Judicious partitions of hypergraphs

B. Bollobás and A.D. Scott

Department of Pure Mathematics
and Mathematical Statistics,
University of Cambridge,
16 Mill Lane, Cambridge, CB2 1SB, England.

Abstract. We prove the asymptotically best possible result that, for every integer $k \geq 2$, every 3-uniform graph with m edges has a vertex-partition into k sets such that each set contains at most $(1+o(1))m/k^3$ edges. We also consider related problems and conjecture a more general result.

§1. Introduction and results.

Given a graph G with m edges and an integer $k \geq 2$, it is easy to see that there is some vertex-partition of G into k sets such that at most m/k edges of G have both vertices in the same set (consider a random partition). Equivalently, G contains a k-partite subgraph with at least $km/(k-1)$ edges. In general, this is close to best possible, as can be seen by considering complete graphs; however, a number of authors have given more precise bounds in terms of the order and size of G. Edwards ([8], [9]), improving upon a result of Erdős (see [10], [11]), proved the best possible result that every graph of order n and size m contains a bipartite subgraph with at least $m/2 + (n-1)/4$ edges. Recently first Erdős, Gyárfás and Kohayakawa [13] and then Alon [2] found considerably simpler proofs and extensions of this result. Andersen, Grant and Linial [1] and Erdős, Faudree, Pach and Spencer [12] have given lower bounds for the maximal size of a k-partite subgraph. The problem of determining the maximal size of a k-partite subgraph is NP-complete (see [14]) and has led to a great deal of work on partitioning algorithms.

A large k-partite subgraph corresponds to a partition into k sets such that the total number of edges contained in the sets is small. However, what happens if we want the number of edges inside *each* set to be small? Thus instead of minimizing one quantity we now seek to minimize k quantities simultaneously. In a random partition into k sets, we expect m/k^2 edges inside each set; we cannot in general demand less than this in every set, since any partition of K_n into k sets has at least $(1+o(1))\binom{n}{2}/k^2$ edges in some set. It was proved in [6] that we can always find a partition with at most about twice m/k^2 edges in each set: every graph G has a vertex-partition into k sets with at most

$$\frac{2}{k(k+1)}m$$

edges in each set. This result is best possible for every $k \geq 2$, as can be seen by considering the complete graph K_{k+1}. However, for fixed k, the bound is good only for small graphs. For large graphs we can get much closer to m/k^2. Indeed, we can demand a partition into k sets with at most

$$\frac{m}{k^2} + cm^{4/5}(\log k)^{2/5}$$

edges in each set. Other bounds are given in [6]: for example, if $\Delta(G) = o(m)$, then there is a partition with $(1+o(1))m/k^2$ edges inside each set and $(1+o(1))2m/k^2$ edges between each pair of sets. The analogous problem for weighted sets was considered by Kató Rényi; a simpler proof was subsequently found by van Lint [18]. Let S be a set with weight function (ie a measure) $w : S \to \mathbb{R}_{\geq 0}$, where $w(S) = 1$. We then ask for the minimum of $\max\{w(X), w(Y)\}$ over partitions $S = X \cup Y$: van Lint gives a best possible bound in terms of the maximum weight $\Delta(S) = \max_{s\in S}\{w(s)\}$.

In this paper we turn to the related problem of finding vertex-partitions of *hypergraphs*, such that each set contains few edges. Let G be an r-uniform hypergraph with m edges. In a random vertex-partition of G into k sets we expect to have m/k^r edges in each set; by considering large complete r-graphs, we see that we cannot demand that every set contain less than m/k^r edges. The results quoted above show that this bound can be achieved asymptotically for graphs ($r = 2$), while van Lint's result gives a precise bound for the problem for weighted sets ($r = 1$). Our main aim here is to prove that the bound m/k^r can be achieved asymptotically in the case $r = 3$. As expected, this problem is much harder for hypergraphs than for graphs.

We shall use a combination of random methods and extremal combinatorics. The idea is always to consider a random k-colouring of the vertices of a hypergraph G and make use of a martingale inequality to show that there is a colouring in which every colour class contains few edges. However, as we shall see (Lemma 7), a straightforward colouring gives us an upper bound of

$$\frac{m}{k^r} + \left(\frac{1}{2}\sum_{v\in V(G)} d(v)^2 \log k\right)^{1/2}$$

edges in each colour class. This bound is only good if the maximum degree of G is small. Thus most of our work will be devoted to dealing with the vertices of large degree. To this end, we seek to colour the vertices of large degree in advance and deterministically in such a way that, extending the colouring randomly to the whole graph, we expect to have at most m/k^r edges in each colour class. Since the randomly coloured vertices do not have large degrees, the martingale inequality then shows that there is a colouring in which we are close to the expectation in every colour class. In fact, it turns out to be easiest to choose one colour class at a time.

