INDUCED CYCLES AND CHROMATIC NUMBER

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ABSTRACT. We prove that, for any pair of integers k, $l \geq 1$, there exists an integer $N(k,l)$ such that every graph with chromatic number at least $N(k,l)$ contains either K_k or an induced odd cycle of length at least 5 or an induced cycle of length at least l.

Given a graph with large chromatic number, it is natural to ask whether it must contain induced subgraphs with particular properties. One possibility is that the graph contains a large clique. If this is not the case, however, are there other graphs that G must then contain as induced subgraphs? For instance, given H and k, does every graph of sufficiently large chromatic number contain either K_k or an induced copy of H? Of course, there are graphs with arbitrarily large chromatic number and girth, so H must be acyclic. Gyárfás [1] and Sumner [8] independently made the beautiful (and difficult) conjecture that for every tree T and integer k there is an integer $f(k,T)$ such that every graph G with $\chi(G) \geq f(k,T)$ contains either K_k or an induced copy of T. Kierstead and Penrice [4], extending a result of Gyárfás, Szemerédi and Tuza [3], have proved the Gyárfás-Sumner conjecture for trees of radius at most two, while Kierstead, Penrice and Trotter [5] have resolved the on-line version of the conjecture. Scott [7] proved a 'topological' version of the conjecture: for every tree T and integer k there is an integer $f(k,T)$ such that every graph G with $\chi(G) \geq f(k,T)$ contains either K_k or an induced copy of a *subdivision* of T. It is also known that for any tree T and integer k, every graph of sufficiently large chromatic number contains either an induced copy of T, an induced $K_{k,k}$ or K_k, see Kierstead and Rödl [6].

Another conjecture relating induced subgraphs to chromatic number is the well-known Strong Perfect Graph Conjecture, which asserts that every graph G such that neither G nor $\overline{G}$ contains an induced odd cycle of length at least 5 must satisfy $\chi(G) = \mathrm{cl}(G)$. In other words, if $\chi(G) \geq \mathrm{cl}(G)$ then G must contain either K_k, or an induced odd cycle of length at least 5, or the complement of such a cycle.

In this note, we are concerned with two conjectures made by Gyárfás [2], which are related both to the Strong Perfect Graph Conjecture and to the Gyárfás-Sumner conjecture.

Conjecture 1. [2] *For every positive integer k there is an integer $N(k)$ such that every graph G with $\chi(G) \geq N(k)$ contains either K_k or an induced odd cycle of length at least 5.*

Conjecture 2. [2] *For every pair of positive integers k and l there is an integer $N(k,l)$ such that every graph with chromatic number at least $N(k,l)$ contains either K_k or an induced cycle of length at least l.*

Our aim here is to prove a result that is a weakened version of both conjectures.

Theorem 1. *For every pair of integers $k, l \geq 1$, there is an integer $N(k,l)$ such that every graph G with chromatic number at least $N(k,l)$ contains either K_k or an induced odd cycle of length at least 5 or an induced cycle of length at least l.*

Let G be a graph and let $d \geq 0$ be an integer. For $v \in V(G)$, define the *ball of radius d around v in G* by

$$B(v,d) = \{w \in V(G) : d_G(w,v) \leq d\},$$

and the *sphere of radius d around v* by

$$S(v,d) = \{w \in V(G) : d_G(w,v) = d\}.$$

Where unambiguous, for $V_0 \subset V(G)$ we write $\chi(V_0)$ instead of $\chi(G[V_0])$. We define the *d-local chromatic number* of G by

$$\chi^{(d)}(G) = \max_{v \in V(G)} \{\chi(B(v,d))\}.$$

As in [7], we prove separate results about graphs with 'large local chromatic number' and graphs with 'small local chromatic number': it turns out that, for a graph with sufficiently large chromatic number, if the chromatic number is 'locally large' then we can find a short induced odd cycle and if the chromatic number is 'locally small' then we can find a long induced cycle.

We begin with two lemmas.

