Polynomial bounds for chromatic number
VI. Adding a four-vertex path

Maria Chudnovsky[1]
Princeton University, Princeton, NJ 08544

Alex Scott[2]
Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour[3]
Princeton University, Princeton, NJ 08544

Sophie Spirkl[4]
University of Waterloo, Waterloo, Ontario N2L3G1, Canada

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Abstract

A class of graphs is χ-*bounded* if there is a function f such that every graph G in the class has chromatic number at most $f(\omega(G))$, where $\omega(G)$ is the clique number of G; the class is *polynomially* χ-*bounded* if f can be taken to be a polynomial. The Gyárfás-Sumner conjecture asserts that, for every forest H, the class of H-free graphs (graphs with no induced copy of H) is χ-bounded. Let us say a forest H is *good* if it satisfies the stronger property that the class of H-free graphs is polynomially χ-bounded.

Very few forests are known to be good: for example, it is open for the five-vertex path. Indeed, it is not even known that if every component of a forest H is good then H is good, and in particular, it was not known that the disjoint union of two four-vertex paths is good. Here we show the latter, and more generally, that if H is good then so is the disjoint union of H and a four-vertex path. We also prove a more general result: if every component of H_1 is good, and H_2 is any path (or broom) then the class of graphs that are both H_1-free and H_2-free is polynomially χ-bounded.

1 Introduction

A class $\mathcal{G}$ of graphs is *hereditary* if it is closed under taking induced subgraphs. We say that a hereditary class $\mathcal{G}$ is *χ-bounded* if there is a function f such that every graph $G \in \mathcal{G}$ has chromatic number at most $f(\omega(G))$, where $\omega(G)$ is the clique number of G; the class $\mathcal{G}$ is *polynomially χ-bounded* if f can be taken to be a polynomial. A graph is *H-free* if it has no induced subgraph isomorphic to H.

The Gyárfás-Sumner conjecture [4, 14] asserts:

1.1 Conjecture: *For every forest H, the class of H-free graphs is χ-bounded.*

There has been a great deal of recent progress on χ-bounded classes (see [9] for a survey), although the Gyárfás-Sumner conjecture remains open. In most cases, proofs of χ-boundedness give fairly fast-growing functions, so it is interesting to ask: when do we get the stronger property of polynomial χ-boundedness?

A provocative conjecture of Louis Esperet [3] asserted that every χ-bounded hereditary class is polynomially χ-bounded. But this was recently disproved by Briański, Davies and Walczak [1]. So the question now is: which hereditary classes are polynomially χ-bounded? For example, when can 1.1 be strengthened to polynomial χ-boundedness? Let us say a graph H is *good* if the class of H-free graphs is polynomially χ-bounded. Very few trees are known to be good: it is easy to show that stars are good, and it was shown in [11] that all trees not containing the five-vertex path P_5 are good. But it is not known whether P_5 is good (although see [12] for the best current bounds for $H = P_5$; and see [13] for the case when H a general tree of radius two).

In the case of χ-boundedness, it is not hard to show that a forest H satisfies the Gyárfás-Sumner conjecture if and only if all its components do. But it has *not* been shown that if every component of a forest H is good then H is good. Indeed, only some very restricted forests are known to be good [8, 10]. One outstanding case was when H is the disjoint union of two copies of the four-vertex path P_4; and this was particularly annoying since the P_4-free graphs are very well-understood and rather trivial.

We will prove the following:

1.2 *If H is a good forest, then the disjoint union of H and P_4 is also good.*

In particular, the disjoint union of two or more copies of P_4 is good. 1.2 is a consequence of the next result, about brooms. A *(k, d)-broom* is a tree obtained from a k-vertex path with one end v by adding d new vertices adjacent to v, and a *broom* is a tree that is a (k, d)-broom for some k, d. It is known that $(3, d)$-brooms are good [6, 11], but this is not known for larger brooms (all of which contain P_5). We will show the following, which implies 1.2:

1.3 *Let H_1 be a forest such that every component of H_1 is good, and let H_2 be either a broom, or the disjoint union of a good forest and a number of paths. Then there is a polynomial ϕ such that $\chi(G) \le \phi(\omega(G))$ for every $\{H_1, H_2\}$-free graph G.*

(*$\{H_1, H_2\}$-free* means both H_1-free and H_2-free.) To deduce 1.2 from 1.3, let H be a good forest, let $H_1 = H_2$ be the disjoint union of H and P_4, and apply 1.3.

