Almost all H-free graphs have the Erdős-Hajnal property

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Abstract

Erdős and Hajnal conjectured that, for every graph H, there exists a constant $\epsilon(H) > 0$ such that every H-free graph G (that is, not containing H as an induced subgraph) must contain a clique or an independent set of size at least $|G|^{\epsilon(H)}$.

We prove that there exists $\epsilon(H)$ such that almost every H-free graph G has this property, meaning that, amongst the H-free graphs with n vertices, the proportion having the property tends to one as $n \to \infty$.

1 Introduction

Szemerédi's Regularity Lemma is a powerful tool with applications in many fields. This paper discusses one of its applications in extremal graph theory.

A class of graphs $\mathcal{P}$ is said to have the *Erdős-Hajnal property* if there is a positive constant $\epsilon = \epsilon(\mathcal{P})$ such that every graph $G \in \mathcal{P}$ contains a *homogeneous* set of size at least $|G|^\epsilon$, where a homogeneous set is either a clique or an independent set. Let Forb(H) be the class of graphs not containing the graph H as an induced subgraph. Erdős and Hajnal [7] conjectured that Forb(H) has the Erdős-Hajnal property.

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Given a class of graphs $\mathcal{P}$, we write $\mathcal{P}^n$ for the members of $\mathcal{P}$ having vertex set $\{1, \ldots, n\}$. In particular, we will focus on $\mathcal{P}_H = \text{Forb}(H)$, which we will sometimes simply write as $\mathcal{P}$. Our intention in this note is to prove the following theorem.

Theorem 1. *For any graph H, there is a sub-class $\mathcal{Q}_H \subset \mathcal{P}_H$ which has the Erdős-Hajnal property, with $|\mathcal{Q}_H^n|/|\mathcal{P}_H^n| \to 1$ as $n \to \infty$.*

The Erdős-Hajnal conjecture itself remains open except in a few cases. Erdős and Hajnal themselves proved that the conjecture holds for those graphs H obtainable recursively from K_1 by disjoint union and complementation. They also proved it for the path of length three. Alon, Pach and Solymosi [3] showed that the class of graphs for which the conjecture holds is closed under replacement; this means if $H, F_1, \ldots, F_k$ satisfy the conjecture and $V(H) = \{v_1, \ldots, v_k\}$, then so does the graph $H(F_1, \ldots, F_k)$, obtained from disjoint copies of $F_1, \ldots, F_k$ by joining every vertex in F_i to every vertex in F_j precisely if $v_i v_j \in E(H)$ (for instance, if H satisfies the conjecture then so does any graph obtained by blowing up the vertices of H into cliques or independent sets). By a very different method, Chudnovsky and Safra [6] proved the conjecture for the *bull*, the self-complementary graph of order 5 comprising a triangle and two pendant edges.

The size of $\mathcal{P}_H^n$ for arbitrary H has received a lot of attention. Letting $\chi(H)$ denote the chromatic number of H, we see that any graph G which can be partitioned into $\chi(H) - 1$ stable sets obviously contains no induced copies of H. Considering a fixed partition of n vertices into $\chi(H) - 1$ parts each of which has size within 1 of $\frac{n}{\chi(H)-1}$, and counting all the graphs where the only edges go between partition elements we see that there are at least $2^{(1-\frac{1}{\chi(H)-1}+o(1))\binom{n}{2}}$ graphs in $\mathcal{P}_H^n$. In the same vein, if G cannot be partitioned into a cliques and b stable sets, then a similar argument shows that $|\mathcal{P}_H^n| \geq 2^{(1-\frac{1}{a+b}+o(1))\binom{n}{2}}$

Prömel and Steger [9] showed that this lower bound is not too far from the truth.

