Max k-cut and judicious k-partitions

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Abstract

Alon, Bollobás, Krivelevich and Sudakov [1] proved that every graph with a large cut has a bipartition in which each vertex class contains correspondingly few edges. We prove an analogous result for partitions into $k \geq 3$ classes; along the way we prove a result for biased bipartitions.

1 Introduction

Let G be a graph with m edges. It is easy to show that G has a cut (or, equivalently, a bipartite subgraph) of size least $m/2$. It is much less obvious (but nevertheless true) that there is a cut of this size such that the remaining edges are roughly evenly distributed between the two sides of the cut: in other words, each vertex class contains no more than (roughly) $m/4$ edges. Now suppose that G has a cut that is much larger than $m/2$. In this case we might hope for more: if G has a cut of size $m/2 + \alpha$, then a near-optimal cut that divides the remaining edges roughly equally between the two vertex classes would have roughly $m/4 - \alpha/2$ edges in each class. Alon, Bollobás, Krivelevich and Sudakov [1] showed that, for α not too large, this is indeed possible (for α large, they proved a complementary result: if $\alpha \geq m/30$, there is a bipartition in which each class contains at most $m/4 - m/100$ edges).

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[‡]Research supported in part by NSF grants DMS-0505550, CNS-0721983 and CCF-0728928, and ARO grant W911NF-06-1-0076

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The aim of this paper is to generalize these results in two directions: we first give results on "biased" cuts, in which edges in the two vertex classes are counted with different weights; we then continue by giving results in partitions into more than two parts. In each case, as with Alon, Bollobás, Krivelevich and Sudakov [1], we obtain matching results for the cases α small and α large.

The remainder of this introduction is divided into two parts. In the first part, we discuss some background to the problem; the second part describes our results and gives a little notation.

1.1 Previous work

For a graph G, let us define

$$f(G) = \max_{V(G)=V_1\dot\cup V_2} e(V_1,V_2) = \max_{V(G)=V_1\dot\cup V_2} \big(m - e(V_1) - e(V_2)\big)$$

to be the maximum size of a cut in G. Then, for $m \geq 1$, we set

$$f(m) = \min_{e(G)=m} f(G).$$

The extremal Max Cut problem asks for the value of $f(m)$, and has been extensively studied. It is easy to see that $f(m) \geq m/2$, for instance by considering random partitions or a suitable greedy algorithm. Edwards [10, 11] showed that

$$f(m) \geq \frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8}, \tag{1}$$

which is sharp for complete graphs of odd order. More precise bounds for other values of m were given by Alon [2], Alon and Halperin [3], and in [5]. From the other side, it is easily seen by considering random graphs $G \in \mathcal{G}(n, 1/2)$ that $f(m) = m/2 + o(m)$.

The Max Cut problem asks for a bipartition in which $e(V_1, V_2)$ is large, and hence $e(V_1) + e(V_2)$ is small. However, it does not place strong constraints on the number of edges in each vertex class separately. Problems in which constraints are placed on all vertex classes simultaneously are known as *judicious partitioning problems* (see [16] and [4] for an overview). In this case, we define a judicious partitioning problem as follows. For a graph G, let

$$g(G) = \min_{V(G)=V_1\dot\cup V_2} \max\{e(V_1), e(V_2)\},$$

and, for $m \geq 1$, set

$$g(m) = \max_{e(G)=m} g(G).$$

Determining the behaviour of $g(m)$ seems significantly harder than analyzing $f(m)$. For instance, proving that $f(m) \sim m/2$ is trivial, but there does not seem to be any simple way to prove that $g(m) \sim m/4$ (which turns out to be true). Bounds on $g(m)$ were proved by several authors, including Porter [12, 13, 14], Porter and Bin Yang [15], and Bollobás and Scott [9]. An analogue of the Edwards bound was finally proved in [7], where it was shown that every graph G with m edges has a bipartition $V(G) = V_1 \cup V_2$ such that

$$\max\{e(V_1), e(V_2)\} \leq \frac{m}{4} + \sqrt{\frac{m}{32} + \frac{1}{256}} - \frac{1}{16} \tag{2}$$

and in addition $e(V_1, V_2)$ satisfies (1). More generally, there is a vertex partition into k classes, each of which contains at most

$$\frac{m}{k^2} + \frac{k-1}{2k^2}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right) \tag{3}$$

edges.

The bounds (2) and (1) are closely related, and it is natural to ask whether graphs with a very large cut (i.e. much larger than that guaranteed by (1)) also have a correspondingly good judicious partition. If G is a graph with m edges, and $f(G) = m/2 + \alpha$, then it is clear that $g(G) \geq m/4 - \alpha/2$, since we cannot do better than a maximum cut with the remaining edges divided equally between the two vertex classes. Alon, Bollobás, Krivelevich and Sudakov [1] showed that it is possible to get pretty close to this bound: if $\alpha \leq m/30$ then

$$g(G) \leq \frac{m}{4} - \frac{\alpha}{2} + 3\sqrt{m} + \frac{10\alpha^2}{m}. \tag{4}$$

For large α, this bound is less useful. However, they also showed the complementary result that if $\alpha \geq m/30$ (and m is sufficiently large) then

$$g(G) \leq \frac{m}{4} - \frac{m}{100}. \tag{5}$$

1.2 Our results

The aim of this paper is to extend the results of Alon, Bollobás, Krivelevich and Sudakov [1] in two directions: to biased partitions, and to partitions into $k \geq 3$ parts.

In Section 2, we give results on biased partitions. For $p \in [0, 1]$ and $q = 1 - p$, define

$$m_p(G) = \min_{V(G)=V_1 \dot\cup V_2} qe(V_1) + pe(V_2)$$

Note that this is a 'biased' generalization of Max Cut: if we take $p = 1/2$ then we get $m_{1/2}(G) = \frac{1}{2}(m - f(G)))$.

