A Proof of a Conjecture of Bondy concerning paths in weighted digraphs

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Abstract. Our aim in this note is to prove a conjecture of Bondy, extending a classical theorem of Dirac to edge-weighted digraphs: if every vertex has out-weight at least 1 then the digraph contains a path of weight at least 1. We also give several related conjectures and results concerning heavy cycles in edge-weighted digraphs.

§1. Introduction

A basic theorem of Dirac [11] states that every graph G with $\delta(G) = d$ contains a path of length d and, for $d \geq 2$, a cycle of length at least $d + 1$; furthermore, if G is 2-connected, then G contains a cycle of length at least $\min\{2d, |G|\}$. Bounds in terms of just the size were given by Erdős and Gallai [12], who proved that every graph G of order n contains a path of length at least $2e(G)/n$ and, provided $e(G) \geq n$, a cycle of length $2e(G)/(n - 1)$.

As conjectured by Bondy and Fan [8], both results of Erdős and Gallai can be generalized to edge-weighted graphs. Frieze, McDiarmid and Reed [14] proved that every weighted graph contains a heavy path.

Theorem A. *Let G be a weighted graph of order n. Then G contains a path of weight at least $2w(G)/n$.*

Bondy and Fan [9] proved the following theorem about heavy cycles.

Theorem B. *Let G be a weighted 2-edge-connected graph of order n. Then G contains a cycle of weight at least $2w(G)/(n - 1)$.*

Dirac's theorem is easily generalized to digraphs: if every vertex in a digraph G has outdegree at least d, then G contains a path of length at least d and a cycle of length at least $d + 1$.

In 1992, Bondy [6] made two other conjectures, extending Dirac's theorem to edge-weighted *digraphs*. First he conjectured that if every vertex in an edge-weighted digraph has outweight at least 1 then the digraph contains a (directed) path of weight at least 1. Secondly, if every vertex in an edge-weighted digraph has outweight at least 1 then the digraph contains a (directed) cycle of weight at least 1. The second conjecture was disproved by T. Spencer of Nebraska.

Our main aim in this note is to find extensions of the theorems of Dirac and of Erdős and Gallai to edge-weighted digraphs. In particular, we prove the first (paths) conjecture of Bondy and give a lower bound on the weight of the heaviest cycle in an edge-weighted digraph in which every vertex has outweight at least

1. We also consider other conditions that could guarantee a heavy cycle in an edge-weighted digraph and make a number of conjectures.

We use standard notation (see, e.g., [3]). We shall consider only loopless digraphs; in other words, if $\langle x, y\rangle$ is a (directed) edge (or *arc*) of G then $x \neq y$. We shall write xy for $\langle x, y\rangle$. An *oriented graph* G is a digraph with no cycles of length two: thus if $xy \in E(G)$ then $yx \notin E(G)$. An *edge-weighting* of a graph or digraph G is a function $w : E(G) \to \mathbb{R}$. We shall only consider edge-weightings with non-negative weights. For $x \in V(G)$, the *inweight* of x is

$$w_{\text{in}}(x) = \sum_{y\in\Gamma^-(x)} w(yx)$$

and the *outweight* of x is

$$w_{\text{out}}(x) = \sum_{y\in\Gamma^+(x)} w(xy).$$

We shall assume that graphs and digraphs have at least one vertex.

§2. Results and problems

We begin by looking for a version of Theorem A for edge-weighted digraphs. If we consider the complete bipartite graph $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ with all edges oriented in the same direction (so that the longest path has length 1) and give weight 1 to every edge, then no path has weight more than $w(G)/\lfloor \frac{n^2}{4}\rfloor$. R.C. O'Brien [18] proved that every digraph G of order n has an edge-partition into $\lfloor \frac{n^2}{4}\rfloor$ paths; therefore some path must weigh at least

$$w(G)/\lfloor \frac{n^2}{4}\rfloor. \tag{1}$$

It can also be shown that, for oriented graphs, there is a path of length at most three with weight at least $w(G)/\lfloor \frac{n^2}{4}\rfloor$.