Definition 2. *The* colouring number *$\tau(H)$ of a graph H is the smallest integer $k \geq 1$ such that, for every integer $0 \leq \ell \leq k$, the vertices of H can be partitioned into ℓ cliques and $k - \ell$ independent sets.*

Prömel and Steger [9] proved:

$$\mathcal{P}_H^n = 2^{(1-\frac{1}{\tau(H)-1}+o(1))\binom{n}{2}}.$$

Let $P(a,b)$ be the class of graphs whose vertices can be partitioned into a cliques and b independent sets. Another way of writing the Prömel-Steger theorem is that, if $\mathcal{P} = \mathrm{Forb}(H)$ then $|\mathcal{P}^n| = 2^{(1-1/t+o(1))\binom{n}{2}}$, where

$$t = \max\{a+b : P(a,b) \subseteq \mathcal{P} \text{ for some } a, b\}.$$

The theorem in this form was extended to all hereditary properties $\mathcal{P}$ by Alexeev [1] and by Bollobás and Thomason [5][1].

For $H = C_4$ and $H = C_5$, Prömel and Steger [8, 10] proved a much sharper result by finding a very well-structured class $\mathcal{Q}$ with $\mathcal{Q} \subset \mathcal{P} = \mathrm{Forb}(H)$ and $|\mathcal{Q}^n|/|\mathcal{P}^n| \to 1$ as $n \to \infty$. Note that our aim in Theorem 1 is to do just this where the class $\mathcal{Q}$ has the Erdős-Hajnal property. In the case $H = C_4$ they showed that $\mathcal{Q} = P(1,1)$ works. Clearly the class $P(1,1)$ has the Erdős-Hajnal property, and indeed if $G \in P(1,1)$ then G can be partitioned into two homogeneous sets, one of which must have size linear in $|G|$. For $H = C_5$ they showed that the class $\mathcal{Q}$ of *generalized split graphs* works. A graph G is a generalized split graph either if its vertices can be partitioned into two classes U and W with $G[U]$ being a disjoint union of cliques and $G[W]$ being a single clique, or else the complement $\overline{G}$ of G has this property. Here again the class $\mathcal{Q}$ has the Erdős-Hajnal property, although since $\mathcal{Q}$ is dominated by graphs in which the cliques in $G[U]$ have size around $\log|G|$, we do not get a partition of G into two pieces each of which contains a homogeneous set of size linear in their order. Nevertheless, almost every generalized split graph does have a linear size homogenous set since in a typical generalized split graph, both W and U will have about half the vertices (see [10]).

It would be of interest to determine for which graphs H the following property holds:

(*) almost every graph in $\mathcal{P}_H$ has a homogeneous set of linear size.

We will not attack this problem here. We do however make a few remarks. First we note that a graph has no induced path on three vertices precisely if it is a disjoint union of cliques. So, as discussed above, for $H = P_3$, we have that for almost every graph G without H as an induced subgraph, the largest homogeneous set in G has size $\Theta(|V(G)|/\log|V(G)|)$.

[1] A class $\mathcal{P}$ of graphs that is closed under induced subgraphs is called *hereditary*. Clearly $\mathrm{Forb}(H)$ is hereditary, as is $\mathrm{Forb}(\mathcal{H})$, the class of graphs none of whose induced subgraphs is in the class $\mathcal{H}$. Every hereditary property $\mathcal{P}$ is of the form $\mathcal{P} = \mathrm{Forb}(\mathcal{H})$ for some $\mathcal{H}$ (just take $\mathcal{H}$ to be those graphs not in $\mathcal{P}$).

We do not know of any graph other than P_3 (and possibly P_4) which do not satisfy (*). These two graphs are exceptional, in that they satisfy $\tau = 2$. We wonder if (*) holds for all other graphs. To this end we note that McDiarmid and Reed claim that for almost every graph H, almost every graph in $\mathcal{P}_H$ contains a homogeneous set of linear size. Finally we note that our approach to proving that a graph H satisfies (*) is to show that almost every graph in $\mathcal{P}_H$ has a partition into $\tau(H) - 1$ linear sized pieces, one of which contains a homogeneous set which has size linear in its order.