Considering a random bipartion where each vertex independently has probability p of being in V_1, we get $\mathbb{E}e(V_1) = p^2 m$ and $\mathbb{E}e(V_2) = q^2 m$. It follows that every graph G with m edges has $m_p(G) \leq pqm$, while complete graphs or not too sparse random graphs show that we can have $m_p(G) = (1 + o(1))pqm$. A corresponding judicious result was proved in [7], where it was shown that there is in fact a bipartition such that there are no more than about $p^2 m$ edges in V_1 and $q^2 m$ edges in V_2. More precisely, there is a bipartition in which

$$e(V_1) \leq p^2 m + h(p, m) \tag{6}$$

and

$$e(V_2) \leq q^2 m + h(p, m), \tag{7}$$

where

$$h(p, m) = pq(\sqrt{m/2 + 1/16} - 1/4).$$

Note that when $p = 1/2$, we recover (2).

Our aim in section 2 is to prove bounds similar to (4) and (5) in this context. Suppose that $m_p(G) = pqm - \alpha$. If $\alpha \leq c(p)m$, we will show in Theorem 1 that there is a bipartition $V(G) = V_1 \cup V_2$ such that V_1 and V_2 satisfy inequalities of form

$$e(V_1) \leq p^2 m - \alpha + O\big(\sqrt{m} + \frac{\alpha^2}{m}\big) \tag{8}$$

and

$$e(V_2) \leq q^2 m - \alpha + O(\big(\sqrt{m} + \frac{\alpha^2}{m}\big). \tag{9}$$

Note that we get α rather than $\alpha/2$ here: this reflects the definition of $m_p(G)$: for example, compare $m_{1/2}(G)$ with $f(G)$.

If $\alpha \geq c(p)m$ then (8) and (9) are no longer useful: we show in Theorem 3 that there is a bipartition $V(G) = V_1 \cup V_2$ such that

$$e(V_1) \leq p^2 m - c^*(p)m$$

and

$$e(V_2) \leq q^2 m - c^*(p)m.$$

More precise statements of these results can be found at Theorems 1 and 3 below.

In Section 3, we turn to partitions into more than 2 pieces. For $k \geq 2$, let us define $\mathrm{mc}_k(G)$ to be the maximum size of a k-cut of G. It is easily seen by considering a random partition that every graph G with m edges has

$$\mathrm{mc}_k(G) \geq \frac{k-1}{k} m.$$

We show (Theorem 5) that if there is a significantly larger cut then we get a very good judicious partition. If

$$\mathrm{mc}_k(G) = \frac{k-1}{k} m + \alpha$$

then the following holds: if $\alpha \leq c(k)m$ then there is a k-cut in which each class has at most

$$\frac{m}{k^2} - \frac{\alpha}{k} + O\big(\sqrt{m} + \frac{\alpha^2}{m}\big) \tag{10}$$

edges (once again, a more precise statement is given below). For $\alpha > c(k)m$ there is (Theorem 8) a k-cut in which each class has at most $m/k^2 - c^*(k)m$ edges. Note that if α is not too large, then (10) is similar to (3), except for the constant in the error term.

In both sections, our proof strategy is to start with a good biased partition or k-cut and then move vertices one at a time out of a 'bad' vertex class while tracking their effect on the distribution of edges. This was used in [7] and refined in [1]. Our strategy is similar to the approach used in [1]. However, there are some additional obstacles that need to be overcome.

Throughout the paper, we use the following notation. Let G be a graph. For $W \subset V(G)$, we write $e(W)$ for the number of edges spanned by W; for disjoint $X, Y \subset V(G)$ we write $e(X, Y)$ for the number of edges $xy \in E(G)$ with $x \in X$ and $y \in Y$.

2 Biased partitions

Let G be a graph with m edges and $p \in [0,1]$, $q = 1-p$. In this section, we consider partitions $V(G) = V_1 \cup V_2$ that minimize $qe(V_1) + pe(V_2)$. Recall that

$$m_p(G) = \min_{V(G)=V_1 \dot\cup V_2} qe(V_1) + pe(V_2).$$

For a random partition in which each vertex independently is placed in V_1 with probability p or in V_2 with probability q, we have $\mathbb{E}(qe(V_1) + pe(V_2)) = pqm$. We shall show that if $m_p(G) = pqe(G) - \alpha$, with $\alpha \gg \sqrt{m}$, then we get a very good judicious partition.

Note that in a partition with $qe(V_1) + pe(V_2)$ minimal, every $v \in V_1$ must satisfy

$$q|\Gamma(v) \cap V_1| \le p|\Gamma(v) \cap V_2|, \tag{11}$$

or else we would have moved v to V_2, and a similar inequality holds for vertices in V_2. We shall refer to (11) as the *local inequality.*

For any partition $V(G) = V_1 \cup V_2$ that satisfies the local inequality, summing over V_1 implies that

$$e(V_1, V_2) \ge \frac{2q}{p} e(V_1)$$

and so

$$\begin{aligned} e(V_2) &= m - e(V_1) - e(V_1, V_2) \\ &\le m - e(V_1) - \frac{2q}{p} e(V_1) \\ &= m - \frac{1+q}{p} e(V_1). \end{aligned}$$

Therefore

$$\begin{aligned} qe(V_1) + pe(V_2) &\le qe(V_1) + pm - (1+q)e(V_1) \\ &= pm - e(V_1). \end{aligned}$$

Thus if $m_p(G) = pqm - \alpha$, and V_1 and V_2 satisfy the local inequality, we have

$$e(V_1) \le p^2 m + \alpha. \tag{12}$$

We begin with a result for α of moderate size, and prove a result for large α later (Theorem 3).