Some notation and terminology: if G is a graph and $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced on X, and we sometimes write $\chi(X)$ for $\chi(G[X])$ and $\omega(X)$ for $\omega(G[X])$. Two disjoint subsets $A, B \subseteq V(G)$ are *complete* if every vertex in A is adjacent to every vertex of B, and *anticomplete* if there is no edge between A, B; and we say a vertex v *is complete to* B if $\{v\}$ is complete to B, and so on. A graph G *contains* a graph H if some induced subgraph of G is isomorphic to H, and such a subgraph is a *copy* of H. The *cone* of a graph H is obtained from H by adding a new vertex adjacent to every vertex of H.

Let us say a graph is 0-*bad* if it is good; and a graph J is β-bad, where $\beta \geq 1$ is an integer, if either J is the disjoint union of two $(\beta - 1)$-bad graphs, or J is the cone of a $(\beta - 1)$-bad graph, or J is $(\beta - 1)$-bad. In general, cones are not forests, so they are not good. Nevertheless, we will prove the following strengthening of 1.3:

1.4 *Let $\beta \geq 0$, let H_1 be a β-bad graph, and let H_2 be either a broom, or the disjoint union of a good forest and a number of paths. Then there is a polynomial ϕ such that $\chi(G) \leq \phi(\omega(G))$ for every $\{H_1, H_2\}$-free graph G.*

This implies several results that were previously known. For instance, in [7] it is proved that:

1.5 *Let H_1 be either*

- *the disjoint union of a complete graph and a good graph, or*
- *the disjoint union of some complete graphs, or*
- *the cone of the disjoint union of some complete graphs.*

Let H_2 be a path. Then there is a polynomial ϕ such that $\chi(G) \leq \phi(\omega(G))$ for every $\{H_1, H_2\}$-free graph G.

Some other results of [7, 8] are also special cases of 1.4.

2 Finding a disjoint union

Suppose that H is the disjoint union of good forests H_1, H_2. Choose c_1, c_2 such that for $i = 1, 2$, every H_i-free graph G satisfies $\chi(G) \leq \omega(G)^{c_i}$. Thus, if G is H-free, we know that there do not exist disjoint, anticomplete subsets $P, Q \subseteq V(G)$ with $\chi(P) > \omega(P)^{c_1}$ and $\chi(Q) > \omega(Q)^{c_2}$; because then $G[P]$ is not H_1-free, and $G[Q]$ is not H_2-free, and the union of a copy of H_1 in $G[P]$ and a copy of H_2 in $G[Q]$ gives a copy of H, which is impossible.

But we do not really need P, Q to be anticomplete. It is enough that $\chi(P) > \omega(P)^{c_1}$, and $\chi(Q) > |H_1|r + \omega(Q)^{c_2}$, where r denotes the maximum over $v \in P$ of the chromatic number of the set of neighbours of v in Q; because then if we choose a copy H_1' of H in $G[P]$, the chromatic number of the set of vertices in Q with no neighbours in $V(H_1')$ is at least $\chi(Q) - |H_1|r > \omega(Q)^{c_2}$, and so this set contains a copy of H_2, a contradiction. In the proof to come later in the paper, this is the only way we will ever use that G is H-free; and so we might as well prove a stronger theorem, replacing the hypothesis that G is H-free with the weaker hypothesis that there is no suitable pair P, Q in G.

Thus we will be excluding pairs of disjoint sets P, Q where $\chi(P)$ is at least some power of $\omega(P)$, and for each vertex in P, its set of neighbours in Q has chromatic number at most some r that is small compared with the chromatic number of Q.

In our proof, it happens that when we find a suitable pair (P, Q), it comes equipped with an extra vertex v that is complete to P and anticomplete to Q; so we might as well prove that there is a "suitable triple" (v, P, Q). Such a thing will also allow us to handle cones.

We denote the set of nonnegative integers by $\mathbb{N}$, and say a function $\phi : \mathbb{N} \to \mathbb{N}$ is *non-decreasing* if $\phi(x) \leq \phi(x')$ for all $x, x' \in \mathbb{N}$ with $x \leq x'$.

Let $\psi : \mathbb{N} \to \mathbb{N}$ be non-decreasing, and let $q \geq 0$ be an integer. We say a (ψ, q)-*scattering* in a graph G is a triple (v, P, Q) where:

- P, Q are disjoint subsets of $V(G)$, and $v \in V(G) \setminus (P \cup Q)$;
- $\{v\}$ is complete to P and anticomplete to Q;
- $\chi(P) > \psi(\omega(P))$; and
- $\chi(Q) > qr + \psi(\omega(Q))$, where r is the maximum, over $v \in P$, of the chromatic number of the set of neighbours of v in Q.