Now clearly a digraph G need not contain cycles. However, even if we demand that G be strongly connected, we cannot do much better than (1). Indeed, let G be the complete tripartite graph with vertex sets V_1, V_2, V_3 such that $|V_1| = \lceil \frac{n-1}{2}\rceil$, $|V_2| = \lfloor \frac{n-1}{2}\rfloor$ and $|V_3| = 1$, and the edges are oriented from V_1 to V_2, from V_2 to

V_3 and from V_3 to V_1. If we give weight 1 to all edges from V_1 to V_2 and weight 0 to the other edges, then the heaviest path has weight $2w(G)/\lceil\frac{(n-1)^2}{4}\rceil$ and the heaviest cycle has weight $w(G)/\lceil\frac{(n-1)^2}{4}\rceil$. More generally, if we take $|V_1| = \lceil\frac{n-k}{2}\rceil$, $|V_2| = \lfloor\frac{n-k}{2}\rfloor$ and $|V_3| = k$, we get a strongly k-connected graph with heaviest path weighing $(k+1)w(G)/\lceil\frac{(n-k)^2}{4}\rceil$ and heaviest cycle weighing $kw(G)/\lceil\frac{(n-k)^2}{4}\rceil$.

In order to guarantee the existence of heavier paths and cycles, we have to impose some conditions on our graphs or our edge-weightings. The natural condition, corresponding to Dirac's minimal degree condition, is that every vertex have large outweight. Our main result, proving a conjecture of Bondy [6], asserts that this condition is indeed sufficient to guarantee a heavy path.

Theorem 1. *Let G be a digraph with edge-weighting w such that every vertex v in G satisfies $w_{\text{out}}(v) \geq 1$. Then G contains a path P such that*

$$w(P) \geq 1. \tag{2}$$

Proof. The key idea of our proof is that, in order to make induction easy, we prove a stronger assertion. Indeed, let G be a digraph with edge-weighting w and let $v_0 \in V(G)$. We prove by induction on $n = |G|$ that if every $v \in V(G) \setminus \{v_0\}$ satisfies $w_{\text{out}}(v) \geq 1$ then G contains a path P such that $w(P) \geq 1$. If $n = 2$, then the result is clear. Suppose $n > 2$ and $v_0 \in V(G)$.

If v_0 has no inedges, then consider the graph $G^* = G \setminus \{v_0\}$ with the same edge-weighting. Every vertex has outweight at least 1, so picking any $u \in V(G^*)$, the conditions of the inductive hypothesis are satisfied, so we can find a path P in G^* with $w(P) \geq 1$, which can also be considered as a path in G.

Otherwise, $d^-(v_0) > 0$. Let uv_0 be an edge with $w(uv_0)$ maximal, ie uv_0 is the heaviest inedge. Let G^* be the digraph $G \setminus \{u, v_0\}$ with an extra vertex x and, for $v \in V(G) \setminus \{u, v_0\}$, an edge from v to x iff $vu \in E(G)$ or $vv_0 \in E(G)$. Thus $\Gamma^+_{G^*}(x) = \varnothing$ and $\Gamma^-_{G^*}(x) = (\Gamma^-_G(u) \cup \Gamma^-_G(v_0)) \setminus \{u, v_0\}$. Let w^* be the weighting obtained by setting $w^* = w$ on $G^* \setminus x$ and, for $vx \in E(G^*)$,

$$w^*(vx) = \begin{cases} w(vu) + w(uv_0) & vu \in E(G) \\ w(vv_0) & \text{otherwise} \end{cases} \tag{3}$$

It is easily checked that G^*, w^* and x satisfy the conditions of the inductive hypothesis. Indeed, for $v \in V(G^*) \setminus \{x\}$, we have $w^*_{\text{out}}(v) = w_{\text{out}}(v) - w(vu) - w(vv_0) + w^*(vx)$. If $vu \in E(G)$, then

$$\begin{aligned} w^*_{\text{out}}(v) &= w_{\text{out}}(v) - w(vu) - w(vv_0) + w(vu) + w(uv_0) \\ &= w_{\text{out}}(v) - w(vv_0) + w(uv_0) \\ &\geq w_{\text{out}}(v) \end{aligned}$$

by maximality of $w(uv_0)$. If $vu \notin E(G)$, then clearly $w^*_{\text{out}}(v) = w_{\text{out}}(v)$. Thus $w^*_{\text{out}}(v) \geq 1$ for all $v \neq x$. Now G^* has fewer vertices than G. Therefore, by our inductive hypothesis, there is a path P^* contained in G^* such that $w^*(P^*) \geq 1$. Now if $x \notin V(P^*)$ then P^* can also be thought of as a path P in G, where $w(P) = w^*(P^*)$, so we have the required path. Otherwise, P^* must end in x, since $d^+_{G^*}(x) = 0$. Suppose the last edge in P^* is vx. We use P^* to define a path P contained in G as follows. P^* is the same as P except for the last vertex. If $vu \in E(G)$ then replace vx with vuv_0; otherwise replace vx with vv_0. In either case, it follows immediately from (3) that $w(P) \geq w^*(P^*) \geq 1$, so we have found the required path. □