Theorem 1. *Let $0 < p < 1$, $q = 1-p$, and let $c(p) = \frac{1}{2}\min\{p^2, q^2\}$. Suppose G is a graph with m edges such that*

$$m_p(G) = pqm - \alpha, \tag{13}$$

where $\alpha \le c(p)m$. Then there is a partition $V(G) = V_1' \cup V_2'$ such that

$$e(V_1') \le p^2 m - \alpha + \sqrt{32mp^2} + \frac{16\alpha^2}{q^3 m} \tag{14}$$

and

$$e(V_2') \le q^2 m - \alpha + \sqrt{32mq^2} + \frac{16\alpha^2}{p^3 m}. \tag{15}$$

Note that this improves on (6) and (7) only in the range $\alpha = O(\min\{p^3, q^3\})m$.

Our main tool in the proof of Theorem 1 is the following.

Lemma 2. *Suppose G has m edges and satisfies (13), where $\alpha \le p^2 m/2$. Suppose $W \subset V = V(G)$ and, for all $v \in W$,*

$$|\Gamma(v) \cap V \setminus W| \ge \frac{q}{p}|\Gamma(v) \cap W|. \tag{16}$$

If $e(W) > p^2 m - \alpha$ then there is $v \in W$ with

$$|\Gamma(v) \cap W| \le \sqrt{32mp^2} \tag{17}$$

and

$$|\Gamma(v) \cap V \setminus W| \le \left(\frac{q}{p} + \frac{8\alpha}{p^3 m}\right)|\Gamma(v) \cap W|. \tag{18}$$

As above, we will refer to inequality (16) as the *local inequality.*

Proof. Define

$$T_1 = \{v \in W : |\Gamma(v) \cap W| > \sqrt{32mp^2}\} \tag{19}$$

and

$$T_2 = \{v \in W : |\Gamma(v) \cap V \setminus W| > \big(\frac{q}{p} + \frac{8\alpha}{p^3 m}\big)|\Gamma(v) \cap W|\}. \tag{20}$$

Summing the inequality satisfied by vertices in (20) over T_2, and summing (16) over the rest of W, we see that

$$\begin{aligned} e(W, V \setminus W) &\ge \frac{q}{p}\sum_{v\in W}|\Gamma(v) \cap W| + \frac{8\alpha}{p^3 m}\sum_{v \in T_2}|\Gamma(v) \cap W| \\ &= \frac{2q}{p}e(W) + \frac{8\alpha}{p^3 m}\sum_{v\in T_2}|\Gamma(v) \cap W|. \end{aligned}$$

Thus

$$\begin{aligned}
qe(W) + pe(V \setminus W) &= qe(W) + p\big(m - e(W) - e(W, V \setminus W)\big) \\
&\le qe(W) + p\left(m - e(W) - \frac{2q}{p}e(W) - \frac{8\alpha}{p^3 m}\sum_{v \in T_2} |\Gamma(v) \cap W|\right) \\
&= pm - e(W) - \frac{8\alpha}{p^2 m}\sum_{v \in T_2} |\Gamma(v) \cap W| \\
&< pqm + \alpha - \frac{8\alpha}{p^2 m}\sum_{v \in T_2} |\Gamma(v) \cap W|.
\end{aligned}$$

Thus, by (13),

$$\frac{8\alpha}{p^2 m}\sum_{v \in T_2} |\Gamma(v) \cap W| < 2\alpha$$

and so

$$\sum_{v \in T_2} |\Gamma(v) \cap W| < \frac{p^2 m}{4}. \tag{21}$$

On the other hand, since W and $V \setminus W$ satisfy the local inequality, by (12) we have

$$e(W) \le p^2 m + \alpha \le 2p^2 m$$

and so

$$\sum_{v \in T_1} |\Gamma(v) \cap W| \le 2e(W) \le 4p^2 m,$$

which, by the definition of T_1, implies

$$|T_1| \le \frac{4p^2 m}{\sqrt{32mp^2}} = \sqrt{\frac{mp^2}{2}}.$$

Thus

$$\sum_{v \in T_1} |\Gamma(v) \cap W| \le e(T_1) + e(W) \le \binom{|T_1|}{2} + e(W) \le \frac{mp^2}{4} + e(W). \tag{22}$$

Since $e(W) > p^2 m - \alpha \ge p^2 m/2$, (22) and (21) give

$$\sum_{v \in T_1 \cup T_2} |\Gamma(v) \cap W| < \frac{p^2 m}{2} + e(W) \le 2e(W)$$

and so $T_1 \cup T_2 \ne W$. The lemma follows immediately. □

We can now turn to the proof of Theorem 1.

Proof of Theorem 1. Let $V_1 \cup V_2$ be a partition with

$$qe(V_1) + pe(V_2) = pqm - \alpha.$$

If (14) and (15) are satisfied for V_1 and V_2, we are done. Otherwise, exchanging p and q if necessary (and noting that this also exchanges (14) and (15)), we may assume that

$$e(V_1) > p^2 m - \alpha.$$

If

$$e(V_1) = p^2 m - \alpha + \lambda$$

then

$$\begin{aligned} pe(V_2) &= pqm - \alpha - qe(V_1) \\ &= pqm - \alpha - qp^2 m + q\alpha - q\lambda \\ &= pq^2 m - p\alpha - q\lambda \end{aligned}$$

and so

$$e(V_2) = q^2 m - \alpha - \frac{q}{p}\lambda. \tag{23}$$

Note that (V_1, V_2) satisfies the local inequality (16) (with $W = V_1$).