Thus we will replace the hypothesis in 1.4 that G is H_1-free and H_1 is β-bad, with the hypothesis that G contains no (ψ, q)-scattering, for appropriate ψ, q. We will show:

2.1 *Let $\psi : \mathbb{N} \to \mathbb{N}$ be a non-decreasing polynomial and let $q \in \mathbb{N}$. Let H_2 be either a broom, or the disjoint union of a good forest and a number of paths. Then there is a polynomial $\phi : \mathbb{N} \to \mathbb{N}$ such that if $\chi(G) > \psi(\omega(G))$ and G contains no (ψ, q)-scattering, then G contains H_2.*

Proof of 1.4, assuming 2.1. We proceed by induction on β. Let H_1 be β-bad, and let H_2 be either a broom, or the disjoint union of a good forest and a number of paths.

If H_1 is good, the result is true, so we assume that H_1 is not good, and therefore $\beta \geq 1$. Thus either H_1 is the disjoint union of two $(\beta - 1)$-bad graphs J_1, J_2, or the cone of a $(\beta - 1)$-bad graph J_1 (and in this case let J_2 be the null graph). From the inductive hypothesis on β, for $i = 1, 2$ there is a non-decreasing polynomial ϕ_i such that if G is H_2-free and J_i-free then $\chi(G) \leq \phi_i(\omega(G))$, and by replacing ϕ_1, ϕ_2 by $\phi_1 + \phi_2$ we may assume that $\phi_1 = \phi_2$.

Let $q = |J_1|$. By 2.1, there is a non-decreasing polynomial ϕ such that if $\chi(G) > \phi(\omega(G))$ and contains no (ϕ_1, q)-scattering, then G contains H_2. We claim that ϕ satisfies 1.4.

Let G be $\{H_1, H_2\}$-free, and suppose that $\chi(G) > \phi(\omega(G))$. Since G is H_2-free, it follows from the choice of ϕ that G contains a (ϕ_1, q)-scattering (w, P, Q) say. Let r be the maximum, over $v \in P$, of the chromatic number of the set of neighbours of v in Q. Since $\chi(P) > \phi_1(\omega(P))$, there is an induced subgraph of $G[P]$ isomorphic to J_1, say J_1'. Hence G contains the cone of J_1, so we may assume that H_1 is the disjoint union of J_1, J_2. The set of vertices in Q with a neighbour in $V(J_1')$ has chromatic number at most $r|J_1|$, and since

$$\chi(Q) > |J_1|r + \phi_2(\omega(Q)),$$

it follows that the set (say Q') of vertices in Q that are anticomplete to J_1' has chromatic number more than $\phi_2(\omega(Q))$. From the choice of ϕ_2, and since G is H_2-free, it follows that $G[Q']$ is not J_2-free; but then, combining this copy of J_2 with J_1', we find a copy of H_1 in G, a contradiction. This proves 1.4. ∎

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. We say a subgraph P of a graph G is *σ-nondominating* if there is a set $X \subseteq V(G) \setminus V(P)$, anticomplete to $V(P)$, with $\chi(X) > \sigma(\omega(X))$. Next we will show that to prove 2.1 it suffices to prove the following:

2.2 *Let $\psi, \sigma : \mathbb{N} \to \mathbb{N}$ be non-decreasing polynomials, and let $q \geq 0$ an integer. Let H be a broom, and let J be a path. Then there is a non-decreasing polynomial $\phi : \mathbb{N} \to \mathbb{N}$ such that if G is a graph, and $\chi(G) > \phi(\omega(G))$, and G contains no (ψ, q)-scattering, then G contains H and a σ-nondominating copy of J.*

Proof of 2.1, assuming 2.2. Let ψ, q, H_2 be as in 2.1. If H_2 is a broom, then 2.1 follows immediately from 2.2 (setting $H = H_2$ and setting J to be some path, for instance the one-vertex path). Thus we assume that H_2 is the disjoint union of a good forest J_1 and a forest J_2 that is a disjoint union of paths. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function such that every J_1-free graph G has chromatic number at most $\sigma(\omega(G))$; and choose a path J such that J_2 is an induced subgraph of J. By 2.2 (setting H to be some broom, for instance with one vertex) there is a non-decreasing polynomial $\phi : \mathbb{N} \to \mathbb{N}$ such that if $\chi(G) > \phi(\omega(G))$ and G contains no (ψ, q)-scattering, then G contains a σ-nondominating copy J' of J.