Lemma 1. *Let $k \geq 2$, $l \geq 2$ and $M \geq 2$ be integers. Then there is an integer $N(k,l,M)$ such that every graph G with $\mathrm{cl}(G) < k$ and $\chi^{(l)}(G) > N(k,l,M)$ contains an induced odd cycle of length between 5 and $2l+1$ or an induced subgraph H such that $\mathrm{cl}(H) < k-1$ and $\chi(H) > M$.*

Proof. We shall prove the result by induction on k. For $k = 2$ the result is trivial. We suppose $k > 2$ and the result is true for smaller values of k. Let G be a graph with $\mathrm{cl}(G) < k$ that contains no induced $C_5, \ldots, C_{2l+1}$ and no induced subgraph G' with $\mathrm{cl}(G') < k-1$ and $\chi(G') > M$. We shall show that $\chi^{(l)}(G) \leq M^{2^l}$.

Suppose that $\chi^{(l)}(G) > M^{2^l}$. Pick $x \in V(G)$ such that $\chi(B(x,l)) > M^{2^l}$. Let $i \geq 1$ be minimal such that

$$\chi(S(x,i)) > M^{2^{i-1}} \tag{1}$$

Since $\chi(B(x,l)) > M^{2^l}$, we have $i \leq l$. Now for $1 \leq j < i$, we have $\chi(S(x,j)) \leq M^{2^{j-1}}$; let χ_j be a proper $M^{2^{j-1}}$-colouring of $G[S(x,j)]$, say with colours from $[M^{2^{j-1}}]$. We partition $S(x,i-1)$ as follows. Let $C = [M] \times [M^2] \times \cdots \times [M^{2^{i-2}}]$. For each $c = (c_1, \ldots, c_{i-1}) \in C$ we say that $v \in S(x,i-1)$ is *good for* c if $\chi_{i-1}(v) = c_{i-1}$ and there is a path $xx_1 \cdots x_{i-2}v$ in G with $x_j \in S(x,j)$ and $\chi_j(x_j) = c_j$, for $j = 1, \ldots, i-2$. Let $(C, <)$ be an arbitrary total ordering of C and define, for $c \in C$,

$$X_c = \{v \in S(x,i-1) : v \text{ is good for } c \text{ and for no } c' < c\}.$$

Then $\{X_c : c \in C\}$ is a partition of $S(x,i-1)$ into independent sets. Now let

$$Y_c = \{v \in S(x,i) : v \in \Gamma(X_c) \text{ and } v \notin \Gamma(X_{c'}) \text{ for } c' < c\}.$$

Thus $\{Y_c : c \in C\}$ is a partition of $S(x,i)$. Since $|C| = \prod_{j=1}^{i-1} M^{2^{j-1}} = M^{2^{i-1}-1}$, it follows from (1) that some Y_c must satisfy $\chi(Y_c) > M$, say for $c = (c_1, \ldots, c_{i-1})$. By the inductive hypothesis, Y_c contains a clique of order $k-1$, say with vertices $v_1, \ldots, v_{k-1}$. (Note that this implies $i > 1$, or else $\mathrm{cl}(G) \geq k$.)

Now, for $j = 1, \ldots, i-1$, let $V_j = \{v \in S(x,j) : \chi_j(v) = c_j\}$. Let $H = G[\bigcup_{j=1}^{i-1} V_j \cup \{x, v_1, \ldots, v_{k-1}\}]$. We claim that H contains an induced odd cycle of length at least 5 and at most $2l+1$.

Suppose first that there exist $1 \leq r < s \leq k-1$ such that neither of $\Gamma_H(v_r) \cap V_{i-1}$ and $\Gamma_H(v_s) \cap V_{i-1}$ is contained in the other. Pick

$$u_r \in (\Gamma_H(v_r) \cap V_{i-1}) \setminus (\Gamma_H(v_s) \cap V_{i-1})$$

and

$$u_s \in (\Gamma_H(v_s) \cap V_{i-1}) \setminus (\Gamma_H(v_r) \cap V_{i-1}).$$

Note that u_r and u_s are not adjacent, since V_{i-1} is an independent set. Let P be a shortest path from u_r to u_s in $\{x, u_r, u_s\} \cup \bigcup_{j=1}^{i-2} V_j$. Since V_j is an independent set for $j = 1, \ldots, i-2$, and u_r and u_s have paths of length at most $l-1$ to x, the path P has even length and

$2 \leq |P| \leq 2l-2$. Thus $v_r u_r P u_s v_s$ is an odd cycle of length at least 5 and at most $2l+1$.