We now successively move vertices from V_1 to V_2, at each step choosing a vertex satisfying (17) and (18). We can find such a vertex, as the local inequality (16) remains true if we remove vertices from V_1 and so we can apply Lemma 2. We continue until we obtain V_1' such that $p^2 m - \alpha \leq e(V_1') \leq p^2 m - \alpha + \sqrt{32mp^2}$ (note that (17) guarantees that our steps are sufficiently small that we don't overshoot). Since we have decreased $e(V_1)$ by at most λ, (18) implies that we have increased $e(V_2)$ by at most

$$\left(\frac{q}{p} + \frac{8\alpha}{p^3 m}\right)\lambda$$

and so, by (23), we end up with V_2' satisfying

$$\begin{aligned} e(V_2') &\leq e(V_2) + \left(\frac{q}{p} + \frac{8\alpha}{p^3 m}\right)\lambda \\ &\leq q^2 m - \alpha + \frac{8\alpha}{p^3 m}\lambda. \end{aligned} \tag{24}$$

By (12) we have $\lambda \leq 2\alpha$, and so the result follows from (24) by taking the partition (V_1', V_2'). □

We now deal with the case when α is large.

Theorem 3. *Let $0 < p < 1$ and $q = 1 - p$. Let $0 < c < \min\{p^2, q^2\}$ and $c^*(p) = \min\{cp/12, cq/12\}$. Suppose that G is a graph with m edges and*

$$m_p(G) = pqm - \alpha, \tag{25}$$

where $\alpha \geq cm$. Then, provided that m is sufficiently large (in terms of c and p), there is a partition $V(G) = V_1 \cup V_2$ such that

$$e(V_1) \leq p^2 m - c^* m \tag{26}$$

$$e(V_2) \leq q^2 m - c^* m. \tag{27}$$

The best fit with Theorem 1 is obtained by specializing to a particular value of c. However, it will be useful in the next section to allow any $c > 0$.

The proof of Theorem 3 is based on the following lemma.

Lemma 4. *Let $0 < p < 1$, $q = 1 - p$, and suppose that $0 < c^* < p^3/9$. Suppose that G is a graph with m edges. Suppose $W \subset V = V(G)$ satisfies*

$$e(W) > p^2 m - c^* m$$

and, for every $w \in W$,

$$q|\Gamma(w) \cap W| \leq p|\Gamma(w) \cap V \setminus W|.$$

Then, provided m is sufficiently large (in terms of p and c^*), either there is $w \in W$ such that

$$|\Gamma(w) \cap W| < c^* m \tag{28}$$

and

$$|\Gamma(w) \cap V \setminus W| \leq \left(\frac{q}{p} + \frac{1}{2}\right)|\Gamma(v) \cap W|, \tag{29}$$

or there is $W' \subset W$ such that $(V_1, V_2) = (W', V \setminus W')$ satisfies

$$e(V_1) \leq p^2 m - c^* m \tag{30}$$

$$e(V_2) \leq q^2 m - c^* m. \tag{31}$$

Proof. Let

$$T_1 = \{v \in W : |\Gamma(v) \cap W| > c^* m\}$$

and

$$T_2 = \left\{ v \in W : |\Gamma(v) \cap V \setminus W| \geq \left(\frac{q}{p} + \frac{1}{2} \right) |\Gamma(v) \cap W| \right\}.$$

Let $T = T_1 \cup T_2$. We consider two cases.

Case 1. $e(T) \geq p^2 m - c^* m$.

Since every graph with m edges has a vertex of degree at most $\sqrt{2m}$, we can delete vertices from $T \subset W$ one at a time until we obtain $V_1 \subset T$ with

$$p^2 m - c^* m - \sqrt{2m} < e(V_1) \leq p^2 m - c^* m. \tag{32}$$

Then, writing $V_2 = V \setminus V_1$, and using the local inequality and the fact that $V_1 \subseteq T$, we have

$$\begin{aligned} e(V_1, V_2) &= \sum_{v \in V_1} |\Gamma(v) \cap V_2| \\ &\geq \sum_{v \in V_1} |\Gamma(v) \cap V \setminus T| \\ &\geq \frac{q}{p} \sum_{v \in V_1} |\Gamma(v) \cap T| + \frac{1}{2} \sum_{v \in V_1 \cap T_2} |\Gamma(v) \cap T| \\ &\geq \frac{2q}{p} e(V_1) + \frac{1}{2} \sum_{v \in V_1 \cap T_2} |\Gamma(v) \cap V_1|. \end{aligned} \tag{33}$$

Now, since $T \subseteq W$, $\sum_{v \in T_1} |\Gamma(v) \cap W| \leq 2e(W) \leq 2m$, and so $|T_1| < 2/c^*$. Thus $e(T_1) < 2/(c^*)^2$, and so

$$\begin{aligned} \sum_{v \in V_1 \cap T_1} |\Gamma(v) \cap V_1| &\leq e(V_1 \cap T_1) + e(V_1) \\ &\leq e(T_1) + e(V_1) \\ &< \frac{2}{(c^*)^2} + e(V_1). \end{aligned}$$

Since $V_1 \subseteq T$, it follows that

$$\begin{aligned} \sum_{v \in V_1 \cap T_2} |\Gamma(v) \cap V_1| &\geq 2e(V_1) - \sum_{v \in V_1 \cap T_1} |\Gamma(v) \cap V_1| \\ &> e(V_1) - \frac{2}{(c^*)^2} \\ &\geq e(V_1)/2, \end{aligned}$$

provided m is sufficiently large. Thus, by (33),

$$e(V_1, V_2) > \left(\frac{2q}{p} + \frac{1}{4}\right) e(V_1)$$

and so, using (32),

$$\begin{aligned} e(V_2) &= m - e(V_1) - e(V_1, V_2) \\ &< m - \big(1 + \frac{2q}{p} + \frac{1}{4}\big) e(V_1) \\ &\leq m - \big(1 + \frac{2q}{p} + \frac{1}{4}\big)\big(p^2 m - c^* m - \sqrt{2m}\big) \\ &= q^2 m - \frac{1}{4} p^2 m + \big(1 + \frac{2q}{p} + \frac{1}{4}\big)\big(c^* m + \sqrt{2m}\big) \\ &< q^2 m - c^* m, \end{aligned}$$

provided m is sufficiently large. Thus (V_1, V_2) satisfies (30) and (31), as required.