This result is best possible: consider, for instance, the complete digraph, where all edges have equal weight. For strongly connected digraphs, however, we can say slightly more about our heavy paths.

Corollary 2. *Let G be a strongly connected digraph with edge-weighting w such that every vertex v in G satisfies $w_{\text{out}}(v) \geq 1$. Then, for every vertex v in G, there is a path P such that $w(P) \geq 1$ and P ends in v.*

Proof. As in the proof of Theorem 1, we prove a stronger assertion. Let G be a digraph with edge-weighting w and let $v \in V(G)$. We prove that if, for every vertex $v' \neq v$, $w_{\text{out}}(v') \geq 1$ and there is a path from v' to v, then there is a path P ending in v such that $w(P) \geq 1$. It is easily checked that this condition is stable under the contraction used in the proof of Theorem 1; the result follows by a similar induction. □

What can we say about cycles? Surprisingly, this question seems to be rather more difficult. Indeed, as remarked above in the introduction, Bondy's conjecture, that if every vertex in an edge-weighted digraph G has outweight at least 1 then G contains a cycle of weight at 1;2cleast 1, is false. We give upper and lower bounds on the minimum possible weight of a heaviest cycle under these conditions. After obtaining the (upper bound) construction below, we discovered from Bondy that the construction had previously been obtained by T. Spencer [7].

For an upper bound, consider the digraph G defined as follows. Let $k,\ l \geq 2$ be fixed integers. Let V be the set of strings of at most l digits from $[k]$, so $V = \{\varnothing, 1, 2, \ldots, k, 11, 12, \ldots\}$. For each string $x_1 \cdots x_p$ with $p < l$, we add edges to $x_1 \cdots x_p i$, for $i = 1, \ldots, k$, each with weight $1/k$ (this includes edges from $\varnothing$ to $1, \ldots, k$). From each string $x_1 \cdots x_l$ we add edges to its initial segments $\varnothing, x_1, x_1x_2, \ldots, x_1 \cdots x_{l-1}$, each with weight $1/l$. Thus every vertex has outweight 1. It is easily seen that a heaviest cycle is given by $\varnothing, x_1, x_1x_2, x_1 \cdots x_l, \varnothing$ for any $x_1 \cdots x_l$, and has weight $(l/k) + (1/l)$. Furthermore, $|V| = 1 + k + k^2 + \cdots + k^l < k^{l+1}$. If we set $k = l^2$ and $|V| = n$, we get that the maximal weight of a cycle is at most

$$\frac{c \log \log n}{\log n} w(G). \tag{4}$$

Let us note that we can make this example bipartite by taking edges from each $x_1 \cdots x_l$ only to $x_1 \cdots x_{l-2}$, $x_1 \cdots x_{l-4}$ and so on. We can also demand that G have girth at least g, for any G, by taking edges from $x_1 \cdots x_l$ to $\varnothing, \ldots, x_1 \cdots x_{l-g+1}$.

For a lower bound, it is easy to see that if every vertex $v \in V(G)$ satisfies $w_{\text{out}} \geq 1$ then we can find a cycle of weight at least $n^{-1/2}/2$. Indeed, we may assume that G is strongly connected, or else replace G with a strongly connected component of G with no outedges. This still satisfies the condition that every vertex has outweight at least 1, and has fewer vertices than G. Now, if any edge xy weighs more than $n^{-1/2}/2$ then we can extend it to a cycle with weight at least $n^{-1/2}/2$. Otherwise, consider the subgraph G^* of G, where we take only those edges that weigh at least $1/2n$. Every vertex has outweight at least $1/2$, and so outdegree at least $\sqrt{n}$, since each edge weighs at most $n^{-1/2}/2$. It follows immediately that there must be a cycle of length at least $\sqrt{n}$, which must weigh at least $\sqrt{n} \cdot (1/2n) = n^{-1/2}/2$.