Clearly, if H is a graph for which we can take $\mathcal{Q}$ to be the union of the $P(a, b) \subset \mathcal{P}_H$ with $a + b = t$, then Theorem 1 and (*) hold for H. Very recently, Balogh and Butterfield [4] have characterized the graphs H for which this is possible: they call such H "critical". The remarks above show that C_4 is critical but C_5 is not. Curiously it turns out that, for $\ell \geq 6$, C_ℓ is not critical if ℓ is even, but C_ℓ is critical if ℓ is odd.

Given that Theorem 1 is weaker than the Erdős-Hajnal conjecture and that it is known in special cases, we aim to give a proof that is short. In particular, we make no effort to optimize $\epsilon(H)$. The results mentioned above all begin with applications of Szemerédi's Regularity Lemma, together with the Erdős-Stone theorem and Ramsey's theorem and perhaps the Erdős-Simonovits stability theorem. (The exception to this is Alexeev [1], who uses only an extension of the Sauer-Shelah lemma.) We shall not use this machinery apart from one of the basic consequences of Szemerédi's Lemma common to all the cited papers. Our proof is based on an observation about partitioning H into $\tau(H) - 1$ sets (Lemma 3). Surprisingly, in order to use the lemma, we need the fact from [6] that the Erdős-Hajnal conjecture is true for the bull.

Because it rests on Lemma 3, our proof of Theorem 1 does not immediately extend to all hereditary properties $\mathcal{P}$. Alon, Balogh, Bollobás and Morris [2] have recently described, for any hereditary property $\mathcal{P}$, a property $\mathcal{Q}$ with $\mathcal{Q} \subset \mathcal{P}$ and $|\mathcal{Q}^n|/|\mathcal{P}^n| \to 1$ as $n \to \infty$. The graphs in $\mathcal{Q}$ have a partition into t sets each of which is "somewhat homogenous", in a well-defined way. However, it is not evident whether the Erdős-Hajnal property can be derived from this description.

2 Proof of Theorem 1

The proof combines the following simple lemma with some, by now standard, regularity lemma machinery.

Lemma 3. *Let $t \geq 1$. Then there is a finite set $\mathcal{F}_t$ of graphs, such that Forb(F) has the Erdős-Hajnal property for all $F \in \mathcal{F}$, and the vertices of any graph H with $\tau(H) = t+1$ can be partitioned into t sets each inducing a graph in $\mathcal{F}_t$.*

Proof. Let H be any graph with $\tau(H) = t$. By the definition of $\tau(H)$, the vertices of H can be partitioned into $t+1$ independent sets $I_1, \ldots, I_{t+1}$ and also into $t+1$ sets $C_1, \ldots, C_{t+1}$ inducing cliques. For any independent set I of H, $|C_i \cap I| \leq 1$ holds for all $1 \leq i \leq t+1$, and so $|I| \leq t+1$. In particular this is true for $|I_j|$ for all j, and so $|H| \leq (t+1)^2$. Symmetrically no clique of G has more than $t+1$ elements.

If $|H| < (t+1)^2$ we may assume that $I_{t+1} = \{v_1, \ldots, v_s\}$ where $s \leq t$. Then put $V_j = I_j \cup \{v_j\}$ for $1 \leq j \leq s$ and put $V_j = I_j$ for $s < j \leq t$.

If $|H| = (t+1)^2$, since $\tau(H) = t+1$ there is a partition of the vertices of H into t independent sets $J_1, \ldots, J_t$ and a set D inducing a clique. Since $|J_j| \leq t+1$ for all j and $|D| \leq t+1$, it follows that all these sets have size exactly $t+1$, so write $D = \{w_1, \ldots, w_{t+1}\}$. Then put $V_j = J_j \cup \{w_j\}$ for $1 \leq j < t$ and let $V_t = J_t \cup \{w_t, w_{t+1}\}$.