We claim that ϕ satisfies 2.1. Thus we must show that if G is H_2-free and contains no (ψ, q)-scattering then $\chi(G) \leq \phi(\omega(G))$. Suppose not. By the choice of f, and since G contains no (ψ, q)-scattering, it follows that G contains a copy J' of J, such that there is a set $X \subseteq V(G)$ with $\chi(X) > \sigma(\omega(X))$ anticomplete to $V(J_1')$. But since $\chi(X) > \sigma(\omega(X))$, it follows that $G[X]$ contains J_1, and since J contains J_2, and $V(J)$ is anticomplete to X, it follows that G contains H_2. This proves 2.1. ∎

We remark that there is an appealing possible strengthening of 2.2, that we could not prove:

2.3 Conjecture: *Let $\psi, \sigma : \mathbb{N} \to \mathbb{N}$ be non-decreasing polynomials, let $q \geq 0$ an integer, and let H be a broom. Then there is a non-decreasing polynomial $\phi : \mathbb{N} \to \mathbb{N}$ such that if G is a graph, and $\chi(G) > \phi(\omega(G))$, and G contains no (ψ, q)-scattering, then G a σ-nondominating copy of H.*

Let us say a graph H is *self-isolating* if for every non-decreasing polynomial $\psi : \mathbb{N} \to \mathbb{N}$, there is a polynomial $\phi : \mathbb{N} \to \mathbb{N}$ with the following property: for every graph G with $\chi(G) > \phi(\omega(G))$, there exists $A \subseteq V(G)$ with $\chi(A) > \psi(\omega(A))$, such that either

- $G[A]$ is H-free, or
- G contains a copy H' of H such that $V(H')$ is disjoint from and anticomplete to A.

Which graphs are self-isolating? It is proved in [10] that stars are self-isolating, and we will show in [2] that complete graphs and complete bipartite graphs are self-isolating. Let us observe that 2.2 implies that:

2.4 *Every path is self-isolating.*

Proof. Let J be a path, and let $\psi : \mathbb{N} \to \mathbb{N}$ be a non-decreasing polynomial. Choose ϕ satisfying 2.2 with $H = J$ and $\sigma = \psi$ and $q = |J|$, and let G be a graph with $\chi(G) > \phi(\omega(G))$. We claim that either there is a ψ-nondominating copy of J in G, or there exists $A \subseteq V(G)$ with $\chi(A) > \psi(\omega(A))$ such that $G[A]$ is J-free. By 2.2 we may assume that there is a (ψ, q)-scattering (w, P, Q) in G. If

$G[P]$ is J-free, the claim holds, so we assume that there is a copy J' of J in $G[P]$. Thus $|J'| = q$. Let r be the maximum over $v \in P$ of the chromatic number of the set of neighbours of v in Q. The set of vertices in Q with a neighbour in $V(J')$ has chromatic number at most $|J'|r = qr$; and $\chi(Q) > \psi(\omega(Q)) + qr$ from the definition of a (ψ, q)-scattering. Consequently J' is ψ-nondominating, and hence J is self-isolating. This proves 2.4. ∎

3 Constructing a horn

Let $d \geq 0$ be an integer. If $A, B \subseteq V(G)$ are disjoint, we say that A is *d-dense* to B if for every vertex $v \in A$, the set of non-neighbours of v in B has chromatic number at most d. Let us say a (d, z)-*horn* in a graph G is a triple (v, A, B) where

- A, B are disjoint subsets of $V(G)$, and $v \in V(G) \setminus (A \cup B)$;
- v is complete to A and anticomplete to B; and
- there is no $Z \subseteq A \cup B$ with $\chi(Z) \leq z$ such that $A \setminus Z$ is d-dense to $B \setminus Z$.

We will need a (d, z)-horn (v, A, B) where z is at least some large function of the clique number of $A \cup B$, and this section produces such a horn. We will use the following well-known version of Ramsey's theorem, proved (for instance) in [10] ($|G|$ denotes the number of vertices of G):

3.1 *Let $x \geq 2$ and $y \geq 1$ be integers. For a graph G, if $|G| \geq x^y$, then G has either a clique of cardinality $x + 1$, or a stable set of cardinality y.*

If $v \in V(G)$, we denote by $N(v)$ or $N_G(v)$ the set of all neighbours of v in G. First, we need a result of Gyárfás [5] (we give the well-known proof, because it is so pretty.)

