

Induced subgraphs of graphs with large chromatic number.
XIV. Excluding a biclique and an induced tree

Alex Scott¹

Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour²

Princeton University, Princeton, NJ 08544

Sophie Spirkl³

University of Waterloo, Waterloo, Ontario N2L3G1, Canada

¹Research supported by EPSRC grant EP/V007327/1.

²Supported by AFOSR grant A9550-19-1-0187, and by NSF grant DMS-1800053.

³We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), [funding reference number RGPIN-2020-03912]. Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG), [numéro de référence RGPIN-2020-03912].

Abstract

Let H be a tree. It was proved by Rödl that graphs that do not contain H as an induced subgraph, and do not contain the complete bipartite graph $K_{t,t}$ as a subgraph, have bounded chromatic number. Kierstead and Penrice strengthened this, showing that such graphs have bounded degeneracy. Here we give a further strengthening, proving that for every tree H , the degeneracy is at most polynomial in t . This answers a question of Bonamy, Pilipczuk, Rzazewski, Thomassé and Walczak.

1 Introduction

The Gyárfás-Sumner conjecture [4, 13] asserts:

1.1 Conjecture: *For every forest H , there is a function f such that $\chi(G) \leq f(\omega(G))$ for every H -free graph G .*

(We use $\chi(G)$ and $\omega(G)$ to denote the chromatic number and the clique number of a graph G , and a graph is H -free if it has no induced subgraph isomorphic to H .) One attractive feature of this conjecture is that it is best possible in a sense: for every graph H that is not a forest, there is no function f as in 1.1 (this is easily shown with a random graph). The conjecture has been proved for some special families of trees (see, for example, [2, 5, 6, 7, 9, 10, 11]) but remains open in general.

A class \mathcal{C} of graphs is χ -bounded if there is a function f such that $\chi(G) \leq f(\omega(G))$ for every graph G that is an induced subgraph of a member of \mathcal{C} (see [12] for a survey). Thus the Gyárfás-Sumner conjecture asserts that, for every forest H , the class of all H -free graphs is χ -bounded. Esperet [3] asked whether, for every χ -bounded class, f can always be chosen to be a polynomial. Neither conjecture has been settled in general.

The complete bipartite graph with parts of cardinality s, t is denoted by $K_{s,t}$. Let us define $\tau(G)$ to be the largest t such that G contains $K_{t,t}$ as a subgraph (not necessarily induced). It was proved by Rödl (mentioned in [8], and see also [6]) that the analogue of the Gyárfás-Sumner conjecture is true if we replace $\omega(G)$ by $\tau(G)$. That is:

1.2 *For every forest H , there is a function f such that $\chi(G) \leq f(\tau(G))$ for every H -free graph G .*

This has the same attractive feature that the result is best possible (in the same sense).

This result was strengthened by Kierstead and Penrice. Let us say a graph G is d -degenerate (where $d \geq 0$ is an integer) if every nonnull subgraph has a vertex of degree at most d ; and the degeneracy $\partial(G)$ of G is the smallest d such that G is d -degenerate. Then $\chi(G) \leq \partial(G) + 1$, and so the following result of Kierstead and Penrice [7] is a strengthening of 1.2:

1.3 *For every forest H , there is a function f such that $\partial(G) \leq f(\tau(G))$ for every H -free graph G .*

What about the analogue of Esperet's question: do 1.2 and 1.3 remain true if we require f to be a polynomial in $\tau(G)$? This question was raised by Bonamy, Bousquet, Pilipczuk, Rzazewski, Thomassé and Walczak in [1], and they proved it when H is a path, that is:

1.4 *For every path H , there exists $c > 0$ such that $\partial(G) \leq \tau(G)^c$ for every H -free graph G .*

In this paper we answer the question completely. Our main result is:

1.5 *For every forest H , there exists $c > 0$ such that $\partial(G) \leq \tau(G)^c$ for every H -free graph G .*

We also look at a related question: what can we say about $\chi(G)$ and $\partial(G)$ if G is H -free and does not contain $K_{s,t}$ as a subgraph? More exactly, if H, s are fixed, how do $\chi(G)$ and $\partial(G)$ depend on t ? We will show that the dependence is in fact linear in t :

1.6 *For every forest H and every integer $s > 0$, there exists $c > 0$ such that for every graph G and every integer $t > 0$, if G is H -free and does not contain $K_{s,t}$ as a subgraph, then $\partial(G) \leq ct$.*

We also prove a weaker result, that under the same hypotheses, $\chi(G) \leq ct$, and for this the bound on c is a small function of s, H .

Finally, there is a second pretty theorem in the paper [1] of Bonamy, Pilipczuk, Rzazewski, Thomassé and Walczak:

1.7 *Let ℓ be an integer; then there exists $c > 0$ such that $\partial(G) \leq \tau(G)^c$ for every graph G with no induced cycle of length at least ℓ .*

We give a new proof of this, simpler than that in [1].

In this paper, all graphs are finite and have no loops or parallel edges. We denote by $|H|$ the number of vertices of a graph H . If $X \subseteq V(G)$, we denote the subgraph of G induced on X by $G[X]$. We use “ G -adjacent” to mean adjacent in G , and “ G -neighbour” to mean a neighbour in G , and so on.

2 Producing a path-induced rooted tree.

We will prove 1.5 in this section and the next. We need to show that if a graph G has degeneracy at least some very large polynomial in t (independent of G), and does not contain $K_{t,t}$ as a subgraph, then it contains any desired tree as an induced subgraph. We will show this in two stages: in this section we will show that G contains a large (with degrees a somewhat smaller polynomial in t) “path-induced” tree, and in the next section we will convert this to the desired induced tree. “Path-induced” means that each path of the tree starting at the root is an induced path of G ; so we should be talking about rooted trees. Let us say this carefully.