With a little more work we can do slightly better.

Theorem 3. *Let G be a digraph with edge-weighting w, such that every vertex v in $V(G)$ satisfies $w_{\mathrm{out}}(v) \geq 1$. Then G contains a cycle C with $w(C) \geq (24n)^{-1/3}$.*

Proof. Let $c = 24^{-1/3}$. We prove the assertion of the theorem by induction on $n = |G|$. As noted above, we may assume that G is strongly connected (by considering a strongly connected component with no outedges). If there is an edge weighing more than $cn^{-1/3}$ then we can extend it to a cycle and we are done. Suppose then that no edge weighs more than $cn^{-1/3}$, and that G contains no cycle of weight $cn^{-1/3}$.

Suppose first that some $v \in V(G)$ satisfies $d^+(v) \geq 6cn^{2/3}$. Starting with the triple $(G_0, w_0, v_0) = (G, w, v)$, consisting of our graph G, edge-weighting w and special vertex v, we shall perform a sequence of contractions to obtain triples $(G_1, w_1, v_1), (G_2, w_2, v_2), \ldots$, where each G_i is a strongly connected digraph with edge weighting w_i such that every vertex except v_i has outweight at least 1.

Given (G_i, w_i, v_i), if there is an edge weighing at least $cn^{-1/3}$ then we can extend it to a cycle of weight at least $cn^{-1/3}$. As we shall note below, this corresponds to a cycle in G with weight at least $cn^{-1/3}$, which is a contradiction. Thus we may assume that no edge of G_i has weight more than $cn^{-1/3}$. Let vv_i be the heaviest edge into v_i ($d^-_{G_i}(v_i) > 0$ since G_i is strongly connected). We define G^* by contracting the edge vv_i: G^* is obtained from G by deleting v and v_i and adding a vertex v_{i+1} with edges from v_{i+1} to $y \in V(G^*)$ iff $v_i y \in E(G_i)$ and from y to v_{i+1} iff $yv \in E(G_i)$ or $yv_i \in E(G_i)$. We define the weighting w^* by $w^* = w_i$ on $G^* \setminus \{v_{i+1}\}$, and $w^*(v_{i+1}y) = w_i(v_i y)$ for $v_{i+1}y \in E(G^*)$ and, for $yv_{i+1} \in E(G^*)$,

$$w^*(yv_{i+1}) = \begin{cases} w_i(yv) + w_i(vv_i) & \text{if } vu \in E(G_i) \\ w_i(vv_i) & \text{if } vu \notin E(G_i) \end{cases} \tag{5}$$

Clearly, no edge in G^* weighs more than $2cn^{-1/3}$, since no edge in G_i weighs more than $cn^{-1/3}$. Furthermore, a cycle in G^* corresponds to a cycle of equal or greater weight in G_i, where we replace an edge yv_{i+1} by yvv_i or yv_i as appropriate. Since all our operations will be contractions of this form and taking subgraphs, any cycle in G_i corresponds to a cycle of equal or greater weight in G. Now let

H be a strongly connected component of $G_i \setminus \{v_{i+1}\}$ (this is well-defined, since $w^*(xy) \leq 2cn^{-1/3} < 1$ for $xy \in E(G^*)$). We define G_{i+1} to be the subgraph of G^* induced by $H \cup \{v_{i+1}\}$ and w_{i+1} to be w^* restricted to this graph. For $y \neq v_{i+1}$, the outweight of y in G_{i+1} is equal to the outweight of y in G^*; it follows from (5) that this is at least as large as the outweight of y in G_i, which is at least 1.

We claim that G_{i+1} is also strongly connected. Indeed, it is enough to show that $d^-(v_{i+1}) > 0$ and $d^+(v_{i+1}) > 0$. If $d^-(v_{i+1}) = 0$, then consider the digraph $G' = G_{i+1} \setminus \{v_{i+1}\}$. Every vertex has outweight at least 1, so G' contains a cycle of weight at least $c|G'|^{-1/3} > cn^{-1/3}$, which corresponds to a cycle in G of weight at least $cn^{-1/3}$, which is a contradiction. Thus $d^-(v_{i+1}) > 0$. If $d^+(v_{i+1}) = 0$, then consider the same digraph G'. Each $y \in V(G')$ has outweight at least $1 - w_{i+1}(yv_{i+1}) \geq 1 - 2cn^{-1/3}$. However, G' contains no vertices from $\Gamma_G^+(v)$ (if $vy \in E(G)$ and $y \in V(G_{i+1})$ then we would have $v_j y \in E(G_j)$ for $j = 0, \ldots, i+1$), and so $|G'| \leq n - d_G^+(v) - 1 < n - 6cn^{2/3}$. Thus, by our inductive hypothesis, G' contains a cycle of weight at least