Our main problem then is to find an appropriate 2-colouring of the vertices of large degree and to specify how the colouring is to be extended randomly. We use a partitioning result for graphs (Lemma 5) to partition the vertices of large degree; the essential difficulty lies in showing that this colouring can be extended randomly in such a fashion that the expected number of edges in each colour class is not too large. This reduces to the problem of maximizing a certain function over a polyhedron, which we achieve by extremal combinatorial arguments (Lemma 4).

We shall deduce our main result from the following more general theorem about partitions of 3-uniform hypergraphs.

Theorem 1. *Let G be a 3-uniform hypergraph with m edges, let $k \geq 2$ be an integer and let $p_1, ...p_k$ be non-negative reals such that $\sum_{i=1}^{k} p_i = 1$. Then there is a partition $V(G) = \bigcup_{i=1}^{k} V_i$ such that, for $i = 1, \ldots, k$,*

$$e(V_i) \leq p_i^3 m + 5m^{6/7}(\log k)^{1/2}.$$

and

$$e(\bigcup_{j=1}^{i} V_j) \leq (\sum_{j=1}^{i} p_j)^3 m + 5m^{6/7}(\log k)^{1/2}.$$

□

Setting $p_1 = \cdots = p_k = 1/k$, we obtain the following result as an immediate corollary.

Corollary 2. *Let G be a 3-uniform hypergraph with m edges, and let $k \geq 2$ be an integer. Then there is a vertex-partition of G into k sets with at most*

$$\frac{m}{k^3} + 5m^{6/7}(\log k)^{1/2}$$

edges in each set. $\square$

Theorem 1 will be proved by repeated application of the following lemma, which enables us to split off one colour class at a time.

Lemma 3. *Let G be a 3-uniform hypergraph with m edges, and let $p_1 \in [0,1]$ and $p_2 = 1 - p_1$. There is a partition $V(G) = V_1 \cup V_2$ such that, for $i = 1, 2$,*

$$e(V_i) \leq p_i^3 m + 5m^{6/7}.$$

$\square$

The engine of our proof of Lemma 3 is the following lemma, which may at first sight seem rather artificial. However, it emerges very naturally from the proof of Lemma 3. Although the result holds for every $k \geq 1$, we are interested mainly in the case when k is an integer.

Lemma 4. *Let a, b, x, y, z, e be non-negative reals and let $k \geq 1$ be a real, such that*

$$z \geq \max\left\{2(k-1)x, \frac{2}{k-1}y\right\} \tag{1}$$

and

$$a + b + x + y + z + e = 1. \tag{2}$$

Then there is some $p \in (0,1)$ such that

$$f_1(a, b, x, y, z, e, p) := p^2 a + px + p^3 e \leq \frac{1}{k^3}$$

and

$$f_2(a, b, x, y, z, e, p) := (1-p)^2 b + (1-p)y + (1-p)^3 e \leq \left(\frac{k-1}{k}\right)^3.$$

$\square$

We shall also need a lemma about partitions of graphs. For a graph G with vertex set V and edge-weighting $w : E(G) \to \mathbb{R}_{\geq 0}$, and disjoint sets $X, Y \subset V$, we write

$$w(X) = \sum_{\{x,y\} \in V^{(2)}} w(xy)$$

and

$$w(X, Y) = \sum_{x \in X, y \in Y} w(xy),$$

where we take $w(xy) = 0$ if $xy \notin E(G)$. The following simple lemma is related to Theorem 1 from [6]; as the proof is straightforward, we give it here.