Case 2. $e(T) < p^2 m - c^* m$.

In this case, there is some vertex $w \in W \setminus T$; this vertex will satisfy the required inequalities.

□

Proof of Theorem 3. Let $V(G) = V_1 \cup V_2$ be a partition such that

$$qe(V_1) + pe(V_2) = pqm - \alpha. \tag{34}$$

If V_1 and V_2 satisfy (26) and (27) then we are done. Otherwise (exchanging V_1 and V_2 and p and q if necessary, and noting that c^* is unchanged) we may assume V_1 fails (26). Suppose that

$$e(V_1) = p^2 m - c^* m + \lambda, \tag{35}$$

so that, by (34),

$$\begin{aligned} e(V_2) &= \frac{1}{p}\big(pqm - \alpha - qe(V_1)\big) \\ &= \frac{1}{p}\big(pq^2 m - \alpha - q\lambda + qc^* m\big) \\ &= q^2 m - \frac{q\lambda + \alpha}{p} + \frac{qc^* m}{p} \end{aligned}$$

Provided m is sufficiently large, we can move vertices from V_1 to V_2 using Lemma 4. At each stage, we either obtain the partition required by the theorem, or by (29) move a vertex that decreases $e(V_1)$ by some integer d and increases $e(V_2)$ by at most $(\frac{q}{p} + \frac{1}{2})d$. We halt when we reach $V_1' \subset V_1$ with

$$p^2m - 2c^*m \leq e(V_1') \leq p^2m - c^*m;$$

here, (28) guarantees that we do stop. We have decreased $e(V_1)$ by

$$e(V_1) - e(V_1') \leq \lambda + c^*m$$

and so, writing $V_2' = V \setminus V_1'$,

$$\begin{aligned} e(V_2') &\leq e(V_2) + \left(\frac{q}{p} + \frac{1}{2}\right)(\lambda + c^*m) \\ &= q^2m - \frac{q\lambda + \alpha}{p} + \frac{qc^*m}{p} + \left(\frac{q}{p} + \frac{1}{2}\right)(\lambda + c^*m) \\ &= q^2m - \frac{1}{p}\alpha + \frac{1}{2}\lambda + \left(\frac{2q}{p} + \frac{1}{2}\right)c^*m. \end{aligned}$$

By (12), (35) and (34), we have $\lambda \leq \alpha + c^*m$, so

$$\begin{aligned} e(V_2') &\leq q^2m - \left(\frac{1}{p} - \frac{1}{2}\right)\alpha + 2\left(\frac{q}{p} + \frac{1}{2}\right)c^*m \\ &\leq q^2m - \frac{1}{2}\alpha + \frac{4}{p}c^*m \\ &< q^2m - c^*m. \end{aligned}$$

Thus (V_1', V_2') will do for our partition. □

3 Partitions into k vertex classes

In this section, we show that graphs with a large k-cut have a good judicious partition into k vertex classes. As in the previous section, we begin with a result for moderate values of α, and then prove a result (Theorem 8) for large α.

Our first result is the following.

Theorem 5. *Let $k \geq 2$. Suppose that G is a graph with m edges such that*

$$\mathrm{mc}_k(G) = \left(1 - \frac{1}{k}\right) m + \alpha, \tag{36}$$

where $\alpha \leq m/k^6$. Then there is a k-cut in which each class has at most

$$\frac{m}{k^2} - \frac{\alpha}{k} + \frac{k^5\alpha^2}{m} + 4\sqrt{m} \tag{37}$$

edges.

Before we prove this result, let us make a few simple observations. Note first that if $\bigcup_{i=1}^k V_i$ is a maximum k-cut of G then, for $i \neq j$ and $v \in V_i$, we have

$$|\Gamma(v) \cap V_j| \geq |\Gamma(v) \cap V_i|, \tag{38}$$

or else we could move v from V_i to V_j to obtain a larger cut. Thus every vertex class V_i satisfies, for all $v \in V_i$, the inequality

$$|\Gamma(v) \cap V \setminus V_i| \geq (k-1)|\Gamma(v) \cap V_i|. \tag{39}$$

Once again, we shall refer to this as the *local inequality.*

Summing (38) over vertices in V_i, we find that

$$e(V_i, V_j) \geq 2e(V_i). \tag{40}$$

It is easily seen (for instance, by considering a random k-cut, or partitioning greedily one vertex at a time) that

$$\mathrm{mc}_k(G) \geq \frac{k-1}{k} e(G). \tag{41}$$

Given a partition of some subset $W \subset V(G)$ into k sets, we can extend greedily to a k-cut of G by adding vertices one at a time to whichever class maximizes the partial cut at each step. We see that, if $H = G[W]$, then

$$\mathrm{mc}_k(G) \geq \mathrm{mc}_k(H) + \frac{k-1}{k}(e(G) - e(H)). \tag{42}$$

We can also obtain a k-cut by choosing one vertex class and then taking a $(k-1)$-cut of the remainder of the graph. In particular, for any $W \subset V = V(G)$,

$$\mathrm{mc}_k(G) \geq e(W, V \setminus W) + \mathrm{mc}_{k-1}(G \setminus W). \tag{43}$$

In addition to these observations, our proof of Theorem 5 will be based on the following two lemmas.