3.2 *Let $k \geq 1$ and $x \geq 0$ be integers. Let G be a connected graph such that $\chi(N(v)) \leq x$ for every vertex v. Let H be a connected induced subgraph of G, and let $v \in V(G) \setminus V(H)$ with a neighbour in $V(H)$. If $\chi(H) > (k-2)x$, there is an induced k-vertex path of G with one end v and all other vertices in $V(H)$.*

Proof. We proceed by induction on k. The result is clear for $k \leq 2$, so we assume that $k \geq 3$. Define J be obtained from H by deleting all vertices in $N(v)$; thus $\chi(J) > (k-3)x > 0$, and so there is a component H' of J with chromatic number more than $(k-3)x$. Let $v' \in N(v) \cap V(H)$ with a neighbour in $V(H')$. From the inductive hypothesis applied to v', H', there is an induced $(k-1)$-vertex path of G with one end v' and all other vertices in $V(H')$. Appending v to this path proves 3.2. ∎

We deduce:

3.3 *Let $\sigma : \mathbb{N} \to \mathbb{N}$ be non-decreasing, let $k, x \geq 1$ be integers, and let G be a graph. If $\chi(N(v)) \leq x$ for every $v \in V(G)$, and $\chi(G) > kx + \sigma(\omega(G))$, then there is a σ-nondominating k-vertex induced path P in G.*

Proof. We may assume that G is connected; choose $v \in V(G)$. Since $\chi(G \setminus v) > kx - 1 \geq (k-2)x$, 3.2 (applied to v and to a component of $G \setminus v$ of maximum chromatic number) implies that G contains a k-vertex induced path P. The set of vertices of G with a neighbour in $V(P)$ has chromatic number at most kx, and the result follows. This proves 3.3. ∎

The next result is also essentially due to Gyárfás (mentioned in [5]):

3.4 *Let H be a (k,s)-broom, and suppose that G is H-free, and $\chi(N(v)) \le x$ for every $v \in V(G)$. Then*

$$\chi(G) \le \max(\omega(G)^{2s}, (2s+1)(x+1) + (k-2)x).$$

Proof. Suppose that $\chi(G) > \max(\omega(G)^{2s}, (2s+1)(t+1) + (k-2)x)$. We may assume that G is connected. If every vertex of G has degree less than $\omega(G)^{2s}$ then $\chi(G) \le \omega(G)^{2s}$, a contradiction, so some vertex v has at least $\omega(G)^{2s}$ neighbours. By 3.1 applied to $G[N(v)]$, there is a stable set S of neighbours of v, with $|S| = 2s$. Let M be the set of all vertices of G that do not belong to $S \cup \{v\}$ and have a neighbour in $S \cup \{v\}$. Thus $\chi(M) \le (2s+1)x$. Let H be a component of $G \setminus (M \cup S \cup \{v\})$ of maximum chromatic number; then $\chi(H) \ge \chi(G) - (2s+1)(x+1) > (k-2)x$. Choose $u \in M \cup S \cup \{v\}$ with a neighbour in $V(H)$. By 3.2 applied to u, H, there is an induced k-vertex path P of G with one end u and all other vertices in $V(H)$. Thus u is the only vertex of P with a neighbour in $M \cup S \cup \{v\}$. If u is adjacent to at least s vertices in S, then the subgraph induced on $V(P)$ and some s of these neighbours is a (k,s)-broom, a contradiction. Thus there exists $S' \subseteq S$ with $|S'| = s$, such that all vertices in S' are nonadjacent to u. If u is adjacent to v, the subgraph induced on $V(P) \cup S \cup \{v\}$ is a $(k+1,s)$-broom, a contradiction. Thus u is adjacent to some $w \in S \setminus S'$, and nonadjacent to v. But then the subgraph induced on $V(P) \cup S \cup \{v,w\}$ is a $(k+2,s)$-broom, a contradiction. This proves 3.4. ∎