Lemma 5. *Let G be a graph with edge-weighting w, let $\lambda > 0$ and let $V(G) = V_1 \cup V_2$ be a partition minimizing $w(V_1) + \lambda w(V_2)$. Then*

$$w(V_1, V_2) \geq \max\left\{\frac{2}{\lambda} w(V_1), 2\lambda w(V_2)\right\}.$$

Proof. Let $V(G) = V_1 \cup V_2$ be a partition minimizing $w(V_1) + \lambda w(V_2)$. Then for every $v \in V_1$ we have

$$\sum_{u \in V_1 \setminus \{v\}} w(vu) \leq \lambda \sum_{u \in V_2} w(vu)$$

or else moving v from V_1 to V_2 would give a better partition. Summing over $u \in V_1$ we get

$$2w(V_1) \leq \lambda w(V_1, V_2)$$

and so

$$w(V_1, V_2) \geq \frac{2}{\lambda} w(V_1).$$

Similarly, for $v \in V_2$, we have

$$\lambda \sum_{u \in V_2 \setminus \{v\}} w(vu) \leq \sum_{u \in V_1} w(vu).$$

Summing over $u \in V_2$, we get

$$w(V_1, V_2) \geq 2\lambda w(V_2).$$

$\square$

We shall also make use of the following immediate consequence of the Azuma-Hoeffding inequality ([3], [15], see also [4], [5], [16], [17]; this is used similarly to Theorem 2 from [6]).

Lemma 6. *Let $X_1, \ldots, X_n$ be independent random variables taking values in $[k] = \{1, \ldots, k\}$, and let $X = (X_1, \ldots, X_n)$. Suppose $f \colon [k]^n \to \mathbb{N}$ satisfies*

$$|f(Y) - f(Y')| \leq d_i$$

whenever the vectors Y and Y' differ only in the ith coordinate. Then for any $t > 0$,

$$\mathbb{P}\left(f(X) - \mathbb{E}(f(X)) \geq t\right) \leq \exp\left(-2t^2 / \sum_{i=1}^{n} d_i^2\right)$$

and

$$\mathbb{P}\left(f(X) - \mathbb{E}(f(X)) \leq -t\right) \leq \exp\left(-2t^2 / \sum_{i=1}^{n} d_i^2\right).$$

$\square$

We shall take X to be a random vertex-colouring with k colours of a hypergraph G and f to be a function defined on the set of k-colourings of $V(G)$. In our applications of Lemma 6, we shall begin with a partial vertex-colouring c of a hypergraph G and extend c randomly to a k-colouring of G: thus X_i will be the constant $c(v_i)$ for $v_i \in V_1$, the set of vertices we have coloured, and an element of $[k]$ (with appropriate distribution) for $v_i \in V \setminus V_1$.

We remark that a weak bound for partitions of a hypergraph follows immediately from Lemma 6.

Lemma 7. *Let G be an r-uniform hypergraph with degree sequence $d_1, \ldots, d_n$ and let $k \geq 2$ be an integer. Then there is a vertex-partition of G into k sets such that each set contains at most*

$$\frac{m}{k^r} + \left(\frac{1}{2} \sum_{i=1}^{n} d_i^2 \log k \right)^{1/2}$$

edges.

Proof. Suppose $V(G) = \{v_1, \ldots, v_n\}$, where $d(v_i) = d_i$. Let $X_1, \ldots, X_n$ be independent random variables, each with uniform distribution on $\{1, \ldots, k\}$ and, for $j = 1, \ldots, k$, let f_j be the number of edges of G contained in the vertex set $\{v_i : X_i = j\}$. Now $\mathbb{E} f_j = m/k^r$, and changing the value of X_i changes f_j by at most d_i. Since f_j is integer-valued, it follows from Lemma 6 that

$$\mathbb{P}\left(f_j > \frac{m}{k^r} + (\frac{1}{2} \sum_{i=1}^{n} d_i^2 \log k)^{1/2} \right) < \frac{1}{k}.$$

Thus there must be some choice of $X_1, \ldots, X_n$ such that $f_j \leq \frac{m}{k^r} + (\frac{1}{2} \sum d_i^2 \log k)^{1/2}$ for $j = 1, \ldots, k$; this corresponds to a partition of $V(G)$ into k sets such that each set satisfies the condition of the lemma. □

We remark that it is easy to prove a similar result showing that the number of edges contained in the union of any j sets differs from the expectation by at most some error term. However, the bound in Lemma 7 can be far worse than the bound in Theorem 1; in particular, Lemma 7 does not give a bound of form $(1 + o(1))m/k^3$.

§2. Proof of Lemma 4.