In each case, we obtain a partition $V(H) = V_1 \cup \cdots \cup V_t$; let F_j be the subgraph induced by V_j. For $j < t$ the graph F_j consists of a star together with isolated vertices. Note that the star $K_{1,s}$ equals $K_2(K_1, \overline{K_s})$ (here we are using the replacement notation from the introduction) and $K_{1,s}$ together with i isolated vertices equals $\overline{K}_2(K_{1,s}, \overline{K_i})$, so each graph F_j, $j < t$ satisfies the Erdős-Hajnal property. Let $F_t = F$ together with i isolated vertices, where F is connected. Then F is obtained from the bull by replacing vertices with (possibly empty) independent sets, and then $F_t = \overline{K}_2(F, \overline{K_i})$. Thus F_t also satisfies the Erdős-Hajnal property.

Finally, let $\mathcal{F}_t$ be the set of all graphs that can arise in the procedure above. $\square$

In the case when $|H| = (t+1)^2$ we could instead have obtained F_t by distributing I_{t+1} amongst $I_1, \ldots, I_t$, but this would require the path of length 4 to satisfy the Erdős-Hajnal property, which is currently unknown. Note that graphs with $\tau(H) = t+1$ and $|H| = (t+1)^2$ do exist, at least if $t+1$ is a prime power. Take $V(H) = \mathbb{F}_{t+1}^2$. Each pair of vertices lies in exactly one line, whose gradient is one of $0, 1, \ldots, t, \infty$. The $t+1$ parallel lines of each gradient form a partition of the vertex set. For finite gradients $m \leq t$ make m of these lines cliques and the other $t+1-m$ lines independent sets; for $m = \infty$ make all the lines cliques.

We remarked earlier that Szemerédi's Lemma is fundamental in the study of the number of H-free graphs. In fact by making use of earlier work we can avoid many of the technicalities involved. Lemma 3.5 in the pioneering work of Prömel and Steger [9] is entirely adequate for our purposes but we borrow instead [5, Theorem 3.1], which is very similar. It is phrased in terms of the density of bipartite graphs, which has the natural meaning, and η-regularity, which has its usual meaning in relation to Szemerédi's Lemma: however, it is not necessary to know this meaning in order to follow the proof of Theorem 1. A *coloured partition* π is a colouring of the edges of the complete graph K_m with colours red, blue, green and grey, where m is the *order* of π, denoted $|\pi|$. Given a graph G and constants $0 < \lambda, \eta < 1$ we say that a partition of the vertices of G into $|\pi|$ classes $V_1, \ldots, V_{|\pi|}$ *satisfies* π with respect to λ and η if $|V_1| \leq |V_2| \leq \ldots \leq |V_{|\pi|}| \leq |V_1| + 1$ and the pair (V_i, V_j) is not η-regular only if ij is grey, and otherwise $0 \leq d(V_i, V_j) \leq \lambda$, $\lambda < d(V_i, V_j) < 1 - \lambda$ or $1 - \lambda \leq d(V_i, V_j) \leq 1$ according as ij is red, green or blue. Here $d(V_i, V_j)$ stands for the density of the bipartite graph between V_i and V_j. We say that G *satisfies* π if there is a partition of G satisfying π.

Szemerédi's Regularity Lemma [11] asserts that, given λ, η and some integer ℓ, there exists an integer $L = L(\ell, \eta)$ such that any graph G satisfies some coloured partition π with respect to λ and η, where $\ell \leq |\pi| < L$ and where π has at most $\eta\binom{|\pi|}{2}$ grey edges. The following proposition, roughly speaking, states that if π has many green edges then G contains every small member of $P(a, b)$.

