := \emptyset$ for $i \in [1,k]$ and $S := \emptyset$. The following properties trivially hold:

(0) $S = \bigcup_{i\in[1,k]} S_i$ and $S \subseteq V_t \cup V_{t+1} \cup \dots \cup V_n$.
(1) X is a (connected) subtree of C rooted at r with (non-root) leaf set S.
(2) $S_i \cap S_j = \emptyset$ for distinct $i, j \in [1,k]$.
(3) S_i is a set of at most $k+1$ neighbours of v_i for $i \in [1,k]$ (and so $|S| \leqslant k(k+1)$).
(4) $|N_{C-V(X)}(v_i)| \geqslant k^3 - 1 - (k-1)|S| > 0$ for $i \in [1,k]$.

Now execute the following algorithm, which maintains properties (0) – (4). Think of V_t as the 'current' layer.

While $|S_i| \leqslant k$ for some $i \in [1,k]$ repeat the following: If $V_t \cap N_{C-V(X)}(v_i) = \emptyset$ for all $i \in [1,k]$ with $|S_i| \leqslant k$, then let $t := t-1$. Properties (0) – (4) are trivially maintained. Otherwise, let w be a vertex in $V_t \cap N_{C-V(X)}(v_i)$ for some $i \in [1,k]$ with $|S_i| \leqslant k$. Since $V_0, V_1, \dots, V_n$ is a BFS layering of C rooted at r and r is in X, there is a path P from w to X consisting of at most one vertex from each of $V_0, \dots, V_t$, and with no internal vertices in X. By (0) and since $w \notin S$, P avoids S. By (1), the endpoint of P in X is not a leaf of X. If P contains at least k vertices in $N_C(v_j)$ for some $j \in [1,k]$, then G contains a k-fan minor and we are done. Now assume that P contains at most $k-1$ vertices in $N_C(v_j)$ for each $j \in [1,k]$. Let $S_i := S_i \cup \{w\}$ and $S := S \cup \{w\}$ and $X := X \cup P$. Now w is a leaf of X, and property (1) is maintained. Properties (0), (2) and (3) are maintained by construction. Property (4) is maintained since $|S|$ increases by 1 and P contains at most $k-1$ vertices in $N_C(v_j)$ for each $j \in [1,k]$.

The algorithm terminates when $|S_i| = k+1$ for each $i \in [1,k]$. Delete $C - V(X)$. Contract $X - S$ (which is connected by (1)) to a single vertex z. Since S is the set of leaves of X, each vertex in S_i is adjacent to both v_i and z. Contract one edge between v_i and S_i for each $i \in [1,k]$. We obtain the k-fat star as a minor. □

Lemma 20. *Let G be a bipartite graph with bipartition A, B, such that at least p vertices in A have degree at least $k|A|$, and every vertex in B has degree at least 2. Then G contains a k-strong H-model for some graph H with at least $p/2$ edges.*

Proof. Let H be the graph with $V(H) := A$ where $vw \in E(H)$ whenever $|N_G(v) \cap N_G(w)| \geqslant k$. Since every vertex in B has degree at least 2, every vertex in A with degree at least $k|A|$ is incident with some edge in H. Thus H has at least $p/2$ edges. By construction, G contains a k-strong H-model. □

For the remainder of this section, let $d := (k+2)k^k(9k^{2k+1}+1)$. A vertex v is *high-degree* if $\deg(v) \geqslant d$, otherwise v is *low-degree*.

Lemma 21. *If a 2-connected graph G has at least $(k+2)k^k$ high-degree vertices, then G contains a k-fat path, a k-fat star, or a k-fan as a minor.*