A *rooted tree* (H, r) consists of a tree H and a vertex r of H called the *root*. A rooted subtree of (H, r) means a rooted tree (J, r) where J is a subtree of H and $r \in V(J)$. The *height* of (H, r) is the length (number of edges) of the longest path of H with one end r . If $u, v \in V(H)$ are adjacent and u lies on the path of H between v, r , we say v is a *child* of u and u is the *parent* of v . The *spread* of H is the maximum over all vertices $u \in V(H)$ of the number of children of u . (Thus the spread is usually one less than the maximum degree.) Let H be a subgraph of G (not necessarily induced), where (H, r) is a rooted tree. We say that (H, r) is a *path-induced rooted subgraph* of G if every path of H with one end r is an induced subgraph of G .

Let $\zeta, \eta \geq 1$. The rooted tree (H, r) is (ζ, η) -uniform if

- every vertex with a child has exactly ζ children;
- every vertex with no child is joined to r by a path of H of length exactly η .

We need two lemmas:

2.1 *Let $k, \zeta, \eta \geq 1$ with $\zeta \geq 2$, and let $(H_1, r_1), \dots, (H_k, r_k)$ be $(k\zeta^{\eta+1}, \eta)$ -uniform rooted trees, each a subgraph of a graph G , such that $r_i \notin V(H_j)$ for all distinct $i, j \in \{1, \dots, k\}$. Then for $1 \leq i \leq k$ there is a (ζ, η) -uniform rooted subtree (H'_i, r_i) of (H_i, r_i) , such that the trees H'_1, \dots, H'_k are pairwise vertex-disjoint.*

Proof. Choose $j \leq k$ maximum such that there are (ζ, η) -uniform rooted subtrees (H'_i, r_i) of (H_i, r_i) for $1 \leq i \leq j$, such that the trees H'_1, \dots, H'_j are pairwise vertex-disjoint. Let $X = V(H'_1) \cup \dots \cup V(H'_j)$. Thus $|X| \leq j\zeta^{\eta+1}$, since each H'_i has

$$1 + \zeta + \zeta^2 + \dots + \zeta^\eta \leq \zeta^{\eta+1}$$

vertices (here we use that $\zeta \geq 2$). Suppose that $j < k$. Then each vertex of (H_{j+1}, r_{j+1}) with a child has at least $(k-j)\zeta^{\eta+1} \geq \zeta^{\eta+1} \geq \zeta$ children not in X ; and since $r_{j+1} \notin X$, it follows that there is a (ζ, η) -uniform rooted subtree (H'_{j+1}, r_{j+1}) of (H_{j+1}, r_{j+1}) vertex-disjoint from X , contrary to the maximality of j . Thus $j = k$, and this proves 2.1. \blacksquare

Let $t, \eta \geq 1$ and $\zeta \geq 2$ be integers. Let (T, r) be a (ζ, η) -uniform rooted tree, where T is a subgraph of G . A vertex $u \in V(G) \setminus V(T)$ is *t-bad* for (T, r) if there is a vertex $w \in V(T)$, with ζ children in (T, r) , such that u is G -adjacent to more than $(t-1)\zeta/t$ of these children. We will often use the following:

2.2 *Let $t, \eta \geq 1$ and $\zeta \geq 2$ be integers. Let (T, r) be a $(t\zeta, \eta)$ -uniform rooted tree, where T is a subgraph of G ; and let $u \in V(G) \setminus V(T)$. If u is not t -bad for (T, r) , then there is a (ζ, η) -uniform rooted subtree (S, r) of (T, r) such that u has no G -neighbour in $V(S)$ except possibly r .*

We omit the proof, which is clear. The second lemma is:

2.3 *Let $t, \eta \geq 1$ and $\zeta \geq 2$ be integers, where t divides ζ . Let G be a graph that does not contain $K_{t,t}$ as a subgraph, and let (T, r) be a (ζ, η) -uniform rooted tree, where T is a subgraph of G . Then at most $\zeta^\eta(t-1)$ vertices in $V(G) \setminus V(T)$ are t -bad for (T, r) .*

Proof. For each $w \in V(T)$ that has ζ children, let C_w be the set of its children in (T, r) . Suppose that there are t distinct vertices u_1, \dots, u_t in $V(G) \setminus V(T)$ such that each is G -adjacent to more than $|C_w|(t-1)/t$ vertices in C_w . Since t divides $|C_w|$, it follows that each u_i has at most $|C_w|/t - 1$ G -non-neighbours in C_w , and so at most $t(|C_w|/t - 1)$ vertices in C_w are G -nonadjacent to one of u_1, \dots, u_t . Consequently at least t vertices in C_w are G -adjacent to all of u_1, \dots, u_t , contradicting that G does not contain $K_{t,t}$ as a subgraph. Thus there are at most $t-1$ vertices in $V(G) \setminus V(T)$ with more than $|C_w|(t-1)/t$ G -neighbours in C_w . So the number of vertices in $V(G) \setminus V(T)$ that are t -bad for (T, r) is at most $(t-1)$ times the number of vertices of T that have children, and so at most $\zeta^\eta(t-1)$ (since $\zeta \geq 2$). This proves 2.3. \blacksquare

The main result of this section is the following:

2.4 *Let $\eta > 0$ be an integer and let $c = (\eta+1)!$. Let $\zeta \geq 2$, and let (H, r) be a rooted tree of height at most η , and spread at most ζ . Let $t \geq 1$ be an integer, and suppose that the graph G does not contain $K_{t,t}$ as a subgraph, and does not contain a rooted tree isomorphic to (H, r) as a path-induced rooted subgraph. Then $\partial(G) \leq (\zeta t)^c$.*

Proof. We may assume that $t \geq 2$. We proceed by induction on η . If $\eta = 1$, it follows that G has maximum degree at most $\zeta - 1$, since it does not contain (H, r) as a path-induced rooted subgraph; and so $\partial(G) \leq \zeta - 1 \leq (\zeta t)^c$ as required. So we may assume that $\eta \geq 2$, and the result holds for all rooted trees with height less than η . Let $c' = \eta!$ and $\zeta' = t\zeta^{\eta+1}$. Let us say a *limb* is a $(\zeta', \eta-1)$ -uniform rooted tree that is a path-induced rooted subgraph of G .