Lemma 6. *Suppose that G is a graph with m edges such that*

$$\mathrm{mc}_k(G) = \frac{k-1}{k}m + \alpha \tag{44}$$

and $W \subset V$ satisfies the local inequality

$$|\Gamma(v) \cap (V \setminus W)| \geq (k-1)|\Gamma(v) \cap W| \tag{45}$$

for all $v \in W$. Then

$$e(W) \leq \frac{m}{k^2} + \frac{k-1}{k}\alpha. \tag{46}$$

Proof. Let $V = V(G)$. Using (43) and (41), we see that

$$\begin{aligned}
\mathrm{mc}_k(G) &\geq e(W, V \setminus W) + \mathrm{mc}_{k-1}(G \setminus W) \\
&\geq e(W, V \setminus W) + \frac{k-2}{k-1}\big(m - e(W) - e(W, V \setminus W)\big) \\
&= \frac{k-2}{k-1}m + \frac{1}{k-1}e(W, V \setminus W) - \frac{k-2}{k-1}e(W).
\end{aligned}$$

Summing (45) over vertices in W, we see $e(W, V \setminus W) \geq 2(k-1)e(W)$. So

$$\begin{aligned}
\mathrm{mc}_k(G) &\geq \frac{k-2}{k-1}m + 2e(W) - \frac{k-2}{k-1}e(W) \\
&= \frac{k-2}{k-1}m + \frac{k}{k-1}e(W).
\end{aligned}$$

The result now follows by a simple calculation. □

The proof of Theorem 5 involves moving certain vertices between the vertex classes of a partition. The fact that we can find suitable vertices is guaranteed by the following lemma.

Lemma 7. *Suppose that $\alpha < m/2k$ and*

$$\mathrm{mc}_k(G) = \frac{k-1}{k}m + \alpha. \tag{47}$$

Suppose that $W \subset V$ and the local inequality (45) *holds for every $v \in W$. If*

$$e(W) \geq \frac{m}{k^2} - \frac{\alpha}{k} \tag{48}$$

then there is a vertex $v \in W$ with

$$|\Gamma(v) \cap W| \leq 4\sqrt{m} \tag{49}$$

and

$$|\Gamma(v) \cap (V \setminus W)| \leq \left(k - 1 + 4k^3 \frac{\alpha}{m}\right) |\Gamma(v) \cap W| \tag{50}$$

Proof. Let

$$\begin{aligned} T_1 &= \{v \in W : |\Gamma(v) \cap W| > 4\sqrt{m}\} \\ T_2 &= \left\{v \in W : |\Gamma(v) \cap (V \setminus W)| > \left(k - 1 + 4k^3 \frac{\alpha}{m}\right) |\Gamma(v) \cap W|\right\} \end{aligned}$$

It is enough to show that $W \setminus (T_1 \cup T_2)$ is nonempty.

By (45), we have $e(W, V \setminus W) \geq 2(k-1)e(W)$ and as $e(W) + e(W, V \setminus W) \leq m$, we get $e(W) \leq m/(2k-1)$. Since $\sum_{v \in T_1} |\Gamma(v) \cap W| \leq 2e(W) \leq 2m/(2k-1)$, we have

$$|T_1| \leq \frac{2e(W)}{4\sqrt{m}} \leq \frac{\sqrt{m}}{2(2k-1)}$$

and so

$$e(T_1) \leq \binom{|T_1|}{2} \leq \frac{m}{8(2k-1)^2}.$$

It follows that

$$\sum_{v \in T_1} |\Gamma(v) \cap W| \leq e(W) + e(T_1) \leq e(W) + \frac{m}{8(2k-1)^2}. \tag{51}$$

We now concentrate on bounding $\sum_{v \in T_2} |\Gamma(v) \cap W|$. Calculating as in the proof of Lemma 6, we have

$$\mathrm{mc}_k(G) \geq \frac{k-2}{k-1} m - \frac{k-2}{k-1} e(W) + \frac{1}{k-1} e(W, V \setminus W). \tag{52}$$

Now (45) and the definition of T_2 imply that

$$\begin{aligned} e(W, V \setminus W) &= \sum_{v \in W} |\Gamma(v) \cap (V \setminus W)| \\ &\geq (k-1) \sum_{v \in W} |\Gamma(v) \cap W| + \frac{4k^3\alpha}{m} \sum_{v \in T_2} |\Gamma(v) \cap W| \\ &= 2(k-1)e(W) + \frac{4k^3\alpha}{m} \sum_{v \in T_2} |\Gamma(v) \cap W|. \end{aligned} \tag{53}$$

It therefore follows from (52) that

$$\mathrm{mc}_k(G) \geq \frac{k-2}{k-1}m + \frac{k}{k-1}e(W) + \frac{4k^3\alpha}{m(k-1)} \sum_{v \in T_2} |\Gamma(v) \cap W|. \tag{54}$$

By (48) the right hand side is at least

$$\frac{k-1}{k}m - \frac{\alpha}{k-1} + \frac{4k^3\alpha}{m(k-1)} \sum_{v \in T_2} |\Gamma(v) \cap W|.$$

But then (47) implies that

$$\frac{4k^3\alpha}{m(k-1)} \sum_{v \in T_2} |\Gamma(v) \cap W| \leq \alpha + \frac{\alpha}{k-1} = \frac{k}{k-1}\alpha,$$

and so

$$\sum_{v \in T_2} |\Gamma(v) \cap W| \leq \frac{m}{4k^2}.$$

It therefore follows from (51) that

$$\sum_{v \in T_1 \cup T_2} |\Gamma(v) \cap W| \leq e(W) + \left(\frac{1}{4k^2} + \frac{1}{8(2k-1)^2} \right) m.$$

Since $\alpha < m/2k$, we have $e(W) > m/2k^2$. Since

$$\frac{1}{4k^2} + \frac{1}{8(2k-1)^2} < \frac{1}{2k^2},$$

we have

$$\sum_{v \in T_1 \cup T_2} |\Gamma(v) \cap W| \leq e(W) + \frac{m}{2k^2} < 2e(W),$$

and so $T_1 \cup T_2 \neq W$, as claimed. □

After this, we are ready to prove Theorem 5.