$$\frac{c(1 - 2cn^{-1/3})}{(n - 6cn^{2/3})^{1/3}} > cn^{-1/3},$$

which corresponds to a cycle of weight at least $cn^{-1/3}$ in G, a contradiction. Thus $d^+(v_{i+1}) > 0$, and so G_{i+1} is strongly connected. However, clearly $|G| = |G_0| > |G_1| > \cdots$, so at some point we reach a contradiction.

Therefore, every vertex $v \in V(G)$ must satisfy $d^+(v) < 6cn^{2/3}$. Let G' be the graph obtained from G by deleting every edge weighing less than $n^{-2/3}/12c$. Then every vertex still has outweight at least $1 - 6cn^{2/3}(n^{-2/3}/12c) = 1/2$. Now no edge weighs more than $cn^{-1/3}$, so every vertex must satisfy

$$d_{G'}^+(v) \geq (1/2)/(cn^{-1/3}) = n^{1/3}/2c.$$

Therefore, G' contains a cycle of length at least $n^{1/3}/2c$, which must weigh at least $(n^{1/3}/2c)(n^{-2/3}/12c) = n^{-1/3}/24c^2 = cn^{-1/3}$, which is a contradiction. □

It seems likely that $n^{-1/3}$ is much too small, and $\log\log n/\log n$ is closer to the truth. However, some new idea will probably be required before such a bound can be achieved. We make the following conjecture.

Conjecture 4. *Let G be a digraph with edge-weighting w such that every vertex in G has outweight at least $1/2$. Then G contains a cycle of weight at least $1/\log|G|$.*

What other condition can we place on G and w to ensure that we have heavy cycles as well as heavy paths? So far we have only restricted outweights; perhaps it is enough to restrict inweights as well. The following conjecture seems natural.

Conjecture 5. *Let G be a digraph with edge-weighting w such that every vertex v in G satisfies*

$$w_{\text{in}}(v) = w_{\text{out}}(v) = 1. \tag{6}$$

Then G contains a cycle C with $w(C) \geq 1$.

For strongly connected graphs we make the stronger conjecture that $w_{\text{in}}(v) \geq 1$ and $w_{\text{out}}(v) \geq 1$ for every v in G would also suffice to guarantee a cycle with weight at least 1. Let us note that if Corollary 2 failed badly then we could construct a counterexample: let G be a digraph with minimal outweight 1 and $v \in V(G)$ a vertex such that no path of weight at least $1/3$ ends in v. Let G' be a copy of G with all edges reversed, and add every edge from G' to G and a single edge from v to its copy v' in G'. If we give weight $1/3$ to all the edges between G' and G then the resulting graph satisfies the conditions of the conjecture but contains no cycle of weight at least 1.

Another question arises when we replace the weight condition (6) by a condition on the structure of G. It seems likely that the following is true.

Conjecture 6. *Let G be a digraph with edge-weighting w such that $d^-(v) = d^+(v)$ for every vertex v in G. Then G contains a cycle of weight at least $cw(G)/(n-1)$, where c is an absolute constant.*

A stronger conjecture (Conjecture 7) will be presented in the next section. The complete weighted digraph on n vertices with all weights equal to $1/(n-1)$ shows that, if true, Conjecture 6 is best possible up to a constant factor.

§3. Related questions and conjectures

One approach to finding heavy cycles in a digraph G is to look for good cycle covers or partitions of G: if we can partition or cover the edges of G with fairly few cycles, then one of the cycles must be fairly heavy. For instance, a weaker form of Theorems A and B follows from a result of Lovász [17], that every graph on n vertices can be covered by $\lfloor n/2 \rfloor$ edge-disjoint paths and cycles. Gallai (see [3]) conjectured that every connected graph of order n can be covered by $\lceil n/2 \rceil$ paths, which would almost give Theorem A. Another result of this type was given by Pyber [19], who proved that every graph of order n can be covered by $n-1$ edges and cycles; this also gives a weaker for of Theorem B.