Let us note first that we may assume $a + x + e > 0$ and $b + y + e > 0$, or else one of $p = \epsilon$ and $p = 1 - \epsilon$ will satisfy the assertion of the theorem for sufficiently small $\epsilon > 0$. Thus we may assume that both f_1 and f_2 are strictly increasing in p. Given a, b, x, y, z, e, let us pick p such that

$$(k-1)^3 f_1 = f_2. \tag{3}$$

It is enough to show that, for any choice of constants satisfying (1) and (2), picking $p \in (0,1)$ such that (3) is satisfied we get $f_1 \leq 1/k^3$.

Now let a, b, x, y, z, e, p be chosen to satisfy (1), (2) and (3); we shall call (a, b, x, y, z, e, p) a *satisfying point*. A satisfying point with f_1 maximal is called a *maximal point*. (Note that this maximum clearly exists and is finite, since we are maximising f_1 over a compact set, namely the subpolyhedron of $[0,1]^7$ given by (1), (2) and (3).) We shall suppose that (a, b, x, y, z, e, p) is, in fact, a maximal point.

If $e > 0$, then $(a + pe, b + (1-p)e, x, y, z, 0, p)$ has the same values for f_1 and f_2, so is also a maximal point; thus we may choose a maximal point with $e = 0$. Furthermore if $u = \max\{2(k-1)x, \frac{2}{k-1}y\} < z$ then, replacing a by $a' = a + z - u$ and z by $z' = u$, and picking p' such that (a', b, x, y, z, e, p') is a satisfying point, we obtain a larger value of f_1, contradicting the maximality of (a, b, x, y, z, e, p). Thus we may assume that $z = \max\{2(k-1)x, \frac{2}{k-1}y\}$.

If $a > 0$ and $b > 0$ then let $\epsilon = \min\{pa, (1-p)b/(k-1)^2\}$. The point

$$\left(a - \frac{\epsilon}{p}, b - \frac{(k-1)^2\epsilon}{1-p}, x + \epsilon, y + (k-1)^2\epsilon, z + 2(k-1)\epsilon, e, p\right)$$

still satisfies (1) and (3); we claim that it also satisfies (2), since

$$\epsilon + (k-1)^2\epsilon + 2(k-1)\epsilon \leq \frac{\epsilon}{p} + \frac{(k-1)^2\epsilon}{1-p}, \tag{4}$$

that is

$$k^2 \leq \frac{1}{p} + \frac{(k-1)^2}{1-p},$$

which is easily seen to hold for $p \in (0,1)$, and with equality when $p = \frac{1}{k}$. If (4) is not satisfied with equality then we may increase a and b (until (2) is satisfied) to obtain a satisfying point with larger f_1, contradicting maximality; thus (4) must hold with equality and so the new sequence satisfies (2). Therefore we may find a maximal point such that $a = 0$ or $b = 0$.

We have remarked that we may assume that $z = \max\{2(k-1)x, \frac{2}{k-1}y\}$. If $z > \frac{2}{k-1}y$ then let $\epsilon = \min\{\frac{k-1}{2}z - y, b\}$. Picking p' such that $(a, b-\epsilon, x, y+\epsilon, z, e, p')$ is a maximal point, we see that we must have $\epsilon = 0$, by the maximality of f_1. Thus we must have either $b = 0$ or $z = \frac{2}{k-1}y$. Similarly, we must have either $a = 0$ or $z = 2(k-1)x$.

Putting these observations together we see that we must be in one of the following cases:

Case 1. $a = 0$, $z = \frac{2}{k-1}y$, $e = 0$.

Case 2. $b = 0$, $z = 2(k-1)x$, $e = 0$.

Case 3. $a = b = e = 0$, $z = \frac{2}{k-1}y$.

Case 4. $a = b = e = 0$, $z = 2(k-1)x$.

We proceed by examining each case in turn, and two subcases. We begin, in the first two cases, by using Lagrange multipliers (see for instance [7], Chapter III, §6) to show that, if (a, b, x, y, z, e, p) is a maximal point then we may find a maximal point such that some additional constraints are satisfied. Let us note that, at the maximum, we do not have $p = 0$ or $p = 1$.