(1) *For each vertex u , there are at most $\zeta - 1$ G -neighbours v of u with the property that there is a limb (J, v) of G such that $u \notin V(J)$ and u is not t -bad for (J, v) .*

Suppose there are ζ such vertices v_1, \dots, v_ζ , and let the corresponding limbs be (J_i, v_i) for $1 \leq i \leq \zeta$.

By 2.2, for $1 \leq i \leq \zeta$, there is a $(\zeta^{\eta+1}, \eta - 1)$ -uniform rooted subtree (J'_i, v_i) of (J_i, v_i) , such that u has no neighbour in $V(J'_i)$ except v_i . By 2.1, there is a $(\zeta, \eta - 1)$ -uniform rooted subtree (H'_i, r_i) of (J'_i, r_i) for $1 \leq i \leq \zeta$, such that the trees H'_1, \dots, H'_k are pairwise vertex-disjoint. But then adding u to the union of these trees gives a (ζ, η) -uniform rooted tree, and it is path-induced in G , and contains a rooted induced subgraph isomorphic to (H, r) , a contradiction. This proves (2).

Let P be the set of vertices v of G such that there is a limb with root v , and let $Q = V(G) \setminus P$. For each $v \in P$, there is at least one limb with root v ; select one, and call it (J_v, v) . For each edge e with at least one end in P , select one such end, and call it the *head* of e .

- Let A be the set of all edges with both ends in Q ;
- Let B be the set of all edges uv of G with head v , such that $u \notin V(J_v)$, and u is not t -bad for (J_v, v) ;
- Let C be the set of all edges uv of G with head v , such that $u \notin V(J_v)$, and u is t -bad for (J_v, v) ;
- Let D be the set of all edges uv of G with head v , such that $u \in V(J_v)$.

Thus every edge of G belongs to exactly one of A, B, C, D . Since $G[Q]$ does not contain a limb, the inductive hypothesis implies that $\partial(G[Q]) \leq (\zeta't)^{c'}$. Consequently

$$|A| \leq (\zeta't)^{c'}|Q| \leq (\zeta't)^{c'}|G|.$$

By (1), for each vertex $u \in V(G)$, there are at most $\zeta - 1$ edges $uv \in B$ with head v ; and so

$$|B| \leq (\zeta - 1)|G|.$$

For each $v \in P$, there are at most $\zeta^{\eta-1}(t - 1)$ edges $uv \in C$ with head v by 2.3, and so

$$|C| \leq \zeta^{\eta-1}(t - 1)|P| \leq \zeta^{\eta-1}(t - 1)|G|.$$

For each $v \in P$, since (J_v, v) is path-induced, there are at most ζ' edges $uv \in D$ with head v , and so

$$|D| \leq \zeta'|P| \leq \zeta'|G|.$$

Summing, we deduce that

$$|E(G)| \leq \left((\zeta't)^{c'} + (\zeta - 1) + \zeta^{\eta-1}(t - 1) + \zeta' \right) |G|,$$

and so some vertex of G has degree at most $2 \left((\zeta't)^{c'} + (\zeta - 1) + \zeta^{\eta-1}(t - 1) + \zeta' \right)$. Since this also holds for every non-null induced subgraph of G , we deduce that

$$\partial(G) \leq 2 \left((\zeta't)^{c'} + (\zeta - 1) + \zeta^{\eta-1}(t - 1) + \zeta' \right).$$

We recall that $\zeta' = t\zeta^{\eta+1}$; and so

$$\partial(G) \leq 2 \left(\zeta^{c'(\eta+1)}t^{2c'} + (\zeta - 1) + \zeta^{\eta-1}(t - 1) + \zeta^{\eta+1}t \right).$$

Of the four terms on the right side, the sum of the second and third is at most the fourth, so

$$\partial(G) \leq 2 \left(\zeta^{c'(\eta+1)} t^{2c'} + 2\zeta^{\eta+1} t \right);$$

and since $c' \geq 2$, the second term here is at most the first, so

$$\partial(G) \leq 4\zeta^{c'(\eta+1)} t^{2c'} \leq \zeta^{c'(\eta+1)} t^{2c'+2}$$

(since we may assume that $t \geq 2$, and so $t^2 \geq 4$). Consequently $\partial(G) \leq \zeta^{c'(\eta+1)} t^{2c'+2}$. But $c = c'(\eta+1)$ and $2c' + 2 \leq c$, and so $\partial(G) \leq (\zeta t)^c$. This proves 2.4. \blacksquare

We remark that 2.4 implies 1.4, and a strengthening:

2.5 *If H is a path, and $t \geq 1$ is an integer, and G is H -free and does not contain $K_{t,t}$ as a subgraph, then $\partial(G) \leq (2t)^{|H|!}$.*

Proof. Let $\zeta = 2$, and $\eta = |E(H)|$. Let r be one end of H . Then G does not contain (H, r) as a path-induced rooted subgraph, and so $\partial(G) \leq (2t)^{|H|!}$ by 2.4. This proves 2.5. \blacksquare

3 Growing a tree

If (T, r) is a rooted tree and $v \in V(T)$, the *height* of v in (T, r) is the number of edges in the path between v, r ; and so the height of (T, r) is the largest of the heights of its vertices. Let (T, r) be a rooted tree, and let (S, r) be a rooted subtree. The graph obtained from T by deleting all the edges of S is disconnected, and each of its components contains a unique vertex of S ; for each $v \in V(S)$, let T_v be the component that contains $v \in V(S)$. We call the rooted tree (T_v, v) the *decoration of S at v in T* .

