Line-width and path-width

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August 22, 2025; revised September 21, 2025

[1]Supported by EPSRC grant EP/X013642/1
[2]Supported by AFOSR grant FA9550-22-1-0234, and by NSF grant DMS-2154169.

Abstract

For finite graphs, path-width is an interesting and useful concept, but if we extend it to infinite graphs in the most obvious way (by making the indexing path infinite), it does not work nicely. The simplest extension that works nicely is to allow the indexing set to be any totally-ordered set, and then the corresponding decomposition is called a “line-decomposition”, and the maximum bag size needed is called “line-width”.

In particular, the indexing set need not be a well-order; but the corresponding decomposition would be easier to use if it was. We show that if a graph has line-width at most k, it admits a well-ordered line-decomposition with width at most $2k$, and this is best possible.

1 Introduction

A *tree-decomposition* of a finite graph G is a pair $(T, (W_t : t \in V(T)))$, where T is a finite tree, and W_t is a subset of $V(G)$ for each $t \in V(T)$ (called a *bag*), such that:

- $G = \bigcup_{t \in V(T)} G[W_t]$, and
- for all $t_1, t_2, t_3 \in V(T)$, if t_2 lies on the path of T between t_1, t_3, then $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$.

The *width* of a tree-decomposition $(T, (W_t : t \in V(T)))$ is the maximum of the numbers $|W_t| - 1$ for $t \in V(T)$, and the *tree-width* of G is the minimum width of a tree-decomposition of G.

Similarly, a *path-decomposition* of a finite graph G is the same except that we require T to be a path; and the *path-width* of a finite graph is the minimum width of a path-decomposition.

We extend tree-width and path-width to infinite graphs by allowing the indexing tree or path to be infinite (thus, for a path-decomposition, the indexing path might be finite, or one-way infinite, or two-way infinite). This seems the most natural extension, and for tree-width it works nicely. In particular, there is a "compactness" theorem of Thomas [6] (and see also [1] for a proof):

1.1 *A graph G has tree-width at most k if and only if every finite subgraph of G has tree-width at most k.*

There are several ways to define infinite analogues of finite trees: for instance, topological trees, or set-theoretic trees. We could choose our favourite one of these definitions (let us call them "supertrees"), and use a supertree as the indexing set, giving "supertree-decompositions" and "supertree-width". But this gives nothing new: any graph with supertree-width at most k also has tree-width at most k. To see this, observe that, whatever definition of supertree we have taken, if G has supertree-width at most k, then all its finite subgraphs have tree-width at most k, and therefore G has tree-width at most k by Thomas' theorem 1.1.

For path-width, the analogue of Thomas' theorem is far from true. For instance, all graphs with finite path-width have only countably many vertices, so a graph with uncountably many vertices and no edges is already a counterexample; all its finite subgraphs have path-width zero, and yet G does not have finite path-width. Even for countable graphs, the analogue is still not true: for instance, any graph with infinitely many infinite connected subgraphs that are pairwise vertex-disjoint does not have finite path-width, and yet its finite subgraphs could all have very small path-width.

Thus, for an infinite graph G, all finite subgraphs having path-width at most k does not imply that G has path-width at most k. But what structure of G does it imply? This was answered in [1] as follows.

A *line* is a nonempty set L together with a linear order $\leq_L$. A *line-decomposition* of G is a pair $(L, (W_t : t \in L))$ where L is a line, and each W_i is a subset of $V(G)$, satisfying:

- $G = \bigcup_{t \in L} G[W_t]$, and
- $W_t \cap W_{t''} \subseteq W_{t'}$ for all $t, t', t'' \in L$ with $t \leq_L t' \leq_L t''$.

The second condition here is called the "betweenness axiom". We define the *width* of such a decomposition to be the maximum of $|B_t| - 1$ over all $t \in L$, if this maximum exists, and ∞ otherwise; and the *line-width* of G is the minimum width of a line-decomposition. (This definition was used in [4, 5], to increase the generality of some results about graphs with bounded path-width. It is also used in [2, 3], where what we call "line-decompositions" are called "linear decompositions".)

With Chudnovsky, we proved in [1] that:

1.2 *A graph G has line-width at most k if and only if every finite subgraph of G has path-width at most k.*

But do we need such general linear orders? Could we restrict the order-type somehow, perhaps in terms of the maximum bag-size? Could we make the order a well-order? This paper is an investigation into these and related questions. In particular, we will show that:

1.3 *If a graph G has finite line-width k, then it admits a line-decomposition $(L, (W_t : t \in L))$ of width at most $2k$ such that L is a well-order.*

We will also show that $2k$ is best possible.