Proof of Theorem 5. We argue by induction on k. Let $(V_1, \ldots, V_k)$ be a maximum cut, and suppose that $e(V_1) \geq \cdots \geq e(V_k)$. If $e(V_1)$ satisfies (37) we are done. Otherwise,

$$e(V_1) = \frac{m}{k^2} - \frac{\alpha}{k} + \lambda, \tag{55}$$

where Lemma 6 implies that $\lambda \le \alpha$.

We proceed by moving vertices one at a time from V_1 to other vertex classes. Suppose we have reached a stage with vertex classes $V_1', \dots, V_k'$ (where $V_1' \subseteq V_1$). Applying Lemma 7, we find a vertex v satisfying (49) and (50), and move v to whichever class V_i', $i > 1$, contains fewest neighbours of v. This decreases $e(V_1')$ by $|\Gamma(v) \cap V_1'| \le 4\sqrt{m}$ and, by (50), decreases the size of the k-cut by at most

$$\begin{aligned}\min_{i>1}\{|\Gamma(v) \cap V_i'| - |\Gamma(v) \cap V_1'|\} &\le \frac{1}{k-1}|\Gamma(v) \cap (V \setminus V_1')| - |\Gamma(v) \cap V_1'| \\ &\le \frac{4k^3\alpha}{m(k-1)}|\Gamma(v) \cap V_1'|. \end{aligned} \tag{56}$$

Since moving v does not affect the local inequality (45), we can continue to move vertices until V_1 is reduced to W_1 with

$$\frac{m}{k^2} - \frac{\alpha}{k} \le e(W_1) \le \frac{m}{k^2} - \frac{\alpha}{k} + 4\sqrt{m}. \tag{57}$$

Note that inequality (49) implies that we do eventually obtain W_1 with $e(W_1)$ in this range.

We end up with a set $W_1 \subseteq V_1$ that satisfies (37), and sets $W_2, \dots, W_k$ with $W_i \supseteq V_i$ for each i. Since (55) and (57) imply that $e(V_1) - e(W_1) \le \lambda \le \alpha$, it follows from (56) that the size of the k-cut we end up with is at least

$$\frac{k-1}{k}m + \alpha - \alpha \cdot \frac{4k^3\alpha}{m(k-1)}. \tag{58}$$

Since $(W_1, \dots, W_k)$ satisfies (57), by (58) we have

$$\begin{aligned}\sum_{i\ge 2} e(W_i) &\le m - \left(\frac{k-1}{k}m + \alpha - \frac{4k^3\alpha^2}{m(k-1)}\right) - e(W_1) \\ &= \frac{m}{k} - \alpha + \frac{4k^3\alpha^2}{m(k-1)} - e(W_1) \\ &\le \frac{k-1}{k^2}m - \frac{k-1}{k}\alpha + \frac{4k^3\alpha^2}{m(k-1)}. \end{aligned} \tag{59}$$

If $k = 2$, this implies (37) immediately. Otherwise, we consider the subgraph $H = G[V \setminus W_1]$, and partition it into $k-1$ classes.

Suppose first that $e(H) \le \left(\frac{k-1}{k}\right)^2 m - \frac{(k-1)^2}{k}\alpha$. We can find a judicious partition of H into $k-1$ classes, each of which satisfies (3). Extending to a k-partition of G by taking W_1 as the kth vertex class gives a partition satisfying (37).

Otherwise, $e(H) > \left(\frac{k-1}{k}\right)^2 m - \frac{(k-1)^2}{k}\alpha$. Note that since V_1 satisfies the local inequality (39), so does W_1, and so $e(W_1, V \setminus W_1) \ge 2(k-1)e(W_1)$. Now

$$\mathrm{mc}_{k-1}(H) \le \frac{k-2}{k-1}e(H) + \frac{k}{k-1}\alpha, \tag{60}$$

or else, using (57), (43) and the local inequality,

$$\begin{aligned}
\mathrm{mc}_k(G) &\ge \mathrm{mc}_{k-1}(H) + e(W_1, V \setminus W_1) \\
&> \frac{k-2}{k-1}\big(m - e(W_1) - e(W_1, V \setminus W_1)\big) + \frac{k}{k-1}\alpha + e(W_1, V \setminus W_1) \\
&= \frac{k-2}{k-1}m + \frac{1}{k-1}e(W_1, V \setminus W_1) - \frac{k-2}{k-1}e(W_1) + \frac{k}{k-1}\alpha \\
&\ge \frac{k-2}{k-1}m + 2e(W_1) - \frac{k-2}{k-1}e(W_1) + \frac{k}{k-1}\alpha \\
&= \frac{k-2}{k-1}m + \frac{k}{k-1}e(W_1) + \frac{k}{k-1}\alpha \\
&\ge \frac{k-2}{k-1}m + \frac{k}{k-1} \cdot \frac{m}{k^2} - \frac{k}{k-1}\frac{\alpha}{k} + \frac{k}{k-1}\alpha \\
&= \frac{k-1}{k}m + \alpha,
\end{aligned}$$

which contradicts (36). Thus, writing

$$\mathrm{mc}_{k-1}(H) = \frac{k-2}{k-1}e(H) + \gamma, \tag{61}$$

by (60) and our assumptions on the size of $e(H)$ and α,

$$\begin{aligned}
\gamma/e(H) &\le \frac{k\alpha/(k-1)}{(k-1)^2 m/k^2 - (k-1)^2\alpha/k} \\
&\le \frac{(m/k^4) \cdot k/(k-1)}{m(k-1)^2/k^2 - (k-1)^2 m/k^5} \\
&\le 1/(k-1)^4.
\end{aligned}$$