Let G be a graph, let (S, r) be a rooted tree, and let $\zeta \geq 2$ and $\eta \geq 1$. We say that (S, r) is (ζ, η) -*decorated* in G if S is an induced subgraph of G with height at most $\eta - 1$, and there is a rooted tree (T, r) with the following properties:

- (S, r) is a rooted subtree of (T, r) , and (T, r) is a path-induced rooted subgraph of G ;
- for each $u \in V(S)$ and $v \in V(T) \setminus V(S)$, if u, v are G -adjacent then they are T -adjacent;
- for each $v \in V(S)$, the decoration of S at v in T is $(\zeta, \eta - h)$ -uniform, where h is the height of v in (S, r) .

Thus, informally, T is obtained from S by attaching to S uniform trees rooted at each vertex of S . Note that T is only required to be path-induced: the various uniform trees that are attached to S might have edges between them.

In view of 2.4, if we have a graph G with huge degeneracy that does not contain $K_{t,t}$, then it contains a (ζ, η) -uniform rooted tree (T, r) as a path-induced rooted subgraph; and consequently there is a one-vertex rooted tree (S, r) that is (ζ, η) -decorated in G . The next result shows that if we start with ζ large enough, then by reducing ζ we can grow S into any larger tree that we wish, and that will prove 1.5.

3.1 Let $\eta, t \geq 1$ and $\zeta \geq 2$ be integers, let G be a graph that does not contain $K_{t,t}$ as a subgraph, and let (S', r) be a (ζ', η) -decorated rooted tree in G , where $\zeta' \geq \zeta^\eta |S'| t^{\eta+1}$. Let $p \in V(S')$ with height in (S', r) less than η . Then there is a G -neighbour q of p , with $q \in V(G) \setminus V(S')$, and with no other G -neighbour in $V(S')$, such that, if S denotes the tree obtained from S' by adding q and the edge pq , then (S, r) is a (ζ, η) -decorated rooted tree in G .

Proof. For each $v \in V(S')$, let $h(v)$ denote the height of v in (S', r) . Since (S', r) is (ζ', η) -decorated in G , it follows that S' is an induced subgraph of G , and there is a rooted tree (T', r) such that

- (S', r) is a rooted subtree of (T', r) , and (T', r) is a path-induced rooted subgraph of G ;
- for each $u \in V(S')$ and $v \in V(T') \setminus V(S')$, if u, v are G -adjacent then they are T' -adjacent;
- for each $v \in V(S')$, the decoration of S' at v in T' is $(\zeta', \eta - h(v))$ -uniform.

For each $v \in V(S')$, let (T_v, v) be the decoration of S' at v in T' . Since T_p is $(\zeta', \eta - h(p))$ -uniform, and $h(p) < \eta$, it follows that p has ζ' children in (T_p, p) . We need to select one of these children, say q , to add to S' , forming S . Any one of them would make a larger induced tree when added to S' , since (S', r) is a (ζ, η) -decorated. But in order to make the new rooted tree (S, r) (ζ, η) -decorated, we will delete from T' all vertices of T' that are G -adjacent and not T' -adjacent to q ; and doing so must not destroy too much of T' .

For each $v \in V(S')$, let (S_v, v) be a $(t\zeta, \eta - h(v))$ -uniform rooted subtree of (T_v, v) . By 2.3, there are at most $(t\zeta)^{\eta-h(v)}(t-1) < t^{\eta+1}\zeta^\eta$ vertices not in $V(S_v)$ that are t -bad for (S_v, v) , and so there fewer than $\zeta^\eta |S'| t^{\eta+1} \leq \zeta'$ children of p in (T_p, p) that are t -bad for one of the rooted trees (S_v, v) ($v \in V(S')$). Hence there is at least one child q of p in (T_p, p) that is t -bad for none of the trees (S_v, v) ($v \in V(S')$). Moreover we claim that we can choose q such that $q \notin V(S_p)$. This is automatic if (S_p, p) has height at least two, since then every child of p in (S_p, p) is bad for (S_p, p) , so we may assume that (S_p, p) has height one, that is, $h(p) = \eta - 1$. Consequently no child of p in (T_p, p) is t -bad for (S_p, p) , and so the number that are t -bad for one of the rooted trees (S_v, v) ($v \in V(S')$) is at most $\zeta' - t^{\eta+1}\zeta^\eta < \zeta' - t\zeta$. This proves that we can choose q such that $q \notin V(S_p)$.

Let Q be the component containing q of the graph obtained from T' by deleting $V(S)$; thus (Q, q) is $(\zeta', \eta - h(p) - 1)$ -uniform, and so we may choose a $(\zeta, \eta - h(p) - 1)$ -uniform rooted subtree (R_q, q) of (Q, q) . Note that q has no neighbours in $V(Q)$ except its neighbours in T' , since (T', r) is path-induced. Since q is not t -bad for any of the rooted trees (S_v, v) ($v \in V(S')$), it follows by 2.2 that for each v there is a $(\zeta, \eta - h(v))$ -uniform rooted subtree (R_v, v) of (S_v, v) such that q has no G -neighbour in $V(R_v)$ except possibly v , and q is G -adjacent to v if and only if they are T' -adjacent (that is, $v = p$), since $v \in V(S')$ and (S', r) is (ζ', η) -decorated. Let S be the tree induced on $V(S') \cup \{q\}$, and let T be the union of T' , the trees R_v ($v \in V(S') \cup \{q\}$) and the edge pq . Then S satisfies the theorem, because the tree T exists. This proves 3.1. ■

We deduce 1.5, which we restate in a strengthened form:

3.2 Let $\eta, t \geq 1$ and $\zeta \geq 2$. For every rooted tree (H, r) with height at most η and spread at most ζ , let $c = (\eta + 3)!|H|$; then $\partial(G) \leq (|H|\zeta t)^c$ for every H -free graph G that does not contain $K_{t,t}$ as a subgraph.