3.5 *Let $\sigma : \mathbb{N} \to \mathbb{N}$ be non-decreasing. Let $k,s,d,z \ge 0$ and $c \ge 2s$ be integers. Let G be a graph such that*

$$\chi(G) > \omega(G)^c;$$
$$\chi(G') \le \omega(G')^c \text{ for every induced subgraph } G' \text{ of } G \text{ with } G' \ne G;$$
$$\omega(G)^c \ge (\omega(G)-1)^c + z + d\omega(G) + 2;$$
$$\omega(G)^c \ge (2s+1)(z+1) + (k-2)z; \text{ and}$$
$$\omega(G)^c \ge kz + \sigma(\omega(G)).$$

Then either

- *G contains a (d,z)-horn; or*
- *G contains a (k,s)-broom, and a σ-nondominating k-vertex path.*

Proof. Suppose that $\chi(N(v)) \le z$ for every vertex $v \in V(G)$. By 3.4, and since

$$\chi(G) > \omega(G)^c \ge \max(\omega(G)^{2s}, (2s+1)(x+1) + (k-2)z)$$

(because $c \ge 2s$), it follows that G contains a (k,s)-broom. By 3.3, since

$$\chi(G) - kz > \sigma(\omega(G)) \ge \sigma(\omega(X)),$$

there is a σ-nondominating k-vertex induced path P in G, and so the second bullet holds.

Thus we assume that $\chi(N(v)) > z$ for some vertex v. Let A be the set of neighbours of v, and $B = V(G) \setminus (A \cup \{v\})$. We claim that (v,A,B) is a (d,z)-horn. Suppose not; then there exists

$Z \subseteq A \cup B$ with $\chi(Z) \le z$, such that $A \setminus Z$ is d-dense to $B \setminus Z$. Let $P \subseteq A \setminus Z$ be a clique with cardinality $p = \omega(A \setminus Z)$. Then $p \ge 1$, since $\chi(Z) \le z < \chi(A)$; and $p < \omega(G)$ since otherwise adding v would give a clique of cardinality $\omega(G)+1$. For each $u \in P$, the set of vertices in $B \setminus Z$ nonadjacent to u has chromatic number at most d, since $A \setminus Z$ is d-dense to $B \setminus Z$; and so the set of vertices in B with a non-neighbour in P has chromatic number at most $pd \le d\omega(G)$. The set of vertices in B complete to P has clique number at most $\omega(G) - p$ and so has chromatic number at most $(\omega - p)^c$. Hence $\chi(B \setminus Z) \le pd + (\omega(G) - p)^c$, and so

$$\chi(G) \le \chi(Z) + \chi(A \setminus Z) + \chi(B \setminus Z) + 1 \le z + p^c + d\omega(G) + (\omega(G) - p)^c + 1.$$

Since $1 \le p \le \omega(G) - 1$, $p^c + (\omega(G) - p)^c \le (\omega(G) - 1)^c + 1$, and so

$$\omega(G)^c < \chi(G) \le z + d\omega(G) + (\omega(G) - 1)^c + 2,$$

a contradiction. This proves 3.5. ∎

4 Making taller horns

In this section we prove 2.2, and hence complete the proofs of 2.1, 1.4, 1.3, and therefore 1.2. If $d, z, \omega \ge 0$ are integers, a graph G is (d, z, ω)-*unsplittable* if there is no partition (A, B, Z) of $V(G)$ such that $\chi(Z) \le z$, and $\chi(A), \chi(B) > d\omega$, and A is d-dense to B. We begin with:

4.1 *If $d, z \ge 0$ are integers, every graph G admits a partition $(D_0, D_1, \dots, D_k)$ of its vertex set with $k \le \omega(G)$ such that $\chi(D_0) \le z\omega(G)$ and $G[D_i]$ is $(d, z, \omega(G))$-unsplittable for $1 \le i \le k$.*

Proof. We may assume that G is not $(d, z, \omega(G))$-unsplittable, and so it admits a partition (D_0, D_1, D_2) such that $\chi(D_0) \le z$, $\chi(D_1), \chi(D_2) > d\omega(G)$, and D_1 is d-dense to D_2. Hence we may choose $k \ge 2$ maximum such that there is a sequence $D_0, D_1, \dots, D_k$ of pairwise disjoint subsets of $V(G)$ with union $V(G)$, and with the following properties:

- $\chi(D_0) \le (k-1)z$
- D_i is d-dense to D_j for $1 \le i < j \le k$; and
- $\chi(D_i) > d\omega(G)$ for $1 \le i \le k$.

We claim:

(1) $k \le \omega(G)$.