Case 1. $a = 0$, $z = \frac{2}{k-1}y$, $e = 0$. Let $y_0 = y/(k-1)^2$. Then $x = 1 - b - y - z = 1 - b - (k^2-1)y_0$. So

$$f_1 = xp = (1 - b - (k^2-1)y_0)p$$

and

$$f_2 = y(1-p) + b(1-p)^2 = (k-1)^2 y_0(1-p) + b(1-p)^2.$$

If (a, b, x, y, z, e, p) is a maximal point then it is a maximum for f_1 subject to $g := (k-1)^3 f_1 - f_2 = 0$, where we consider both f_1 and g as functions of b, y_0

and p. Note that $\frac{\partial g}{\partial b} \neq 0$. Now, using Lagrange multipliers (we maximise $f_1 - \lambda g$ and choose λ suitably), we have $\frac{\partial f_1}{\partial b} = \lambda \frac{\partial g}{\partial b}$, so

$$-p = \lambda \left((k-1)^3(-p) - (1-p)^2 \right).$$

Since $\frac{\partial f_1}{\partial y_0} = \lambda \frac{\partial g}{\partial y_0}$, we have

$$-(k^2-1)p = \lambda \left(-(k-1)^3(k^2-1)p - (k-1)^2(1-p) \right).$$

Since $p \neq 0$ and so $\lambda \neq 0$, we get

$$-(k-1)^3(k^2-1)p - (k-1)^2(1-p) = -(k^2-1)(k-1)^3 p - (k^2-1)(1-p)^2.$$

Simplifying gives

$$p = \frac{2}{k+1}$$

and so

$$1 - p = \frac{k-1}{k+1}.$$

Now let $\epsilon = b/(k^2-1)$. The sequence $(0, 0, x, y + (k-1)^2\epsilon, z + (2k-2)\epsilon, 0, p)$ leaves f_1 unchanged, and f_2 is also unchanged, since

$$\begin{aligned}
(1-p)(y + (k-1)^2\epsilon) &= (1-p)y + \frac{k-1}{k+1}(k-1)^2 \frac{b}{k^2-1} \\
&= (1-p)y + \left(\frac{k-1}{k+1} \right)^2 b \\
&= (1-p)y + (1-p)^2 b.
\end{aligned}$$

Thus there is either a maximal point with $b = 0$ (Case 3), or some additional constraint is satisfied with equality: we have $z = 2(k-1)x$ (Case 5).

Case 2. $b = 0$, $z = 2(k-1)x$, $e = 0$. We have $y = 1 - a - x - z = 1 - a - (2k-1)x$, so

$$f_1 = px + p^2 a$$

and

$$f_2 = (1-p)y = (1-p)(1 - a - (2k-1)x).$$

As before, we set $g = (k-1)^3 f_1 - f_2$. We are maximising f_1 subject to $g = 0$. We consider f_1 and g as functions of x, a and p; note that $\frac{\partial g}{\partial x} \neq 0$. Now, using Lagrange multipliers, $\frac{\partial f_1}{\partial x} = \lambda \frac{\partial g}{\partial x}$ gives

$$p = \lambda \left((k-1)^3 p + (2k-1)(1-p) \right)$$

and $\frac{\partial f_1}{\partial a} = \lambda \frac{\partial g}{\partial a}$ gives

$$p^2 = \lambda \left((k-1)^3 p^2 + (1-p) \right).$$

Since $p \neq 0$, we have $\lambda \neq 0$ and so

$$(k-1)^3 p^2 + (1-p) = p((k-1)^3 p + (2k-1)(1-p)),$$

which gives

$$p = \frac{1}{2k-1}.$$

Now let $\epsilon = a/(2k-1)$. The sequence $(0, 0, x+\epsilon, y, z+2(k-1)\epsilon, 0, p)$ has the same values of f_1 and f_2 as $(a, 0, x, y, z, 0, p)$. Thus we may assume that either $a = 0$ (Case 4) or some other constraint is satisfied with equality: $z = \frac{2}{k-1} y$ (Case 6).

The remaining four cases, which are quite similar to each other, are all quite straightforward: in each case, we show that under a given set of conditions on the parameters a, b, e, x, y, there is a choice of p that satisfies the assertions of the lemma.