Proof. It suffices to prove the statement for all trees H , and it is helpful to assign a root s to H , so (H, s) is a rooted tree. Choose $\eta \geq 1$ and $\zeta \geq 2$ such that (H, s) has height at most η and spread at most ζ . Let H have k vertices. Define $\zeta_k = \zeta$, and for $i = k-1, k-2, \dots, 1$ let $\zeta_i = i\zeta_{i+1}^\eta t^{\eta+1}$.

Let G be an H -free graph that does not contain $K_{t,t}$ as a subgraph. Suppose that G contains a one-vertex rooted tree that is (ζ_1, η) -decorated in G . Choose a maximal rooted subtree (F, s) of (H, s) such that there is a rooted subtree (S, r) of G , isomorphic to (F, s) , such that (S, r) is (ζ_i, η) -decorated in G , where $i = |F|$. By 3.1, $i = k$; and so G contains an induced subgraph isomorphic to H , a contradiction.

Thus G contains no one-vertex rooted tree that is (ζ_1, η) -decorated in G . Hence G contains no (ζ_1, η) -uniform rooted tree as a path-induced rooted subgraph, and so by 2.4 (applied with (H, r) replaced by a (ζ_1, η) -uniform rooted tree), $\partial(G) \leq (\zeta_1 t)^d$ where $d = (\eta + 1)!$.

Now $\zeta_k = \zeta$, and $\zeta_{k-1} = (k-1)\zeta^\eta t^{\eta+1}$. For all i with $1 \leq i \leq k-2$, $\zeta_{i+1} \geq i t^{\eta+1}$, and so $\zeta_i = i\zeta_{i+1}^\eta t^{\eta+1} \leq \zeta_{i+1}^{\eta+1}$. Consequently

$$\zeta_1 \leq \zeta_{k-1}^{(k-2)(\eta+1)} \leq (k\zeta^\eta t^{\eta+1})^{(k-2)(\eta+1)} \leq (k\zeta t)^{(k-2)(\eta+1)^2}.$$

So $\partial(G) \leq (k\zeta t)^c$ where $c = (k-2)(\eta+1)^2(\eta+1)! + (\eta+1)! \leq (\eta+3)!k$. This proves 3.2. \blacksquare

4 Excluding $K_{s,t}$

In this section we prove 1.6, and before that we prove a weaker statement, with $\partial(G)$ replaced by $\chi(G)$. For the latter we need the following lemma:

4.1 *Let J be a digraph such that every vertex has outdegree at most k . Then the undirected graph underlying J has chromatic number at most $2k + 1$.*

Proof. Let G be the undirected graph underlying J . Since every subgraph of G has the property that its edges can be directed so that it has outdegree at most k , it follows that every such subgraph H has at most $k|H|$ edges; and therefore (if it is non-null) has a vertex of degree at most $2k$. Consequently G is $2k$ -degenerate, and so is $(2k+1)$ -colourable. This proves 4.1. \blacksquare

We use 4.1 to prove the following (which we include here because the proof gives a relatively small constant c , although the fact that some c exists follows from 1.6):

4.2 *Let H be a tree and $s \geq 1$ an integer, and let $c = (2s|H|)^{s+|H|}$. Then for every H -free graph G and every integer $t \geq 1$, if G does not contain $K_{s,t}$ as a subgraph then $\chi(G) \leq ct$.*

Proof. We will prove this by induction on $|H|$ (for the same value of s). Let H be a tree and $s \geq 0$ an integer, and suppose the theorem holds for all smaller trees and the same value of s . We may assume that $|H| \geq 3$, since the theorem is true for trees with at most two vertices; let $p \in V(H)$ have degree one, and let q be its H -neighbour. Let H' be obtained by deleting p from H . Let $c' = (2s|H'|)^{s+|H'|}$. We observe that

$$(1) \quad c \geq \max\left((|H|-2)^{s-1}, (s-1)(|H|-2), (2(s-2)(|H|-2)+1)c'+1\right).$$

Let $t \geq 1$ be an integer, and let G be an H -free graph not containing $K_{s,t}$ as a subgraph. We will show that $\chi(G) \leq ct$. Suppose that this is false, and choose a minimal induced subgraph G' of G with $\chi(G') > ct$. It follows that every vertex of G' has degree at least ct (since c is an integer).

Let $v \in V(G')$. We say a subset $X \subseteq V(G') \setminus \{v\}$ is a v -bag if there is an isomorphism from H' to $G[X \cup \{v\}]$ that maps q to v . (Thus each v -bag has cardinality $|H| - 2$.)

Let $v \in V(G')$, and suppose that there are $s - 1$ pairwise disjoint v -bags, say X_1, \dots, X_{s-1} . Since G is H -free, every G -neighbour u of v either belongs to X_i or has a G -neighbour in X_i , for $1 \leq i \leq s - 1$. In particular, every G -neighbour u of v not in $X_1 \cup \dots \cup X_{s-1}$ has a G -neighbour in each of X_1, \dots, X_{s-1} . But for each choice of $x_i \in X_i$ ($1 \leq i \leq s - 1$) there are at most $t - 1$ G -neighbours of v G -adjacent to each of x_1, \dots, x_{s-1} (since they are also all adjacent to v , and G has no $K_{s,t}$ subgraph). Consequently there are at most $(t - 1)(|H| - 2)^{s-1}$ G -neighbours of v not in $X_1 \cup \dots \cup X_{s-1}$; and hence

$$(s - 1)(|H| - 2) + (t - 1)(|H| - 2)^{s-1} > ct.$$

Since $ct = c + c(t - 1)$, and $(s - 1)(|H| - 2) < c$, and $(t - 1)(|H| - 2)^{s-1} \leq c(t - 1)$, this contradicts (1); so there is no such choice of X_1, \dots, X_{s-1} .