Proof. Let A be a set of exactly $(k+2)k^k$ high-degree vertices in G. Let $C_1, \dots, C_p$ be the components of $G - A$. Say (v, C_j) is a *heavy pair* if $v \in A$ and v has at least $3k^{k+1}$ neighbours in C_j. Since $3k^{k+1} \geqslant k^3$, by Lemma 19, if some C_j is in at least k heavy pairs, then G contains a k-fan or k-fat star as a minor, and we are done. Now assume that each C_j is in fewer than k heavy pairs. Let h be the total number of heavy pairs. Then there is a set P of at least h/k heavy pairs containing at most one heavy pair for each component C_j. For each such heavy pair (v, C_j), by Lemma 18 with $\ell = 3k^k$, $G[V(C_j) \cup \{v\}]$ contains a k-fan as a minor (and we are done) or a $K_{2,3k^k}$ minor, where $G[\{v\}]$ is the subgraph corresponding to one of the vertices in the colour class of size 2 in $K_{2,3k^k}$. We obtain a $3k^k$-strong H-model for some graph H, where $|E(H)| = |P| \geqslant h/k$. If $h/k \geqslant k^k$, then we are done by Lemma 17. Now assume that $h < k^{k+1}$. In particular, the number of vertices in A that are in a heavy pair is less than k^{k+1}. Let A' be the set of vertices in A in no heavy pair; thus $|A'| \geqslant 2k^k$. Let H be the bipartite graph with bipartition A, B, where there is one vertex w_j in B for each component C_j, and $v \in A$ is adjacent to $w_j \in B$ if and only if v is adjacent to some vertex in C_j. In H, every vertex in A' has degree at least $(d - |A|)/3k^{k+1}$, which is at least $3k^k|A|$. (Note that d is defined so that this property holds.) Since G is 2-connected, each C_j is adjacent to at least two vertices in A. Thus every vertex in B has degree at least 2 in H. By Lemma 20, H contains a $3k^k$-strong model of a graph with at least $|A'|/2 \geqslant k^k$ edges. By Lemma 17 we are done. $\square$

Lemma 22. *Let $V_0, V_1, \dots$ be a BFS layering in a connected graph G. If $G[V_i \cup V_{i+1} \cup \dots \cup V_{i+c}]$ contains a path on at least k^{c+1} vertices for some $i, c \geqslant 0$, then G contains a k-fan minor.*

Proof. We proceed by induction on c. Let P be a path in $G[V_i \cup V_{i+1} \cup \dots \cup V_{i+c}]$ on k^{c+1} vertices. First suppose that P contains k vertices $v_1, \dots, v_k$ in V_i (which must happen in the base case $c = 0$). Each vertex v_i has a neighbour in V_{i-1}. Thus, contracting $G[V_0 \cup \dots \cup V_{i-1}]$ into a single vertex and contracting P between v_i and v_{i+1} to an edge (for $i \in [1, k-1]$) gives a k-fan minor. Now assume that P contains at most $k-1$ vertices in V_i and $c \geqslant 1$. Thus $P - V_i$ has at least $k^{c+1} - (k-1)$ vertices and at most k components. Thus some component of $P - V_i$ has at least $\lceil (k^{c+1} - k + 1)/k \rceil = k^c$ vertices and is contained in $G[V_{i+1} \cup V_{i+2} \cup \dots \cup V_{i+c}]$. By induction, G contains a k-fan minor. $\square$

Say a vertex v in a coloured graph is *properly* coloured if no neighbour of v gets the same colour as v.

Lemma 23. *Let G be a 2-connected graph with no k-fan, k-fat star or k-fat path as a minor. Let h be the number of high-degree vertices in G. Let r be a vertex in G. Then G is 2-colourable with clustering at most $d^{k^{3(k+2)k^k}}$. Moreover, if $h = 0$ then we can additionally demand that r is properly coloured.*

Proof. Let $V_0, V_1, \dots$ be the BFS layering of G starting at r.

First suppose that $h = 0$. Colour each vertex $v \in V_i$ by $i \bmod 2$. Then r is properly coloured. Every monochromatic component is contained in some V_i. Suppose that some component X of $G[V_i]$ has at least d^k vertices. Thus $i \geqslant 1$. Since G and thus X has maximum degree at most d, X contains a path of k vertices. Contracting $G[V_0 \cup \dots \cup V_{i-1}]$ into a single vertex gives a k-fan minor. This contradiction shows that the 2-colouring has clustering at most d^k.

Now assume that $h \geqslant 1$. By Lemma 21, $h \leqslant (k+2)k^k$. Colour all the high-degree vertices black. Let I be the set of integers $i \geqslant 0$ such that V_i contains a high-degree vertex. Colour all the low-degree vertices in $\bigcup\{V_i : i \in I\}$ white.