Case 3. $a = b = e = 0$, $z = \frac{2}{k-1} y$. Then

$$f_1 = px = p(1-y-z) = p\left(1 - \frac{k+1}{k-1} y\right)$$

and

$$f_2 = (1-p)y.$$

Clearly $y > 0$. Now if $f_2 > (k-1)^3/k^3$ then $1 - p > (k-1)^3/k^3 y$, so

$$p < 1 - \frac{(k-1)^3}{k^3 y}.$$

We shall show that, in this case, we cannot have $(k-1)^3 f_1 = f_2$. It suffices to show that $f_1 \le 1/k^3$, which would follow from

$$\left(1 - \frac{(k-1)^3}{k^3 y}\right)\left(1 - \frac{k+1}{k-1}y\right) \le \frac{1}{k^3},$$

which is equivalent to

$$(k^3 y - (k-1)^3)(k - 1 - (k+1)y) - (k-1)y \le 0.$$

Now setting $y = (k-1)^2/k^2$ gives equality; differentiating with respect to y gives a maximum at

$$y = \frac{(k-1)^2(2k+1)}{2k^2(k+1)} < \frac{(k-1)^2}{k^2},$$

so it is enough to show that $y \ge (k-1)^2/k^2$. Since $x \le z/2(k-1) = y/(k-1)^2$, we have

$$1 = x + z + y \le \frac{y}{(k-1)^2} + \frac{2y}{k-1} + y = \frac{k^2}{(k-1)^2}y,$$

so $y \ge (k-1)^2/k^2$.

Case 4. $a = b = e = 0$, $z = 2(k-1)x$. Then

$$f_1 = px$$

and

$$f_2 = (1-p)(1-(2k-1)x).$$

As in Case 3, if $f_1 > 1/k^3$ then $p > 1/k^3 x$, so $1 - p < 1 - (1/k^3 x)$, and it is therefore enough to show

$$\left(1 - \frac{1}{k^3 x}\right)(1 - (2k-1)x) \le \frac{(k-1)^3}{k^3};$$

in other words

$$(k^3 x - 1)(1 - (2k-1)x) - (k-1)^3 x \le 0.$$

This is satisfied with equality when $x = 1/k^2$ and, since $k \geq 1$, the left hand side has a maximum when

$$x = \frac{3k-1}{2k^2(2k+1)} < \frac{1}{k^2}.$$

Thus it suffices to show $x \geq 1/k^2$, which follows from $z = 2(k-1)x$, $y \leq (k-1)z/2 = (k-1)^2 x$ and $x + z + y = 1$.

Case 5. $a = e = 0$, $z = \frac{2}{k-1} y = 2(k-1)x$. Then $b = 1 - k^2 x$, so

$$f_1 = px$$

and

$$f_2 = (1-p)^2(1-k^2x) + (1-p)(k-1)^2 x.$$

Arguing as above, if $f_1 > 1/k^3$ then $p > 1/k^3 x$, so $1 - p < 1 - (1/k^3 x)$. It is enough to show that

$$\left(1 - \frac{1}{k^3 x}\right)^2 (1 - k^2 x) + \left(1 - \frac{1}{k^3 x}\right)(k-1)^2 x \leq \frac{(k-1)^3}{k^3}.$$

Multiplying by $k^6 x^2$ and simplifying, this is equivalent to

$$(k^2 x - 1)^2 (1 + k^2 x - 2k^3 x) \leq 0,$$

which holds for

$$x \geq \frac{1}{k^2(2k-1)}.$$

Otherwise, since $k^2(2k-1) > k^3$, we have $x < 1/k^3$, so we could not have had $f_1 > 1/k^3$.

Case 6. $b = e = 0$, $z = \frac{2}{k-1} y = 2(k-1)x$. Then $a = 1 - k^2 x$, so

$$f_1 = p^2(1 - k^2 x) + px$$

and

$$f_2 = (1-p)(k-1)^2 x.$$

If $f_2 > (k-1)^3/k^3$ then $1 - p > (k-1)/k^3 x$, so

$$p < 1 - \frac{k-1}{k^3 x}.$$

It is therefore enough to show that

$$\left(1 - \frac{k-1}{k^3 x}\right)^2 (1 - k^2 x) + \left(1 - \frac{k-1}{k^3 x}\right) \leq \frac{1}{k^3}.$$

Multiplying by $k^6 x^2$ and simplifying, this is equivalent to

$$-(k-1)(k^3 x + k^2 x - k - 1)(k^2 x - 1)^2 \leq 0,$$

which is true for $x \leq 1/k^2$. However, $x \leq 1/k^2$ follows easily from the conditions on a, x, y, z. $\square$

§3. Proof of Lemma 3.