Suppose that $k > \omega(G)$, and define $d_i \in D_i$ for $1 \le i \le \omega(G) + 1$ inductively as follows. Let $1 \le i \le \omega(G) + 1$, and suppose that $d_1, \dots, d_{i-1}$ have been defined, all pairwise adjacent. The set of vertices in D_i that have a non-neighbour among $d_1, \dots, d_{i-1}$ has chromatic number at most

$$(i-1)d \le d\omega(G) < \chi(D_i),$$

and so some vertex $d_i \in D_i$ is adjacent to all of $d_1, \dots, d_{i-1}$. This completes the inductive definition. But then $\{d_1, \dots, d_{\omega(G)+1}\}$ is a clique of G, contradicting the definition of $\omega(G)$. This proves (1).

(2) *For* $1 \le i \le k$, $G[D_i]$ *is* $(d, z, \omega(G))$*-unsplittable.*

Suppose that (A, B, Z) is a partition of D_i such that $\chi(Z) \le z$, and $\chi(A), \chi(B) > d\omega(G)$, and A is d-dense to B. Then the sequence

$$(D_0 \cup Z, D_1, \dots, D_{i-1}, A, B, D_{i+1}, \dots, D_k)$$

contradicts the maximality of k. This proves (2).

From (1), (2), this proves 4.1. ∎

Let (v, A, B) be a (d, z)-horn in a graph G, and let $k \ge 1$ be an integer. We say that (v, A, B) is *k-tall* if there is an induced path P in G with k vertices, with one end v, such that $V(P) \setminus \{v\}$ is disjoint from and anticomplete to $A \cup B$. Thus every (d, z)-horn is 1-tall. We use 4.1 to prove a result which is the heart of the paper:

4.2 *Let* G *be a graph, let* $d, z, d', z', q \ge 0$ *be integers, and let* $\psi : \mathbb{N} \to \mathbb{N}$ *be non-decreasing, satisfying:*

$$z \ge \big(2\psi(\omega(G)) + (1+q)z' + qd'\omega(G)\big)\,\omega(G)$$
$$d \ge \big(z' + d'\omega(G)\big)\,\omega(G).$$

Let (v, A, B) *be an* ℓ*-tall* (d, z)*-horn in a graph* G*, for some* $\ell \ge 1$*. Then either*

- *there exist* $P \subseteq A$ *and* $Q \subseteq B$ *such that* (v, P, Q) *is a* (ψ, q)*-scattering; or*
- *there exist* $v' \in A$ *and disjoint subsets* A', B' *of* B *such that* (v', A', B') *is an* $(\ell + 1)$*-tall* (d', z')*-horn.*

Proof. Let $p = \psi(\omega(G))$. By 4.1, B admits a partition $(D_0, D_1, \dots, D_k)$ with $k \le \omega(G)$ such that $\chi(D_0) \le z'\omega(G)$ and $G[D_i]$ is $(d', z', \omega(G))$-unsplittable for $1 \le i \le k$. For $1 \le i \le k$, if $\chi(D_i) \le q(z' + d'\omega(G)) + p$ let $P_i = \emptyset$, and if $\chi(D_i) > q(z' + d'\omega(G)) + p$ let P_i be the set of vertices $a \in A$ such that $\chi(U) \le z' + d'\omega(G)$, where U is the set of neighbours of a in D_i. Let $P = P_1 \cup \cdots \cup P_k$. For $1 \le i \le k$, we may assume that $\chi(P_i) \le p$, for otherwise the first bullet of the theorem holds; and consequently $\chi(P) \le p\omega(G)$.

Let Z be the union of P, D_0, and all the sets D_i with $1 \le i \le k$ such that

$$\chi(D_i) \le q(z' + d'\omega(G)) + p.$$

Consequently

$$\chi(Z) \le 2p\omega(G) + z'\omega(G) + q(z' + d'\omega(G))\omega(G) \le z.$$

Since (v, A, B) is a (d, z)-horn, it follows that $A \setminus Z$ is not d-dense to $B \setminus Z$; and so there exists $v' \in A \setminus P$ such that the set of vertices in $B \setminus Z$ that are nonadjacent to v' has chromatic number more than d. Since $B \setminus Z$ is the union of the sets D_i with $\chi(D_i) > q(z' + d'\omega(G)) + p$, there exists $i \in \{1, \dots, k\}$ with $\chi(D_i) \ge q(z' + d'\omega(G)) + p$ such that the set B' of vertices in D_i nonadjacent to v' has chromatic number more than $d/\omega(G)$. Since $v' \notin P$, the set A' of neighbours of v' in D_i has chromatic number more than $d'\omega(G) + z'$.