Let $V_i, V_{i+1}, \dots, V_{i+c}$ be a maximal sequence of layers with no high-degree vertices, where $c \geqslant 0$. Thus V_{i-1} is empty or contains a high-degree vertex. Similarly, V_{i+c+1} is empty or contains a high-degree vertex. If c is even, then colour $V_i \cup V_{i+2} \cup \dots \cup V_{i+c}$ white and colour $V_{i+1} \cup V_{i+3} \cup \dots \cup V_{i+c-1}$ black. If c is odd, then colour $V_i \cup V_{i+2} \cup \dots \cup V_{i+c-1}$ and V_{i+c} white, and colour $V_{i+1} \cup V_{i+3} \cup \dots \cup V_{i+c-2}$ black. Note that if $c \geqslant 2$ then at least one of $V_{i+1}, \dots, V_{i+c-1}$ is black.

We now show that each black component X has bounded size. If X contains some high-degree vertex, then every vertex in X is high-degree and $|X| \leqslant h \leqslant (k+2)k^k$. Now assume that X contains no high-degree vertices. Say X intersects V_j. Since each black layer is preceded by and followed by a white layer, X is contained in V_j. Every vertex in X has degree at most d in G. Thus if X has at least d^k vertices, then X contains a path of length k, and contracting $V_0 \cup \dots \cup V_{j-1}$ to a single vertex gives a k-fan. Hence X has at most d^k vertices.

Finally, let X be a white component. Then X is contained within at most $3h \leqslant 3(k+2)k^k$ consecutive layers (since in the notation above, if all of $V_i, V_{i+1}, \dots, V_{i+c}$ are white, then $c \leqslant 1$). Suppose that $|X| \geqslant d^{k^{3(k+2)k^k}}$. Since X has maximum degree at most d, X contains a path of length $k^{3(k+2)k^k}$. Thus, Lemma 22 with $c + 1 = 3(k+2)k^k$ implies that G contains a k-fan minor. Hence $|X| \leqslant d^{k^{3(k+2)k^k}}$. □

We now complete the proof of Theorem 14.

Lemma 24. *Let G be a graph with no k-fan, no k-fat path, and no k-fat star as a minor. Then G is 2-colourable with clustering $kd^{k^{3(k+2)k^k}}$.*

Proof. We may assume that G is connected. Let r be a vertex of G. If B is a block of G containing r, then consider B to be rooted at r. If B is a block of G not containing r, then consider B to be rooted at the unique vertex in B that separates B from r. Say (B, v) is a *high-degree pair* if B is a block of G and v has high-degree in B. Note that one vertex might be in several high-degree pairs.

Suppose that some vertex v is in at least k high-degree pairs with blocks $B_1, \dots, B_k$. By Lemma 18, each B_i contains a $K_{2,k+1}$ minor rooted at v. Contracting one edge incident with

v in each $K_{2,k+1}$ gives a k-fat star as a minor. Now assume that each vertex is in fewer than k high-degree pairs.

Colour each block B in non-decreasing order of the distance in G from r to the root of B. Let B be a block of G rooted at v (possibly equal to r). Then v is already coloured in the parent block of B. Let h_B be the number of high-degree pairs involving B. By Lemma 23, B is 2-colourable with clustering at most $d^{k^{3(k+2)k^k}}$, such that if $h_B = 0$ then v is properly coloured. Permute the colours in B so that the colour assigned to v matches the colour assigned to v by the parent block. Then the monochromatic component containing v is contained within the parent block of B along with those blocks rooted at v that form a high-degree pair with v. As shown above, there are at most k such blocks. Thus each monochromatic component has at most $kd^{k^{3(k+2)k^k}}$ vertices. □

4 Excluding a Fat Star

This section considers colourings of graphs excluding a fat star. We need the following more general lemma.