Choose an integer r maximum such that there are r pairwise disjoint v -bags, say X_1, \dots, X_r . Consequently $r \leq s - 2$. Let $Y_v = X_1 \cup \dots \cup X_r$; then from the maximality of r , $X \cap Y_v \neq \emptyset$ for every v -bag X . Moreover $|Y_v| \leq (s - 2)(|H| - 2)$.

Let J be the digraph with vertex set $V(G')$ in which every vertex in Y_v is J -adjacent from v , for each $v \in V(G')$. Thus J has maximum outdegree at most $(s - 2)(|H| - 2)$, and so by 4.1, the undirected graph J' underlying J has chromatic number at most $2(s - 2)(|H| - 2) + 1$; and so $V(G') = V(J')$ can be partitioned into $2(s - 2)(|H| - 2) + 1$ sets each of which is a stable set of J' . Let Z be one of these sets. Then $G[Z]$ is H' -free (because otherwise there would be a vertex $v \in Z$, and a subset $X \subseteq Z \setminus \{v\}$, and an isomorphism from H' to $G[X \cup \{v\}]$ mapping q to v , and hence with $X \cap Y_v \neq \emptyset$; but no vertex of Y_v belongs to Z , since Z is stable in J'). From the inductive hypothesis, $\chi(Z) \leq c't$, and hence

$$ct < \chi(G) = \chi(G') \leq (2(s - 2)(|H| - 2) + 1)c't$$

contrary to (1). This proves 4.2. ▀

To prove 1.6, we will need the following strengthening of 1.3, also proved in [7]:

4.3 *For every forest H , and every integer $s > 0$, there is a tree S such that for every H -free graph G , if G contains S as a subgraph, then G contains $K_{s,s}$ as a subgraph.*

Now we prove 1.6, which we restate:

4.4 *For every forest H and every integer $s > 0$, there exists $c > 0$ such that for every graph G and every integer $t > 0$, if G is H -free and does not contain $K_{s,t}$ as a subgraph, then $\partial(G) < ct$.*

Proof. Let S be as in 4.3, and let $c = |S|^s$; we will show that c satisfies the theorem. Let $t > 0$ be an integer, and let G be an H -free graph that does not contain $K_{s,t}$ as a subgraph. Suppose that $\partial(G) \geq ct$, and choose G minimal with these properties: then every vertex of G has degree at least ct .

(1) Let R be a tree. If every vertex of G has degree at least $t|R|^s$, then G contains a subgraph T isomorphic to R , and $V(T)$ can be ordered as $\{t_1, \dots, t_n\}$, such that for $1 \leq i \leq n$, t_i is G -adjacent to at most $s - 1$ of t_1, \dots, t_{i-1} .

We prove this by induction on $|R|$. We may assume that $|R| > 1$; let $p \in V(R)$ have degree one in R , and let q be its R -neighbour. Let R' be obtained from R by deleting p . From the inductive hypothesis, G contains a subgraph T' isomorphic to R' , and its vertex set can be ordered as $\{t_1, \dots, t_{n-1}\}$, such that for $1 \leq i \leq n - 1$, t_i is G -adjacent to at most $s - 1$ of t_1, \dots, t_{i-1} . Choose $v \in V(T')$ such that some isomorphism from R' to T' maps q to v . If some G -neighbour u of v does not belong to $V(T')$ and has at most $s - 1$ G -neighbours in $V(T')$, then we may set $t_n = u$ as required; so we may assume that every G -neighbour u of v in G either belongs to $V(T')$ or has at least s G -neighbours in $V(T')$. Let $X \subseteq V(T')$ with $|X| = s$. If there are at least t vertices in $V(G)$ that are G -adjacent to every vertex in X , then G contains $K_{s,t}$ as a subgraph, a contradiction. So for each such X , there are at most $t - 1$ vertices in $V(G)$ that are G -adjacent to every vertex in X . Since there are most $|R'|^s$ choices of X , there are at most $(t - 1)|R'|^s$ vertices in $V(G) \setminus V(T')$ that have at least s G -neighbours in $V(T')$. Consequently v has at most $(t - 1)|R'|^s$ G -neighbours not in $V(T')$. But it has at most $|R'|$ G -neighbours in $V(T')$ and so the degree of v in G is at most $(t - 1)|R'|^s + |R'| < t|R|^s$. This proves (1).

Each vertex of G has degree at least $ct = t|S|^s$; let us apply (1) taking $R = S$. We deduce that G contains a subgraph T isomorphic to S , and its vertex set can be ordered as $\{t_1, \dots, t_n\}$, such that for $1 \leq i \leq n$, t_i is G -adjacent in G to at most $s - 1$ of t_1, \dots, t_{i-1} . By 4.3, $G[V(T)]$ contains $K_{s,s}$ as a subgraph. Choose i maximum such that t_i belongs to this subgraph; then t_i is G -adjacent to at least s vertices that are earlier in the ordering, a contradiction. This proves 4.4. ■

5 Long holes

There is another result in the paper by Bonamy et al. [1]:

5.1 Let $\ell \geq 2$ be an integer; then there exists $c > 0$ that $\partial(G) \leq \tau(G)^c$ for every graph G with no induced cycle of length at least ℓ .

In this section we give a simpler proof of this result.