At the end of the paper, we look at a different question. We will show that, after a little tidying, every line-decomposition of width $\leq k$ of a graph G can be built by substitution and concatenation, starting from so-called "prime" line-decompositions, which are much simpler.

2 Tidying a line-decomposition

If L is a line, an *interval* of L is a nonempty subset $I \subseteq L$ such that if $r, s, t \in L$ with $r \leq_L s \leq_L t$, and $r, t \in I$, then $s \in I$. An interval I is *initial* if $I \neq L$ and for all $s, t \in L$ with $s \leq_L t$, if $t \in I$ then $s \in I$. (Note that if I is an initial interval then $I, L \setminus I$ are both nonempty, and so $L \setminus I$ is an initial interval of the line obtained from L by reversing its order.)

If $(L, (W_t : t \in L))$ is a line-decomposition of G, and I is an interval, we define

$$W(I) = \bigcup_{t \in I} W_t.$$

If I is an initial interval, then $W(I) \cap W(L \setminus I)$ is called the *I-split*; every path between $W(I), W(L \setminus I)$ has a vertex in this set. We say X is a *split* if X is the I-split for some initial interval I.

Let us see first some easy things we can do to make a line-decomposition nicer without increasing its width. Let $(L, (W_t : t \in L))$ be a line-decomposition of G, of finite width. If there are distinct $s, s' \in L$ with $W_s = W_{s'}$, we could remove one of s, s' from L and still have a line-decomposition. In general, we can partition L into equivalence classes under the relation $s \equiv s'$ if $W_s = W_{s'}$, and choose one element of each equivalence class. Let L' be the set of chosen elements, ordered by the order inherited from L. Then $(L', (W_t : t \in L'))$ is still a line-decomposition of G, with the same width, and now there are no repeated bags. So we could assume that all the sets W_t $(t \in L)$ are distinct.

Now let L' be the set of all $s \in L$ such that there is no $s' \in L$ with $s' \neq s$ and with $W_s \subseteq W_{s'}$; then $(L', (W_t : t \in L'))$ is also a line-decomposition of G, with the same width. So we could assume that no W_s is a subset of another.

Let us add edges to G, forming G', in such a way that every edge we add has both ends in some W_t, and every W_t becomes a clique. The same line-decomposition is also a line-decomposition of G', and any line-decomposition of the new graph is also a line-decomposition of G; so we might as well work with the new graph in place of G.

Thus, so far we are assuming:

- G is non-null;
- $(L, (W_t : t \in L))$ is a line-decomposition of finite width;
- for all distinct $s, t \in L$, $W_s \not\subseteq W_t$; and
- W_t is a clique of G for each $t \in L$.

Let us say a line-decomposition $(L, (W_t : t \in L))$ is *tidy* if it satisfies these conditions. If $(L, (W_t : t \in L))$ is tidy, then each W_t is a clique, and every clique of G is a subset of some W_t (this is an easy exercise); and consequently, each W_t is a maximal clique of G, and every maximal clique of G is one of the sets W_t. Thus, the sets W_t are precisely the maximal cliques of G, and in particular they are determined by G; the only freedom in choosing a tidy line-decomposition is choosing the order $\leq_L L$, that is, choosing a linear order of the set of maximal cliques that satisfies the betweenness axiom. (See [1, 7] for when such an order exists.) Thus, the cardinality of L is the cardinality of the set of maximal cliques (that is, $|G|$ if G is infinite). There can be many different such orders; for instance, if G is a "star" graph, a tree in which all edges have a common end, then the maximal cliques of G are just the edges, and can be arranged in any linear order to make a line-decomposition.

3 Well-ordered line-decompositions

A line L is a *well-order* if there is no infinite sequence $t_1, t_2, \ldots$ of distinct elements of L such that $t_{i+1} \leq_L t_i$ for each $i \geq 1$; and a line-decomposition $(L, (W_t : t \in L))$ is a *wo-decomposition* if L is a well-order. The minimum width of a wo-decomposition of G is called the *wo-width* of G.

Our objective is to show that the wo-width of G is at most twice the line-width of G. Let us see first that this is best possible.