Let $Z' \subseteq D_i$ with $\chi(Z') \leq z'$. Thus $\chi(A' \setminus Z') \geq \chi(A') - \chi(Z') \geq d'\omega(G)$; and $\chi(B' \setminus Z') \geq d/\omega(G) - z' \geq d'\omega(G)$. Since $G[D_i]$ is $(d', z', \omega(G))$-unsplittable, it follows that $A' \setminus Z'$ is not d'-dense to $B' \setminus Z'$. This proves that (v', A', B') is a (d', z')-horn.

Since (v, A, B) is ℓ-tall, there is an ℓ-vertex induced path P of G with one end v, such that $V(P) \setminus \{v\}$ is disjoint from and anticomplete to $A \cup B$. Then $P' = G[V(P) \cup \{v'\}]$ is an $(\ell+1)$-vertex path, and since $V(P)$ is anticomplete to B and hence to $A' \cup B'$, it follows that (v', A', B') is $(\ell+1)$-tall, and so the second bullet of the theorem holds. This proves 4.2. ∎

Now we prove 2.2, which we restate:

4.3 *Let $k, s \geq 1$ and $q \geq 0$ be integers, and let $\psi, \sigma : \mathbb{N} \to \mathbb{N}$ be non-decreasing polynomials. Then there exists an integer $c \geq 0$ such that if G is a graph with $\chi(G) > \omega(G)^c$, and G contains no (ψ, q)-scattering, then G contains a (k, s)-broom and a σ-nondominating k-vertex path.*

Proof. Let $\zeta_k : \mathbb{N} \to \mathbb{N}$ be the polynomial defined by $\zeta_k(x) = \sigma(x) + x^s$, and let $\delta_k(x) = 0$. For $i = k-1, \ldots, 1$, define polynomials $\zeta_i, \delta_i : \mathbb{N} \to \mathbb{N}$ by

$$\zeta_i(x) = 2x\psi(x) + (1+q)x\zeta_{i+1}(x) + x\delta_{i+1}(x)$$
$$\delta_i(x) = x\zeta_{i+1}(x) + x^2\delta_{i+1}(x).$$

Choose an integer $c \geq 2s$ such that

$$x^c \geq (x-1)^c + \zeta_1(x) + x\delta_1(x) + 2$$
$$x^c \geq (2s+1)(\zeta_1(x)+1) + (k-2)\zeta_1(x), \text{ and}$$
$$x^c \geq k\zeta_1(x) + \sigma(x)$$

for all integers $x \geq 2$. We claim that c satisfies 4.3. To see this, let G be a graph with $\chi(G) > \omega(G)^c$, and suppose that G contains no (ψ, q)-scattering. We must show that G contains a (k, s)-broom and a σ-nondominating k-vertex path. We show this by induction on $|G|$. If there is an induced subgraph G' of G with $G' \neq G$ and $\chi(G') > \omega(G')^c$, then G' contains no (ψ, q)-scattering, and from the inductive hypothesis, G' contains a (k, s)-broom and a σ-nondominating k-vertex path, and hence so does G, as required. We may assume then that there is no such G'. Since $\chi(G) > \omega(G)^c$, it follows that $\omega(G) \geq 2$, and so the five displayed inequalities of 3.5 hold with z, d replaced by $\zeta_1(\omega(G)), \delta_1(\omega(G))$ respectively. From 3.5, we may assume that G contains a $(\delta_1(\omega(G)), \zeta_1(\omega(G)))$-horn, which is therefore 1-tall.

From 4.2, it follows that for $i = 2, \ldots, k$, G contains an i-tall $(\delta_i(\omega(G)), \zeta_i(\omega(G)))$-horn, and so contains a k-tall $(0, z)$-horn (v, A, B) say, where $z = \zeta_k(\omega(G))$. Since this horn is k-tall, there is a k-vertex induced path P of G with one end v, such that $V(P) \setminus \{v\}$ is disjoint from and anticomplete to $A \cup B$. From the definition of a $(0, z)$-horn, $\chi(A), \chi(B) > z$. Since $\chi(A) > z \geq \omega(A)^s$, 3.1 implies that there is a stable set $S \subseteq A$ with $|S| = s$, and so $G[V(P) \cup S]$ is a (k, s)-broom. Since $\chi(B) > z > \sigma(\omega(B))$, and $V(P)$ is anticomplete to B, P is σ-nondominating. This proves 4.3. ∎

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