Lemma 25. *For every planar graph H,*

$$\chi_\star(\mathcal{M}_H) \leqslant 2\,\chi_\Delta(\mathcal{M}_H).$$

Proof. The grid minor theorem of Robertson and Seymour [19] says that every graph in $\mathcal{M}_H$ has tree-width at most some function $w(H)$. (Chekuri and Chuzhoy [2] recently showed that w can be taken to be polynomial in $|V(H)|$.) Alon, Ding, Oporowski, and Vertigan [1] observed that every graph with tree-width w and maximum degree Δ is 2-colourable with clustering $24w\Delta$. Let $k := \chi_\Delta(\mathcal{M}_H)$. That is, every H-minor-free graph G is k-colourable with monochromatic components of maximum degree at most some function $d(H)$. Apply the above result of Alon et al. [1] to each monochromatic component. Thus G is $2k$-colourable with clustering $24\,w(H)\,d(H)$. Hence $\chi_\star(\mathcal{M}_H) \leqslant 2k$. □

A variant of Lemma 25 holds for arbitrary graphs H with "2" replaced by "3". The proof uses a result of Liu and Oum [14] in place of the result of Alon et al. [1]; see [5, 24].

Theorem 26. *For $k \geqslant 3$, the clustered chromatic number of the class of graphs with no k-fat star minor equals 4.*

Proof. As illustrated in Figure 2, the k-fat star is planar. Ossona de Mendez et al. [17] proved that graphs with no k-fat star minor are 2-colourable with defect $O(k^{13})$. Thus, Lemma 25 implies that the clustered chromatic number of the class of graphs with no k-fat star is at most 4. To obtain a bound on the clustering, note that a result of Leaf and Seymour [13] implies that every graph with no k-fat star minor has tree-width $O(k^2)$. It follows from the proof of Lemma 25 that every graph with no k-fat star minor is 4-colourable with clustering $O(k^{15})$. Since the 3-fat star is $C\langle 3,3\rangle$, Lemma 13 implies that for $k \geqslant 3$, the clustered chromatic number of the class of graphs with no k-fat star minor is at least 4. □

Every graph H with $\overline{\mathrm{td}}(H) \leqslant 3$ is a subgraph of the k-fat star for some $k \leqslant |V(H)|$. Thus Theorem 26 implies Conjecture 4 in the case of connected tree-depth 3.

Corollary 27. *For every graph H with $\overline{\mathrm{td}}(H) \leqslant 3$,*

$$\chi_\star(\mathcal{M}_H) \leqslant 4.$$

We can push this result further.

Theorem 28. *For every graph H with $\mathrm{td}(H) \leqslant 3$,*

$$\chi_\star(\mathcal{M}_H) \leqslant 5.$$

Proof. Say H has p components. Each component of H is a subgraph of the k-fat star for some $k \leqslant |V(H)|$. Let H' consist of p pairwise disjoint copies of the k-fat star. Let G be an H-minor-free graph. Thus G is also H'-minor-free. By the Grid Minor Theorem of Robertson and Seymour [19] and since H' is planar, G has treewidth at most $w = w(H')$. By Theorem 5, there is a set X of at most $(p-1)(w-1)$ vertices in G, such that $G - X$ contains no k-fat star as a minor. By Theorem 26, $G - X$ is 4-colourable with clustering at most some function of H. Assign vertices in X a fifth colour. Thus G is 5-colourable with clustering at most some function of H. $\square$

5 A Conjecture about Clustered Colouring

We now formulate a conjecture about the clustered chromatic number of an arbitrary minor-closed class of graphs. Consider the following recursively defined class of graphs. Let $\mathcal{X}_{1,c} := \{P_{c+1}, K_{1,c}\}$. For $k \geqslant 2$, let $\mathcal{X}_{k,c}$ be the set of graphs obtained by the following three operations. For the first two operations, consider an arbitrary graph $G \in \mathcal{X}_{k-1,c}$.

- Let G' be the graph obtained from c disjoint copies of G by adding one dominant vertex. Then G' is in $\mathcal{X}_{k,c}$.
- Let G^+ be the graph obtained from G as follows: for each k-clique D in G, add a stable set of $k(c-1)+1$ vertices complete to D. Then G^+ is in $\mathcal{X}_{k,c}$.
- If $k \geqslant 3$ and $G \in \mathcal{X}_{k-2,c}$, then let G^{++} be the graph obtained from G as follows: for each $(k-1)$-clique D in G, add a path of $(c^2-1)(k-1)+(c+1)$ vertices complete to D. Then G^{++} is in $\mathcal{X}_{k,c}$.