Proposition 4 ([5, Theorem 3.1]). *Let $t, h \in \mathbb{N}$ and $0 < \lambda, \nu < 1$ be given. Then there exist positive constants ℓ_0, η_0, and n_0 with the following property. Let π be a coloured partition with $|\pi| \geq \ell_0$, having at most $\eta_0\binom{|\pi|}{2}$ grey edges and at least $(1 - 1/t + \nu)\binom{|\pi|}{2}$ green edges. Then there are integers a and b with $a + b = t + 1$, such that every graph of order at least n_0 that satisfies π with respect to λ and η_0 contains every member of $P(a, b)$ with at most h vertices as an induced subgraph.*

With our lemma and the regularity lemma in hand, we are in a position to finish the proof of Theorem 1.

Proof of Theorem 1. Let $\mathcal{P} = \mathrm{Forb}(H)$ and let $t = \tau(H) - 1$. Note that, as in the proof of Lemma 3, $|H| \leq (t+1)^2$, so we may assume $t \geq 1$. Furthermore, if $\tau(H) = 2$ then (as the Erdős-Hajnal property holds for K_2, $K_2 \cup K_1$, P_4 and C_4) we are done, so we may assume $t \geq 2$. By the definition of $\tau(H)$ there exist a, b with $a + b = t$ and $P(a, b) \subset \mathcal{P}$. The graphs of order n formed by adding edges between a cliques and b independent sets are all in $\mathcal{P}$, and hence $|\mathcal{P}^n| \geq 2^{(1-1/t+o(1))\binom{n}{2}}$.

Let $\mathcal{F}_t$ be the class given by Lemma 3. For each F in $\mathcal{F}_t$ let ϵ_F be such that for every graph G, either G contains F as an induced subgraph or G contains a homogenous set of size at least $|G|^{\epsilon_F}$. Let ϵ be the minimum of the ϵ_F.

Let $\mathcal{Q}$ be the class of graphs $G \in \mathcal{P}$ containing a homogeneous set of size $|G|^{\epsilon/2}$, so $\mathcal{Q}$ has the Erdős-Hajnal property. Let $\mathcal{R} = \mathcal{P} - \mathcal{Q}$. We shall show that $|\mathcal{R}^n|/|\mathcal{P}^n| \to 0$ as $n \to \infty$; that is, $|\mathcal{Q}^n|/|\mathcal{P}^n| \to 1$ as required.

Our approach is straightforward. Szemerédi's Regularity Lemma tells us that for large enough n, every graph in $\mathcal{R}^n$ satisfies one of a certain class of coloured partitions. We count the number of elements of $\mathcal{R}^n$ by summing the number of elements satisfying each partition. Forthwith the details.

We set $h = |H|$ and pick λ, ν to certify certain inequalities given below. Choose η_0, ℓ_0 and n_0 satisfying Proposition 4 with respect to h, t, λ, and ν. We choose $\ell \geq \ell_0$ and $\eta \leq \eta_0$ which satisfy some inequalities given below. We choose L satisfying the Szemerédi Regularity Lemma for this choice of λ, η and ℓ. Let $G \in \mathcal{P}^n$ for some $n \geq n_0$. Then G satisfies some π with respect to λ, η where $\ell \leq |\pi| < L$ and π has at most $\eta\binom{|\pi|}{2}$ grey edges.

We want to bound the number of graphs of $\mathcal{R}^n$ satisfying a particular partition π with respect to a given λ and η. We actually bound the number of partitions of graphs in $\mathcal{R}^n$ which satisfy π, which is larger. We do so by summing over each partition of V, the number of graphs in $\mathcal{R}^n$ for which this partition satisfies π with respect to λ, η. If V_i, V_j corresponds to a green edge or a grey edge then there are at most $2^{|V_i||V_j|}$ ways to join V_i to V_j. But for red and blue edges there are at most $2^{c|V_i||V_j|}$ where $c \to 0$ as $\lambda \to 0$. Furthermore, there are at most $\binom{n}{2}/|\pi|$ edges within the partition classes. So letting n_g be the proportion of green edges of the partition we see that the total number of graphs G such that this partition satisfies π is at most

$$2^{(n_g+\eta+c+\frac{1}{|\pi|})\binom{n}{2}}. \tag{1}$$

Since we know that $\mathcal{P}^n$ has at least $2^{(1-\frac{1}{t}+o(1))\binom{n}{2}}$ elements, an easy computation yields that for large n the number of graphs satisfying partitions with $n_g < 1 - \frac{1}{t} - \eta - 2c - \frac{1}{\ell}$ is $o(|\mathcal{P}^n|)$.