Applying the inductive hypothesis to H, we obtain a partition $W_2', \dots, W_k'$ with

$$\max_{i>1} e(W_i') \le \frac{e(H)}{(k-1)^2} - \frac{\gamma}{k-1} + (k-1)^5 \frac{\gamma^2}{e(H)} + 4\sqrt{e(H)} \tag{62}$$

Now, by (61),

$$\begin{aligned} \frac{e(H)}{(k-1)^2} - \frac{\gamma}{k-1} &= \frac{1}{k-1}\left(e(H) - \mathrm{mc}_{k-1}(H)\right) \\ &\le \frac{1}{k-1} \sum_{i\ge 2} e(W_i) \end{aligned} \tag{63}$$

and, since $\gamma \le \frac{k}{k-1}\alpha$ (by (60)) and $e(H) \ge \left(\frac{k-1}{k}\right)^2 m - (k-1)^2\alpha/k \ge \left(\frac{k-1}{k}\right)^2 m - (k-1)^2 m/k^5$,

$$\begin{aligned} \frac{\gamma^2}{e(H)} &\le \left(\frac{k}{k-1}\right)^2 \frac{\alpha^2}{m} \frac{1}{(k-1)^2/k^2 - (k-1)^2/k^5} \\ &= \frac{\alpha^2}{m} \cdot \frac{k^7}{(k-1)^4(k^3-1)}. \end{aligned}$$

It follows from (62), (63) and (59) that

$$\begin{aligned} \max_{i\ge 1} e(W_i') &\le \frac{1}{k-1}\sum_{i\ge 2} e(W_i) + (k-1)^5 \frac{\gamma^2}{e(H)} + 4\sqrt{e(H)} \\ &\le \frac{m}{k^2} - \frac{\alpha}{k} + \frac{4k^3}{(k-1)^2}\frac{\alpha^2}{m} + \frac{(k-1)^5 k^7}{(k-1)^4(k^3-1)}\frac{\alpha^2}{m} + 4\sqrt{m} \\ &\le \frac{m}{k^2} - \frac{\alpha}{k} + k^5 \frac{\alpha^2}{m} + 4\sqrt{m}, \end{aligned}$$

for $k \ge 3$. The result now follows immediately by taking the partition $W_1, W_2', \dots, W_k'$. □

Finally, we turn to the case when the maximum k-cut is very large. As in Alon, Bollobás, Krivelevich and Sudakov [1], we use a rather cruder argument.

Theorem 8. *Let $k \ge 2$. Suppose that G is a graph with m edges such that*

$$\mathrm{mc}_k(G) = \frac{k-1}{k} m + \alpha,$$

where $\alpha > m/k^6$. Then, provided that m is sufficently large (in terms of k), there is a partition of $V(G)$ into k sets, each of which contains at most

$$\frac{m}{k^2} - \frac{m}{12k^{10}} \tag{64}$$

edges.

Proof. Let $(V_1, \ldots, V_k)$ be a cut of size $(k-1)m/k + \alpha$. Let $i \in \{1, \ldots, k\}$ be chosen uniformly at random, and consider the partition $(V_i, V \setminus V_i)$. Then, writing $m' = \sum_{j=1}^{k} e(V_j) = m/k - \alpha$ and $p = 1 - q = 1/k$, we have

$$\begin{aligned}
\mathbb{E}(qe(V_i) + pe(V \setminus V_i)) &= q\frac{1}{k}m' + p\left(\frac{k-1}{k}m' + \frac{\binom{k-1}{2}}{\binom{k}{2}}(m - m')\right) \\
&= \frac{2k-2}{k^2}m' + \frac{k-2}{k^2}(m - m') \\
&= \frac{k-2}{k^2}m + \frac{1}{k}m' \\
&= \frac{k-1}{k^2}m - \frac{\alpha}{k} \\
&= pqm - \frac{\alpha}{k}.
\end{aligned}$$

Suppose that $m_{1/k}(G) = pqm - \alpha'$. Since $\alpha' > m/k^7$, we can apply Theorem 3 with $p = 1/k$ and $c = 1/k^7$ to get a bipartition $V(G) = V_1' \cup V_2'$ with $e(V_1) \leq m/k^2 - m/12k^8$ and $e(V_2') \leq (k-1)^2 m/k^2 - m/12k^8$. We refine the partition by splitting V_2 into $k - 1$ pieces satisfying (3) (for the $(k-1)$-partite case). Providing m is sufficiently large (in terms of k), we obtain a partition of $V(G)$ satisfying (64). □

4 Conclusion

It seems likely that our constants could be improved significantly. It would be interesting to have sharper constants both when δ is small (for instance, in (37)), and when δ is large (for instance, in (64)). Particularly when $\delta = \Omega(m)$, all the bounds are rather crude, and it would be very interesting to know the correct dependence of the error term on δ, and to have some idea of the extremal graphs.

It would be very interesting to prove analogous results for hypergraphs (see, for instance, [6] and [8] for results on judicious partitions of hypergraphs).

Finally, it would also be of interest to consider bisections instead of cuts. More specifically, for a graph G, let

$$b(G) = \max\{e(V_1, V_2) : V(G) = V_1 \sqcup V_2, \big| |V_1| - |V_2| \big| \leq 1\}$$

be the maximum size of a bisection of G, and let $g_b(G)$ be the minimum of $\max\{e(V_1), e(V_2)\}$ over bisections of G. What can be said about the relationship between $b(G)$ and $g_b(G)$? Note that the star $K_{1,n-1}$ has $b(K_{1,n-1}) = \lceil n/2 \rceil \sim e(K_{1,n-1})/2$, while $g_b(K_{1,n-1}) = \lfloor n/2 \rfloor - 1 \sim e(K_{1,n-1})/2$, which is about as bad as it could be. But what about graphs with bisections much larger than $m/2$?

Acknowledgement. We would like to thank the referees for their careful reading of the paper.

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