A vertex-coloured graph is *rainbow* if every vertex receives a distinct colour.

Lemma 29. *For every $c \geqslant 1$ and $k \geqslant 2$, for every graph $G \in \mathcal{X}_{k,c}$, every colouring of G with clustering c contains a rainbow K_{k+1}. In particular, no graph in $\mathcal{X}_{k,c}$ is k-colourable with clustering c.*

Proof. We proceed by induction on $k \geqslant 1$. In the case $k = 1$, every colouring of P_{c+1} or $K_{1,c}$ with clustering c contains an edge whose endpoints receive distinct colours, and we are done. Now assume the claim for $k-1$ and for $k-2$ (if $k \geqslant 3$).

Let $G \in \mathscr{X}_{k-1,c}$. Consider a colouring of G' with clustering c. Say the dominant vertex v is blue. At most $c-1$ copies of G contain a blue vertex. Thus, some copy of G has no blue vertex. By induction, this copy of G contains a rainbow K_k. With v we obtain a rainbow K_{k+1}.

Now consider a colouring of G^+ with clustering c. By induction, the copy of G in G^+ contains a clique $w_1, \dots, w_k$ receiving distinct colours. Let S be the set of $k(c-1)+1$ vertices adjacent to $w_1, \dots, w_k$ in G^+. At most $c-1$ vertices in S receive the same colour as w_i. Thus some vertex in S receives a colour distinct from the colours assigned to $w_1, \dots, w_k$. Hence G^+ contains a rainbow K_{k+1}.

Now suppose $k \geqslant 3$ and $G \in \mathscr{X}_{k-2,c}$. Consider a colouring of G^{++} with clustering c. By induction, the copy of G in G^{++} contains a clique $w_1, \dots, w_{k-1}$ receiving distinct colours. Let P be the path of $(c^2-1)(k-1) + (c+1)$ vertices in G^{++} complete to $w_1, \dots, w_{k-1}$. Let X_i be the set of vertices in P assigned the same colour as w_i, and let $X := \bigcup_i X_i$. Thus $|X_i| \leqslant c-1$ and $|X| \leqslant (c-1)(k-1)$. Hence $P - X$ has at most $(c-1)(k-1)+1$ components, and $|V(P-X)| \geqslant (c^2-1)(k-1) + (c+1) - (c-1)(k-1) = c\big((c-1)(k-1)+1\big)+1$. Some component of $P-X$ has at least $c+1$ vertices, and therefore contains a bichromatic edge xy. Then $\{w_1, \dots, w_{k-1}\} \cup \{x, y\}$ induces a rainbow K_{k+1} in G^{++}. □

We conjecture that a minor-closed class that excludes every graph in $\mathscr{X}_{k,c}$ for some c is k-colourable with bounded clustering. More precisely:

Conjecture 30. *For every minor-closed class $\mathscr{M}$ of graphs,*

$$\chi_\star(\mathscr{M}) = \min\{k : \exists c \ \mathscr{M} \cap \mathscr{X}_{k,c} = \emptyset\}.$$

Several comments about Conjecture 30 are in order:

- To prove the lower bound in Conjecture 30, let k be the minimum integer such that $\mathscr{M} \cap \mathscr{X}_{k,c} = \emptyset$ for some integer c. Thus for every integer c some graph $G \in \mathscr{X}_{k-1,c}$ is in $\mathscr{M}$. By Lemma 29, G has no $(k-1)$-colouring with clustering c. Thus $\chi_\star(\mathscr{M}) \geqslant k$.
- Note that the $k=1$ case of Conjecture 30 is trivial: a graph is 1-colourable with bounded clustering if and only if each component has bounded size, which holds if and only if every path has bounded length and every vertex has bounded degree.
- We note that Theorem 14 implies Conjecture 30 with $k=2$. If $G = P_{c+1}$ then G' contains the $(c+1)$-fan and G^+ contains the $(c+1)$-fat path. If $G = K_{1,c}$ then G' and G^+ are both the c-fat-star. Thus, if a minor-closed class $\mathscr{M}$ excludes every graph in $\mathscr{X}_{2,c}$ for some c, then $\mathscr{M}$ excludes the $(c+1)$-fan, the $(c+1)$-fat-path, and the c-fat-star. Then $\chi_\star(\mathscr{M}) \leqslant 2$ by Theorem 14.