Let G be a 3-uniform graph with m edges. Let $p_1 = 1 - p_2$ be as in the statement of the lemma; we may assume $p_1 \neq 0, 1$, or the lemma is trivial. Let $c = (648 \ln 2/m^5)^{1/7}$, let V_1 be the set of $\lfloor cm \rfloor$ vertices of highest degree (note that $cm \leq n$), and let $V_2 = V(G) \setminus V_1$. For $x, y \in V_1$, let

$$w_v(x) = |\{e \in E(G) : e \cap V_1 = \{x\}\}|/m$$

and

$$w_e(x, y) = |\{e \in E(G) : e \cap V_1 = \{x, y\}\}|/m.$$

Thus $w_v(x)$ is the proportion of edges in G that meet V_1 in $\{x\}$ and $w_e(x, y)$ is the proportion of edges that meet V_1 in $\{x, y\}$.

Let $\alpha = 1/p_1$. Given the edge-weighting w_e of the complete graph on V_1, it follows from Lemma 5 that we can find some partition $V_1 = X \cup Y$ such that

$$w_e(X, Y) \geq \max\{2(\alpha - 1) w_e(X), \frac{2}{\alpha - 1} w_e(Y)\}.$$

Let us pick such a partition, and define

$$a = w_v(X)$$
$$b = w_v(Y)$$
$$x = w_e(X)$$
$$y = w_e(Y)$$
$$z = w_e(X, Y)$$

and

$$e = 1 - a - b - x - y - z.$$

Now colour X with colour 1 and Y with colour 2, and extend the colouring to $V = V(G)$, where each vertex in V_2 is independently coloured 1 with probability p and 2 with probability $1 - p$. (More formally, let $V_2 = \{v_1, \ldots, v_s\}$ and let $X_1, \ldots, X_s$ be independent random variables with $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 2) = p$.) Since $em \geq e(V_2)$, we expect to have at most

$$e(X) + (p^2 a + px + p^3 e)m$$

monochromatic edges of colour 1 and

$$e(Y) + ((1-p)^2 b + (1-p)y + (1-p)^3 e)m$$

monochromatic edges of colour 2. We note that $\max\{e(X), e(Y)\} \leq \binom{|V_1|}{3} \leq (cm)^3/6$.

Letting f_i be the number of monochromatic edges of colour i, it follows from Lemma 4 that we can pick p such that

$$\mathbb{E}(f_1) \leq e(X) + \frac{m}{\alpha^3} = e(X) + p_1^3 m$$

and

$$\mathbb{E}(f_2) \leq e(Y) + m\left(\frac{\alpha - 1}{\alpha}\right)^3 = e(Y) + p_2^3 m.$$

Now changing the colour of a vertex v_j (ie changing the value of X_j) changes f_1 and f_2 by at most $d(v_j)$. It follows from Lemma 6 that, for $i = 1, 2$,

$$\mathbb{P}(f_i > \mathbb{E}f_i + t) \leq \exp(-t^2/2 \sum_{v \in V_2} d(v)^2).$$

Since $|V_i| = \lfloor cm \rfloor$, we have $d(v) \leq 3/c$ for $v \in V_2$. Thus

$$\sum_{v \in V_2} d(v)^2 \leq (3m/(3/c))(3/c)^2 = 9m/c.$$

Taking $t = (18m \ln 2/c)^{1/2}$, we have

$$P(f_1 > \mathbb{E}f_1 + t \text{ or } f_2 > \mathbb{E}f_2 + t) < 1,$$

so there must be some colouring with at most

$$p_i^3 m + \max\{e(X), e(Y)\} + t \leq p_i^3 m + \frac{(cm)^3}{6} + (18m \log 2/c)^{1/2}$$

monochromatic edges in the ith colour, $i = 1, 2$. Setting $c = (648 \log 2/m^5)^{1/7}$, we see that that we can demand that f_1 and f_2 exceed $p_1^3 m$ and $p_2^3 m$ respectively by at most

$$6m^{6/7}(\log 2)^{-3/7} 648^{-1/14} < 5m^{6/7}.$$

□

§4. Proof of Theorem 1.

Let V_1, V_2, w_v, w_e and e be as in the proof of Lemma 3. We show that we can find a partition $V_1 = \bigcup_{i=1}^k W_i$ and $q_1, \ldots, q_k \in [0, 1]$ such that $\sum_{i=1}^k q_i = 1$ and, for $i = 1, \ldots, k$,

$$q_i^2 w_v(W_i) + q_i w_e(W_i) + q_i^3 e \leq p_i^3.$$

The theorem then follows by an easy calculation similar to that in the proof of Lemma 3.