In counting these graph partitions satisfying π for which n_g is larger, we need to exploit the fact that we are only counting graphs in $\mathcal{R}^n$. Since $H \notin P(a,b)$, Proposition 4 therefore implies that we need only consider π with at most $(1 - 1/t + \nu)\binom{|\pi|}{2}$ green edges.

The proof of the following two lemmas is postponed to the end of the proof.

Lemma 5. *If π contains s edge disjoint cliques of size t all of whose edges are green then the total number of graphs in $\mathcal{R}^n$ for which a given partition satisfies π is at most $2^{(n_g - \frac{s}{16t^2|\pi|^2} + \eta + c + \frac{1}{|\pi|})\binom{n}{2}}$.*

Lemma 6. *If n_g exceeds $1 - \frac{1}{t-1} + \frac{1}{2t^2}$ and $|\pi|$ is sufficiently large then there are at least $|\pi|^2/4t^4$ edge disjoint cliques of size t in π all of whose edges are green.*

With these two lemmas in hand, it is straightforward to prove the theorem. We choose λ so that c is less than $\frac{1}{1000t^6}$. We choose $\nu = \frac{1}{1000t^6}$. We choose ℓ such that $\ell \geq \max(l_0, 1000t^6)$ and Lemma 5 applies, and we choose $\eta \leq \min(\eta_0, \frac{1}{1000t^6})$.

Now, we have that $\eta + c + \frac{1}{|\pi|} \leq \frac{3}{1000t^6}$. So, if π is a partition for which $n_g < 1 - \frac{1}{t-1} + \frac{1}{2t^2}$ then there are at most $2^{(1-\frac{1}{t-1}+\frac{2}{3t^2})\binom{n}{2}}$ graphs G in $\mathcal{R}^n$ for which a given partition satisfies π. On the other hand for any partition π with $n_g \geq 1 - \frac{1}{t-1} + \frac{1}{2t^2}$ we know by Proposition 4 that $n_g \leq 1 - \frac{1}{t} + \nu$. Hence, combining Lemmas 6 and 5, we see that at most $2^{(1-\frac{1}{t}-\frac{1}{64t^6}+\frac{4}{1000t^6})\binom{n}{2}}$ graphs in $\mathcal{R}^n$ satisfy π. So, in either case, there are at most $2^{(1-\frac{1}{t}-\frac{1}{100t^6})\binom{n}{2}}$ graphs in $\mathcal{R}^n$ for which a particular partition satisfies π. But the number of choices for π is independent of n, and the number of partitions of the vertex set is L^n which is $o(2^{\binom{n}{2}})$. So for large n, the number of elements of $\mathcal{R}^n$ is $o(2^{(1-\frac{1}{t})\binom{n}{2}})$ and hence $|\mathcal{R}^n| = o(|\mathcal{P}^n|)$. This completes the proof of the lemma and the theorem. □

It remains only to prove the two lemmas.

The second one is straightforward. We simply greedily rip out the edges of a green clique of size t in π until no such cliques remain. Turan's theorem tells us that when we stop, only $(1 - \frac{1}{t-1})\binom{|\pi|}{2}$ green edges can remain. But by assumption π has at least $(1 - \frac{1}{t-1} + \frac{1}{2t^2})\binom{|\pi|}{2}$ green edges. So we must have ripped out at least $(1 + o(1))\binom{|\pi|}{2}/2t^2$ edges and hence at least $(1 + o(1))\binom{|\pi|}{2}/t^4$ cliques which is at least $\binom{|\pi|}{2}/4t^4$ for large enough $|\pi|$.