Of course, we are equally happy with multiple covers. For instance, a *perfect path double cover* $\mathcal{P}$ of a graph G is a family of $|G|$ paths such that every edge of G is contained in exactly two paths and every vertex of G is an endvertex for exactly two paths. Bondy [5] conjectured that every graph has a perfect path double cover, and this was proved by Hao Li [16]. This implies Theorem A, since

$$\sum_{P \in \mathcal{P}} w(P) = 2w(G),$$

and so some path in $\mathcal{P}$ must weigh at least $2w(G)/|\mathcal{P}| = 2w(G)/|G|$.

Possibly the best known conjecture about cycle covers for graphs is the cycle double cover conjecture of Szekeres [22] and Seymour [21], which asserts that for every bridgeless graph G there exists a collection of cycles that covers every edge of G exactly twice. Bondy [4] makes the stronger conjecture that every 2-edge-connected graph has a cycle double cover with at most $n-1$ cycles. This would clearly imply Theorem B. Seymour (see [9]) has proved the slightly weaker assertion (which also implies Theorem B) that for every 2-edge-connected graph G there is a collection $\mathbf{C}$ of cycles and a collection $\{\alpha_C : C \in \mathbf{C}\}$ of positive reals such that $G = \sum_{C \in \mathbf{C}} \alpha_C C$ and $\sum_{C \in \mathbf{C}} \alpha_C \leq (n-1)/2$.

Similar questions can be asked for digraphs. Clearly, a digraph G can be partitioned into cycles iff $d^+(v) = d^-(v)$ for every vertex v in G. A digraph satisfying this condition is called *eulerian.* Meyniel (see [1], [2]) conjectured that every such digraph has a partition into at most $n-1$ cycles. This was shown to be false by

Dean [10], who conjectured that every eulerian digraph can be decomposed into at most $\frac{8n}{3} - 3$ cycles, and every eulerian oriented graph can be decomposed into at most $2n - 3$ cycles. It would be useful to prove even the following weak version of this conjecture.

Conjecture 7. *There exists an integer k such that every eulerian digraph G has an edge decomposition into at most $k(n-1)$ cycles.*

Let us note that this would imply a weaker version of Theorem B, namely that every weighted 2-edge-connected graph of order n contains a cycle of weight at least $w(G)/k(n-1)$ (see [20]).

Conjecture 7 clearly implies Conjecture 6. However, in order to get Conjecture 4 we would need a weighted result. For instance, given a collection $\mathcal{C}$ of cycles, for each v in G let $m_{\mathcal{C}}$ denote the minimum number of times that any outedge from v is covered by $\mathcal{C}$. Then it would be enough to prove that, for some collection $\mathcal{C}$ of cycles,

$$\sum_{v \in V(G)} m_{\mathcal{C}}(v) \geq |\mathcal{C}|.$$

Finally, let us remark that the cycle cover problem for digraphs in general is not very hard. Indeed, we have the following easy but best possible result.

Theorem 8. *Every strongly connected digraph of order n has an edge-covering with at most $\binom{n}{2}$ cycles. This is best possible for all n.*

Proof. Finding a cover with $\binom{n}{2}$ cycles is easy. For each pair of vertices $\{x, y\}$ we define a cycle $C_{\{x,y\}}$. If both xy and yx are edges then take the two-cycle they generate. If only one is an edge, say xy, then since G is strongly connected, we can extend xy to a cycle. Let $C_{\{x,y\}}$ be any cycle through the edge xy. If neither edge is present, then pick an arbitrary cycle. We have defined $\binom{n}{2}$ cycles, and these cover $E(G)$.

To see that the result is best possible, consider the directed path $x_1, \ldots, x_n$ with additional directed edges $\{x_i x_j : i > j\}$. Any cycle in this digraph contains at most one edge from the set $\{x_ix_j : i > j\}$, so any cycle cover requires at least $|\{x_ix_j : i > j\}| = \binom{n}{2}$ cycles. □

This is again in sharp contrast to the situation for graphs: as mentioned above, Pyber [19] has proved that every graph of order n can be covered by $n-1$ edges and cycles; therefore every 2-edge-connected graph of order n can be covered by $n-1$ cycles.

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