For $k = 2$ the result was proved in the proof of Lemma 3. Let us assume that $k > 2$ and the result holds for $k-1$. We proceed by picking the vertex classes one at a time. We can find a partition $V_1 = X \cup Y$ and $q_k \in [0,1]$ such that

$$q_k^2 w_v(X) + q_k w_e(X) + q_k^3 e \leq p_k^3$$

and

$$(1-q_k)^2 w_v(Y) + (1-q_k) w_e(Y) + (1-q_k)^3 e \leq (1-p_k)^3.$$

Define, for $S \subset Y$,

$$\begin{aligned} w_v^*(S) &= (1-q_k)^2 w_v(S)/(1-p_k)^3 \\ w_e^*(S) &= (1-q_k) w_e(S)/(1-p_k)^3 \\ e^* &= (1-q_k)^3 e/(1-p_k)^3 \end{aligned}$$

and, for $i = 1, \ldots, k-1$,

$$r_i = p_i/(1-p_k).$$

Then

$$w_v^*(Y) + w_e^*(Y) + e \leq 1$$

and

$$p_1^* + \ldots + p_{k-1}^* = 1.$$

By the inductive hypothesis, we can find some partition $Y = W_1 \cup \cdots \cup W_{k-1}$ and non-negative reals $s_1, \ldots, s_{k-1}$ such that $s_1 + \cdots + s_{k-1} = 1$ and, for $i = 1, \ldots, k-1$,

$$s_i^2 w_v^*(W_i) + s_i w_e^*(V_i) + s_i^3 e^* \leq r_i^3.$$

Now consider the partition $V_i = W_1 \cup \cdots \cup W_k$ and, for $i = 1, \ldots, k-1$, set $q_i = s_i(1-q_k)$. Note that $\sum_{i=1}^k q_i = 1$ and $q_k^2 w_v(W_k) + q_k w_e(W_k) + q_k^3 e \leq p_k^3$. Then, for $i < k$,

$$\begin{aligned} q_i^2 w_v(W_i) + q_i w_e(W_i) + q_i^3 e &= s_i^2(1-q_k)^2 w_v(W_i) + s_i(1-q_k) w_e(W_i) \\ &\quad + s_i^3(1-q_k)^3 e \\ &= (1-p_k)^3 (s_i^2 w_v^*(W_i) + s_i w_e^*(W_i) + s_i^3 e^*) \\ &\leq (1-p_k)^3 r_i^3 \\ &= p_i^3. \end{aligned}$$

$\square$

Perhaps a similar argument could be used to deal with r-uniform graphs for $r > 3$. However, the proof of Lemma 3 relied on a good partitioning result for graphs (Lemma 5) for which we do not yet have any good hypergraph generalisation.

§5. Conclusion and open problems.

It seems likely that Theorem 1 should hold in general for hypergraphs. Indeed, we make the following conjecture.

Conjecture 8. *Let $r \geq 3$ and $k \geq 2$ be fixed integers. Then every 3-uniform hypergraph with m edges has a vertex-partition into k sets with at most*

$$\frac{m}{k^r} + o(m)$$

edges in each set.

In fact the $o(m)$ error term may not be needed. For the case $k = 2$ and $r = 3$, we have not been able to find an 3-uniform hypergraph which we could not partition into two sets such that each set contains at most $m/8$ edges, and we conjecture that such a partition always exists. This may even hold more generally, and we are tempted to make the following conjecture.

Conjecture 9. *Let $r \geq 3$ be an integer. Then every r-uniform hypergraph with m edges has a vertex-partition into 2 sets with at most $m/2^r$ edges in each set.*

In other words, there is a vertex partition into two sets such that each set contains no more edges than the expected number of edges in a random partition. We note that, for graphs, while we can demand $(1+o(1))m/4$ edges, any complete graph would be a counterexample to the conjecture for $r = 2$. Thus, if true, this conjecture would be rather surprising.

There are many related questions that can be asked. In general, for integers $k > i$ and r, and an r-uniform hypergraph G with n vertices and m edges, what is the minimal t such that we can find a vertex-partition of G into k sets with at most t edges in the union of any i sets?

Finally, we mention a conjecture due to Bollobás and Thomason: every r-uniform hypergraph with m edges has a vertex-partition into r sets such that every vertex class meets at least $rm/(2r-1)$ edges.

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