- We now relate Conjectures 4 and 30. Fix a graph H. Conjecture 30 says that the clustered chromatic number of $\mathcal{M}_H$ equals the minimum integer k such that for some integer c, every graph in $\mathcal{X}_{k,c}$ contains H as a minor. Let $k := \overline{\operatorname{td}}(H) \geqslant 2$. First observe that every graph in $\mathcal{X}_{2k-2,c}$ contains a $C\langle k,c\rangle$ minor. Thus, for a suitable value of c, every graph in $\mathcal{X}_{2k,c}$ contains H as a minor. Hence, Conjecture 30 implies Conjecture 4.
- Consider the case of excluding the complete bipartite graph $K_{s,t}$ as a minor for $s \leqslant t$? Van den Heuvel and Wood [24] proved the lower bound, $\chi_\star(\mathcal{M}_{K_{s,t}}) \geqslant s+1$ for $t \geqslant \max\{s,3\}$. Their construction is a special case of the construction above. Conjecture 30 says that $\chi_\star(\mathcal{M}_{K_{s,t}}) = s+1$.

6 An Alternative View

We conclude the paper with alternative versions of Conjectures 2 and 30 that shift the focus to characterizing minimal minor-closed classes of given defective and clustered chromatic number.

We start with the definitions. Let H and G be two vertex-disjoint graphs, and let $S \subseteq V(G)$. Let G' be obtained from $G \cup H$ by joining every vertex of S to every vertex of H by an edge. Then we say that G' is obtained from G by *taking a join with H along S*. Let $\mathcal{H}$ be a class of graphs. We say that a graph G' is an *$\mathcal{H}$-decoration* of a graph G, if G' is obtained from G by repeatedly taking joins with graphs in $\mathcal{H}$ along cliques of G. For a class of graphs $\mathcal{G}$, let $\mathcal{G} \wedge \mathcal{H}$ denote the class of all minors of $\mathcal{H}$-decorations of graphs in $\mathcal{G}$. One can routinely verify that the $\wedge$ operation is associative. The examples below show that it is not always commutative.

First, we introduce notation for some minor-closed classes that will serve as the basis for our constructions. Let $\mathcal{I}$ denote the class of graphs on at most one vertex, let $\mathcal{O}$ denote the class of edgeless graphs, and let $\mathcal{P}$ denote the class of linear forests (that is, subgraphs of paths). Let $\mathcal{T}_d$ denote the class of all graphs of tree-depth at most d. Then $\mathcal{T}_1$ is a class of all edgeless graphs. It follows from the alternative definition of tree-depth given in [15, Section 6.1] that $\mathcal{T}_{d+1} = \mathcal{O} \wedge \mathcal{T}_d$.

The operations used in Conjecture 30 can be described as follows.

- Adding a vertex adjacent to several copies of graphs in the class $\mathcal{G}$ (and taking all possible minors) produces the class $\mathcal{I} \wedge \mathcal{G}$.
- Adding stable sets complete to cliques in graphs in $\mathcal{G}$ produces the class $\mathcal{G} \wedge \mathcal{I}$.
- Adding paths complete to cliques in graphs in $\mathcal{G}$ produces the class $\mathcal{G} \wedge \mathcal{P}$.

Note that by definition $\mathcal{G} \wedge \mathcal{H}$ is a minor-closed class for any pair of minor-closed classes $\mathcal{G}$ and $\mathcal{H}$.

We next present an analogue of Lemma 29 using the notions introduced above. A class of graphs $\mathcal{G}$ is *k-cluster rainbow* (respectively, *k-defect rainbow*) if for every c there exists $G \in \mathcal{G}$ such that every colouring of G with clustering (respectively, defect) at most c contains a rainbow clique of size k. For example, $\mathscr{I}$ is 1-cluster rainbow and 1-defect rainbow, $\mathscr{P}$ is 2-cluster rainbow, but not 2-defect rainbow. Note that if a class of graphs $\mathscr{G}$ is k-cluster rainbow, then clearly $\chi_\star(\mathscr{G}) \geqslant k$. Similarly, if $\mathscr{G}$ is k-defect rainbow, then $\chi_\Delta(\mathscr{G}) \geqslant k$.