Let $\eta, t \geq 1$ be integers. We say a rooted tree (H, r) is (t, η) -tapering if (H, r) has height η , and every vertex $v \in V(H)$ of height $i < \eta$ has exactly $t^{\eta-i}$ children. For each $v \in V(H)$, let $h(v)$ be its height in (H, r) .

Let G be a graph. A map ϕ from $V(H)$ into $V(G)$ is a (t, η) -infusion of (H, r) into G if

- for all distinct $u, v \in V(H)$, if $u, v \in V(H)$ are H -adjacent then $\phi(u), \phi(v)$ are distinct and G -adjacent;
- for each $u \in V(H)$, if v, w are distinct children of u in (H, r) , then $\phi(v) \neq \phi(w)$;
- for every path P of H with one end r , the vertices $\phi(v)$ ($v \in V(P)$) are all distinct; and

- for every path P of H with one end r , and for all distinct $u, v \in V(P)$, $\phi(u), \phi(v)$ are G -adjacent if and only if u, v are H -adjacent.

Let ϕ be a (t, η) -infusion into G . We define $V(\phi) = \{\phi(v) : v \in V(H)\}$, and we define the *root* of ϕ to be $\phi(r)$. We say $u \in V(G)$ is *t-bad* for ϕ if there exists $v \in V(H)$ with $h(v) < \eta$, such that u is distinct from and G -adjacent to $\phi(w)$ for more than $(t-1)t^{\eta-h(v)-1}$ children w of v in (H, r) . Then we have:

5.2 *Let $t, \eta \geq 1$ be integers, let (H, r) be a (t, η) -tapering rotted tree, let G be a graph not containing $K_{t,t}$ as a subgraph, and let ϕ be a (t, η) -infusion of (H, r) into G . There are at most t^{η^2} vertices in G that are t-bad for ϕ .*

The proof is like that for 2.3, using that H has at most $t^{\eta-1}$ vertices that have children, and we omit it.

The next result strengthens 1.7:

5.3 *Let $\eta \geq 2$ be an integer, and let G be a graph with no induced cycle of length more than η . For every integer $t \geq 1$, if G does not contain $K_{t,t}$ as a subgraph then $\partial(G) \leq t^{\eta^2}$.*

Proof. We may assume that $t \geq 2$. Let $t \geq 1$ be an integer, and let G be a graph with no induced cycle of length more than η that does not contain $K_{t,t}$. Let (H, r) be a (t, η) -tapering rooted tree (not necessarily contained in G).

(1) *If $u \in V(G)$ and v_i is a G -neighbour of u for $1 \leq i \leq t^\eta$, all distinct, and for each i there is a (t, η) -infusion of (H, r) into G with root v_i , such that $u \notin V(\phi_i)$, and u is not t-bad for ϕ_i , then there is a (t, η) -infusion of (H, r) into G , with root u .*

Let (H', r) be a $(t, \eta - 1)$ -tapering rooted subtree of (H, r) . It follows (analogously to 2.2) that for $1 \leq i \leq t^\eta$, there is a $(t, \eta - 1)$ -infusion ϕ'_i of (H', r) into G such that u has no G -neighbour in $V(\phi'_i)$ except v_i . Let us number the components of $H \setminus \{r\}$ as H_1, \dots, H_{t^η} . Let $\psi(r) = v$, and for $1 \leq i \leq t^\eta$ and each $v \in V(H_i)$, define $\psi(v) = \phi'_i(w)$ where w is the parent of v in (H, r) . Then ψ is a (t, η) -infusion of (H, r) into G , with root v . This proves (1).

In these circumstances we say that ψ , constructed as in the proof of (1), is *derived from* the sequence $(\phi_i : 1 \leq i \leq t^\eta)$.

If P is a path of H with length η and one end r , and ϕ is a (t, η) -infusion of (H, r) into G , then ϕ maps P to an induced path $\phi(P)$ of G with length η and with one end the root of ϕ . We call $\phi(P)$ a *column* of ϕ . We observe that if ψ is derived from $(\phi_i : 1 \leq i \leq t^\eta)$ as above, then for every column Q of ψ , there is a column Q' of one of $\phi_i (1 \leq i \leq t^\eta)$, say of ϕ'_i , such that $Q \setminus \psi(r)$ is a subpath of Q' . Let us call (ϕ', Q') a *shift* of (ϕ, Q) .

Let \mathcal{A}_1 be the set of all (t, η) -infusions of (H, r) into G . Inductively for $i > 1$, let \mathcal{A}_i be the set of all (t, η) -infusions ϕ such that for some choice of $\phi_i (1 \leq i \leq t^\eta) \in \mathcal{A}_{i-1}$, ϕ is derived from $(\phi_i : 1 \leq i \leq t^\eta)$. Thus $\mathcal{A}_i \subseteq \mathcal{A}_{i-1}$ for each i . There are two cases: either \mathcal{A}_i is empty for some i , or it remains nonempty for all values of i . Suppose first that \mathcal{A}_i is nonempty for all i , and let \mathcal{A} be the intersection of all the sets $\mathcal{A}_i (i \geq 1)$. Choose $\phi_1 \in \mathcal{A}$, and let Q_1 be a column of ϕ_1 . Since ϕ_1 is derived from some members of \mathcal{A} , there exists $\phi_2 \in \mathcal{A}$ with root u_2 , and a column Q_2 of ϕ_2 , such that

(ϕ_2, Q_2) is a shift of (ϕ_1, Q_1) . Similarly we can choose an infinite sequence (ϕ_i, Q_i) ($i = 1, 2, 3, \dots$) such that each $\phi_i \in \mathcal{A}$ and each (ϕ_i, Q_i) is a shift of its predecessor. Let v_i be the root of ϕ_i for each i . Then $v_i, v_{i+1}, \dots, v_{i+\eta}$ are the vertices in order of Q_i for each i ; and so form an induced path of G . Since G is finite, there exists $j > 0$ such that v_j is adjacent to one of v_1, \dots, v_{j-2} ; choose a minimum such value of j , and choose $i \leq j - 2$ maximum such that v_i, v_j are adjacent. Then $\{v_i, \dots, v_j\}$ induces a cycle of G of length more than η , a contradiction.