Now suppose that for $1 \leq r < s \leq k-1$ we have one of $\Gamma_H(v_r) \cap V_{i-1}$ and $\Gamma_H(v_s) \cap V_{i-1}$ contained in the other. For $r < s$, orient the edge $v_r v_s$ from v_r to v_s if $\Gamma_H(v_r) \cap V_{i-1} \subset \Gamma_H(v_s) \cap V_{i-1}$; otherwise orient it from v_s to v_r. This gives a transitive orientation of the clique with vertices $v_1, \ldots, v_{k-1}$. Let v_r be the vertex with indegree zero. Then $\Gamma_H(v_r) \cap V_{i-1} \subset \Gamma_H(v_s) \cap V_{i-1}$ for $s = 1, \ldots, k-1$. Picking $v \in \Gamma_H(v_r) \cap V_{i-1}$, we see that the clique $\{v_1, \ldots, v_{k-1}\}$ is contained in $\Gamma(v)$, a contradiction since $\mathrm{cl}(G) < k$. □

Lemma 2. *Let $l \geq 1$ be an integer and let G be a graph such that $\chi(G) > 2\chi^{(l)}(G)$. Then G contains an induced cycle of length at least $2l$.*

Proof. We may assume G is connected, or else work with a component of maximal chromatic number. Let x be any vertex in G and let i be minimal such that $\chi(S(x,i)) > \chi^{(l)}(G)$. Such an i must exist or else $\chi(G) \leq 2\chi^{(l)}(G)$. Clearly $i > l$. Let K be a connected component of $S(x,i)$ with $\chi(K) = \chi(S(x,i))$, and let v be any vertex in $S(x,i-1)$ such that $\Gamma(v) \cap V(K) \neq \emptyset$. Now $V(K) \not\subset B(v,l)$, since $\chi(K) = \chi(S(x,i)) > \chi^{(l)}(G) \geq \chi(B(v,l))$, so pick $y \in K \setminus B(v,l)$ and $z \in \Gamma(y) \cap S(x,i-1)$. Then $d_G(v,y) > l$, so $d_G(v,z) \geq l$. Let P_1 be a shortest path from v to z in $S(x,i) \cup \{v,z\}$, and let P_2 be a shortest path from z to v in $\{v,z\} \cup B(x,i-2)$. Then vP_1zP_2 is a cycle of length at least $2l$. □

Putting Lemma 1 and Lemma 2 together, we arrive immediately at a proof of Theorem 1.

Proof of Theorem 1. The proof, which follows easily from Lemmas 1 and 2, proceeds by induction on k. For $k = 2$ the result is trivial. Suppose $k > 2$ and we have proved the result for smaller values of k. Let $N(k,l) = 2N(k,l,N(k-1,l)) + 1$ and let G be a graph with $\chi(G) \geq N(k,l)$. Then it follows from Lemma 2 that G contains an induced cycle of length at least $2l$ or else $\chi^{(l)}(G) > N(k,l,N(k-1,l))$. In the latter case, it follows from Lemma 1 that G contains an induced odd cycle of length at least 5 or an induced subgraph H such that $\chi(H) > N(k-1,l)$ and $\mathrm{cl}(H) < k-1$. The assertion of the theorem then follows from the inductive hypothesis. □

It would be interesting to improve upon either Lemma 2 or Lemma 3, since each of them goes part of the way towards one of the two conjectures of Gyárfás mentioned above.

More generally, for what families $\mathcal{C}$ of cycles is it the case that every graph of large chromatic number contains either a large clique or an induced cycle from $\mathcal{C}$? Clearly $\mathcal{C}$ cannot be finite, since there are graphs of arbitrarily high girth and chromatic number. Moreover, it is not the case that any infinite set will do, since we can take the disjoint union of graphs $G_0, G_1, \ldots$ with $\chi(G_{i+1}) > 2\chi(G_i)$ and $g(G_{i+1}) > 2|G_i| + 1$ to obtain a graph with infinite chromatic number that contains no induced cycles from the set $\{C_{|G_i|} : i \in \mathbb{N}\}$. Is it the case that the set of cycle lengths in $\mathcal{C}$ must have positive upper density in the integers?

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