The proof of the following lemma parallels the proof of Lemma 29; we present it for completeness.

Lemma 31. *Let $\mathscr{G}, \mathscr{H}$ be graph classes, such that $\mathscr{G}$ is k-cluster rainbow and $\mathscr{H}$ is ℓ-cluster rainbow. Then $\mathscr{G} \wedge \mathscr{H}$ is $(k+\ell)$-cluster rainbow.*

Proof. Fix c, and let $G \in \mathscr{G}$ and $H \in \mathscr{H}$ be such that every colouring of G with clustering at most c contains a rainbow clique of size k, and every colouring of H with clustering at most c contains a rainbow clique of size ℓ. Let J be obtained from G by taking a join of G with H, $(c-1)k+1$ times along every clique S of G. Then $J \in \mathscr{G} \wedge \mathscr{H}$ by definition. It remains to show that every colouring $\phi : V(J) \to C$ of J for some set of colours C with clustering at most c contains a rainbow clique of size $k+\ell$. By the choice of J there exists a clique S in G of size k, which is rainbow in ϕ. Let $H_1, H_2, \dots, H_r$ be copies of H glued along S to G. By the choice of H, for every i there exists a clique S_i of size ℓ in H_i that is rainbow in ϕ. Suppose for a contradiction that $S \cup S_i$ is not rainbow for any i. Then there exists $s \in S$ with a neighbour of the same colour in S_i for at least c choices of i. Thus s belongs to a monochromatic component of size at least $c+1$ in ϕ, a contradiction. $\square$

Note that an analogue of Lemma 31 also holds for defective colourings. The proof is identical.

Let $\mathscr{G}$ be a graph class obtained by taking a wedge-product of v copies of $\mathscr{I}$ and p copies of $\mathscr{P}$ in some order such that $v + 2p = k+1$. Then we say that $\mathscr{G}$ is *k-cluster critical.* By Lemma 31 the clustered chromatic number of a k-cluster critical class is at least $k+1$. (In fact, it is not difficult to see that equality holds.) For example, the class $\mathscr{I} \wedge \mathscr{P}$ of minors of fans, the class $\mathscr{I} \wedge \mathscr{I} \wedge \mathscr{I}$ of minors of fat stars, and the class $\mathscr{P} \wedge \mathscr{I}$ of minors of fat paths are all possible 2-cluster critical classes. Thus Theorem 14 is equivalent to the statement that $\chi_\star(\mathscr{G}) \leqslant 2$ if and only if $\mathscr{G}$ contains no 2-cluster critical class.

The discussion above implies that for all k and c every graph in $\mathscr{X}_{k,c}$ is a member of some k-cluster critical class. Conversely, for all n, k there exists c such that for every graph $G \in \mathscr{X}_{k,c}$ there exists a k-cluster critical class $\mathscr{G}$ such that $\mathscr{X}_{k,c}$ contains as minors all graphs in $\mathscr{G}$ on at most n vertices. Thus Conjecture 30 can be reformulated as follows.

Conjecture 32. *Let $\mathscr{M}$ be a minor-closed class of graphs and $k \geqslant 0$ an integer. Then $\chi_\star(\mathscr{G}) \geqslant k+1$ if and only if $\mathscr{G} \nsubseteq \mathscr{M}$ for some k-cluster critical class $\mathscr{G}$.*

Similarly, note that the k-term $\wedge$-product $\wedge^k \mathscr{I} = \mathscr{I} \wedge \mathscr{I} \wedge \ldots \wedge \mathscr{I}$ is the class of minors of connected graphs of tree-depth k and therefore the following conjecture is equivalent to Conjecture 2.

Conjecture 33. *Let $\mathscr{M}$ be a minor-closed class of graphs and $k \geqslant 0$ an integer. Then $\chi_\Delta(\mathscr{G}) \geqslant k+1$ if and only if $\wedge^{k+1}\mathscr{I} \nsubseteq \mathscr{M}$.*

Acknowledgement. This research was initiated at the *2017 Barbados Graph Theory Workshop* held at the Bellairs Research Institute. Thanks to the workshop participants for creating a stimulating working environment.

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