So the second case holds, that is, \mathcal{A}_i is empty for some i . Choose $k + 1$ minimum such that $\mathcal{A}_{k+1} = \emptyset$. For $1 \leq i \leq k$ let X_i be the set of all vertices v such that v is the root of a member of \mathcal{A}_i and not the root of any member of \mathcal{A}_{i+1} . Thus the sets X_1, \dots, X_k are pairwise disjoint. Let X_0 be the set of vertices that are not the root of any member of \mathcal{A}_1 ; so the sets X_0, \dots, X_k form a partition of $V(G)$. For each edge e of G with an end in one of X_1, \dots, X_k , choose i maximum such that e has an end in X_i , let v be an end of e in X_i , and call v the *head* of e . For each $v \in X_i$, choose $\phi_v \in \mathcal{A}_i$ with root v . (Thus $\phi_v \notin \mathcal{A}_{i+1}$ from the definition of X_i .) Define

- A is the set of all edges of G with both ends in X_0 ;
- B is the set of all edges uv with head v such that $u \notin V(\phi_v)$ and u is not bad for ϕ_v ;
- C is the set of all edges uv with head v such that $u \notin V(\phi_v)$ and u is bad for ϕ_v ;
- D is the set of all edges uv with head v such that $u \in V(\phi_v)$.

Since there is no (t, η) -infusion of (H, r) into $G[X_0]$, it follows that $G[X_0]$ does not contain a (ζ, η) -uniform tree as a path-induced rooted subgraph, where $\zeta = t^\eta$, and so $\partial(G[X_0]) \leq (\zeta t)^{(\eta+1)!}$ from 2.4. Hence

$$|A| \leq (\zeta t)^{(\eta+1)!} |G|.$$

For each $u \in V(G)$, with $u \in X_i$ say, there do not exist t^η neighbours v of u such that uv has head v and belongs to B , since there is no (t, η) -infusion of (H, r) with root u that is derived from members of \mathcal{A}_i . Hence

$$|B| \leq t^\eta |G|.$$

For each $v \in V(G)$, there are at most t^{η^2} neighbours u of v such that the edge uv has head v and belongs to C , by 5.2; so

$$|C| \leq t^{\eta^2} |G|.$$

Finally, for each $v \in V(G)$, there are at most t^η neighbours u of v such that the edge uv has head v and belongs to D ; so

$$|D| \leq t^\eta |G|.$$

Summing, we obtain

$$|E(G)| \leq \left((t^{\eta+1})^{(\eta+1)!} + t^\eta + t^{\eta^2} + t^\eta \right) |G| \leq \left(t^{(\eta+2)!} + t^{\eta^2} \right) |G| \leq t^{7\eta^2} / 2.$$

Consequently $\partial(G) \leq t^{7\eta^2}$. This proves 5.3. ■

Acknowledgement

We would like to express our thanks to Andras Gyárfás, who clarified the somewhat confusing history of the authorship of 1.2 for us.

References

- [1] M. Bonamy, N. Bousquet, M. Pilipczuk, P. Rzazewski, S. Thomassé and B. Walczak, “Degeneracy of P_t -free and $C_{\geq t}$ -free graphs with no large complete bipartite subgraphs”, [arXiv:2012.03686](#).
- [2] M. Chudnovsky, A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. XII. Distant stars”, *J. Graph Theory* **92** (2019), 237–254, [arXiv:1711.08612](#).
- [3] L. Esperet, *Graph Colorings, Flows and Perfect Matchings*, Habilitation thesis, Université Grenoble Alpes (2017), 24.
- [4] A. Gyárfás, “On Ramsey covering-numbers”, in *Infinite and Finite Sets, Vol. II* (Colloq., Keszthely, 1973), *Coll. Math. Soc. János Bolyai* **10**, 801–816.
- [5] A. Gyárfás, “Problems from the world surrounding perfect graphs”, *Proceedings of the International Conference on Combinatorial Analysis and its Applications*, (Pokrzywna, 1985), *Zastos. Mat.* **19** (1987), 413–441.
- [6] A. Gyárfás, E. Szemerédi and Zs. Tuza, “Induced subtrees in graphs of large chromatic number”, *Discrete Math.* **30** (1980), 235–344.
- [7] H. A. Kierstead and S.G. Penrice, “Radius two trees specify χ -bounded classes”, *J. Graph Theory* **18** (1994), 119–129.
- [8] H. A. Kierstead and V. Rödl, “Applications of hypergraph coloring to coloring graphs not inducing certain trees”, *Discrete Math.* **150** (1996), 187–193.
- [9] H. A. Kierstead and Y. Zhu, “Radius three trees in graphs with large chromatic number”, *SIAM J. Disc. Math.* **17** (2004), 571–581.
- [10] A. Scott, “Induced trees in graphs of large chromatic number”, *J. Graph Theory* **24** (1997), 297–311.
- [11] A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. XIII. New brooms”, *European J. Combinatorics* **84** (2020), 103024, [arXiv:1807.03768](#).
- [12] A. Scott and P. Seymour, “A survey of χ -boundedness”, *J. Graph Theory* **95** (2020), 473–504, [arXiv:1812.07500](#).
- [13] D. P. Sumner, “Subtrees of a graph and chromatic number”, in *The Theory and Applications of Graphs*, (G. Chartrand, ed.), John Wiley & Sons, New York (1981), 557–576.