With the proof of Lemma 6 completed, we turn to the proof of Lemma 5.

Proof of Lemma 5. For each green clique C, we let m_C be the sum of $|V_i||V_j|$ over every two partition classes V_i and V_j in C. We claim that for each of the s cliques, having fixed the edges within the partition classes, there are at most $2^{m_C - \frac{n^2}{16t^2|\pi|^2}}$ ways to pick the edges within the green edges of the clique. Combining this with our earlier analysis leading to (1) then yields the desired result.

It remains to prove the claim. By Lemma 3, there is a partition $V(H) = V_1 \cup \cdots \cup V_t$ such that each V_i induces a subgraph F_i that belongs to $\mathcal{F}_t$. Let G be any graph in $\mathcal{R}^n$ and let C be a green clique, say with vertices corresponding to $V_1, \ldots, V_t$. Let p be a prime between $n/4t|\pi|$ and $n/3t|\pi|$ (which can be found provided n is large enough). Because G contains no large homogeneous set, and each F_i satisfies the Erdős-Hajnal property with constant $\epsilon_{F_i} \geq \epsilon$, we can pick out p vertex-disjoint copies $F_i^{(1)}, \ldots, F_i^{(p)}$ of F_i in $G[V_i]$ for each i. For $1 \leq r, s \leq p$, consider the t-tuple $(F_1^{(r)}, F_2^{(r+s)} \ldots, F_t^{(r+(t-1)s)})$, where indices are taken modulo p. For each t-tuple, there is at least one way to join the classes to obtain a copy of H; on the other hand, as p is prime, no pair of t-tuples coincide in more than one coordinate, and so no edge between classes is spanned by more than one t-tuple. As there are p^2 t-tuples, it follows that the number of ways of filling in the edges between $V_1, \ldots, V_t$ is at most $2^{m_C - p^2} \leq 2^{m_c - n^2/16t^2|\pi|^2}$. The desired result follows. □

This completes the proof of the lemma and our theorem.

References

[1] V.E. Alekseev, On the entropy values of hereditary classes of graphs, *Discrete Math. Appl.* **3** (1993) 191-199.

[2] N. Alon, J. Balogh, B. Bollobás and R. Morris, The structure of almost all graphs in a hereditary property (preprint).

[3] N. Alon, J. Pach and J. Solymosi, Ramsey-type theorems with forbidden subgraphs, *Combinatorica* **21** (2001), 155–170.

[4] J. Balogh and J. Butterfield, Excluding induced subgraphs: critical graphs (preprint).

[5] B. Bollobás and A. Thomason, Hereditary and monotone properties of graphs, in "The Mathematics of Paul Erdős II" (R.L. Graham and J. Nešetřil, eds.) *Algorithms and Combinatorics* **14** Springer-Verlag (1997) 70–78.

[6] M. Chudnovsky and S. Safra, The Erdős-Hajnal conjecture for bull-free graphs, *J. Combin. Theory Ser. B* **98** (2008), 1301–1310.

[7] P. Erdős and A. Hajnal, Ramsey-type theorems, *Discrete Applied Mathematics* **25** (1989), 37–52

[8] H.-J. Prömel and A. Steger, Excluding induced subgraphs: quadrilaterals, *Random Structures and Algorithms* **2** (1991), 55–71.

[9] H.-J. Prömel and A. Steger, Excluding induced subgraphs III: extremal graphs, *Random Structures and Algorithms* **3** (1992), 19–31.

[10] H.-J. Prömel and A. Steger, Almost all Berge graphs are perfect, *Combinatorics, Probability and Computing* **1** (1992), 53–79.

[11] E. Szemerédi, Regular partitions of graphs, *in* Problèmes combinatoires et théorie des graphes, Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976, CNRS, Paris, 1978, pp. 399–401.