3.1 *For each integer $k \geq 1$, there is a graph with line-width at most k and with wo-width at least $2k$.*

Proof. Let G be the graph with vertices the set of all integers, where for all distinct integers i, j, we say i, j are adjacent if $|j - i| \leq k$. Let $B_i = \{i, i+1, \ldots, i+k\}$ for each $i \in \mathbb{Z}$; then $(B_i : i \in \mathbb{Z})$ is a line-decomposition of G of width k. Now suppose that $(L, (W_t : t \in L))$ is a wo-decomposition of G of width less than $2k$, and so $|W_t| \leq 2k$ for each $t \in L$. Every clique of G is a subset of some W_t. Since L is a well-order, there exists $t \in L$ minimum such that W_t includes B_i for some integer i. Choose $t', t'' \in L$ such that $B_{i+k+1} \subseteq W_{t'}$ and $B_{i-k-1} \subseteq W_{t''}$. Thus $t \leq_L t', t''$ from the choice of t. Since $B_i, B_{i+k+1}, B_{i-k-1}$ are pairwise disjoint and have size $k+1$, no W_t includes two of them, since each W_t has size at most $2k$. In particular, t, t', t'' are all different. There is a matching M of size k between B_i, B_{i+k+1}, joining j and $j+k$ for $i+1 \leq j \leq i+k$. Since $|W_{t''}| \leq 2k$, and $B_{i-k-1} \subseteq W_{t''}$, there is an edge of M with neither end in $W_{t''}$; and so it is not the case that $t \leq_L t'' \leq_L t'$. But similarly, there is a matching of size k between B_i, B_{i-k-1}, and so it is not the case that $t \leq_L t' \leq_L t''$, a contradiction. This proves 3.1. ∎

Let L be a line. We say that L is *integral* if L has order-type that of a subset of the set of integers, that is, there is a function $\phi : L \to \mathbb{Z}$ such that if $s, t \in L$ are distinct and $s \leq t$ then $\phi(s) < \phi(t)$.

3.2 *Let L be a line, and suppose that for all $r, t \in L$ with $r \leq t$, there are only finitely many $s \in L$ with $r \leq s \leq t$. Then L is integral.*

Proof. Choose $r \in L$ and define $\phi(r) = 0$. For each $t \in L \setminus \{r\}$ with $t > r$, let $\phi(t)$ be the number of $s \in L$ with $r < s \leq t$. If $t < r$, let $-\phi(t)$ be the number of $s \in L$ with $r > s \geq t$. Then ϕ satisfies the theorem. This proves 3.2. ∎

For a split S, there may be many different initial intervals I such that S is the I-split. For instance, when G is the star graph, and $v \in V(G)$ is an end of every edge of G, then $\{v\}$ is the I-split for every initial interval I, for every tidy line-decomposition.

Let $(L, (W_t : t \in L))$ be a line-decomposition of G, and let S_1, S_2 be splits. We say S_1 is *before* S_2 if $I_1 \subseteq I_2$ for all choices of initial intervals I_1, I_2 such that S_1, S_2 are the I_1-split and I_2-split respectively.

3.3 *Let $(L, (W_t : t \in L))$ be a line-decomposition of G, and let S_1, S_2 be distinct splits of the same size. Then either S_1 is before S_2, or S_2 is before S_1.*

Proof. For $i = 1, 2$, let $\mathcal{I}_i$ be the set of all initial intervals I such that S_i is the I-split.

(1) *For all $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$, if $I_1 \subseteq I_2$ then $I_1 \subseteq I_2'$ for all $I_2' \in \mathcal{I}_2$, and $I_1' \subseteq I_2$ for all $I_1' \in \mathcal{I}_1$.*

Suppose that there exists $I_2' \in \mathcal{I}_2$ such that $I_1 \not\subseteq I_2'$. Thus, $I_2' \subseteq I_1$. Since $S_1 \neq S_2$ and $|S_1| = |S_2|$, there exists $v \in S_2 \setminus S_1$. Since $v \in S_2$ and S_2 is the I_2-split, it follows that $v \in W(L \setminus I_2) \subseteq W(L \setminus I_1)$; and similarly, since S_2 is the I_2'-split, $v \in W(I_2') \subseteq W(I_1)$. But then $v \in W(I_1) \cap W(L \setminus I_2) = S_1$, a contradiction. This proves the first statement of (1), and the second is proved similarly. This proves (1).

Choose $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$; then one of I_1, I_2 is a subset of the other, and by exchanging S_1, S_2 if necessary we may assume that $I_1 \subseteq I_2$. Now let $I_1' \in \mathcal{I}_1$ and $I_2' \in \mathcal{I}_2$. By the first statement of (1), we have $I_1 \subseteq I_2'$, and so by the second statement of (1), with I_2 replaced by I_2', we have $I_1' \subseteq I_2'$. This proves that S_1 is before S_2, and so proves 3.3. ∎

As we said earlier, one split can be the I-split for many initial intervals I; but we can enumerate the *distinct* splits of minimum size by the following:

3.4 *Let $(L, (W_t : t \in L))$ be a tidy line-decomposition of G, such that for some m, every split has size at least m. Then we can number the splits with size exactly m as $\{S_i : i \in K\}$, where K is an interval of $\mathbb{Z}$, in such a way that S_i is before S_j for $i, j \in K$ with $i < j$.*

Proof. Let $\mathcal{S}$ be the set of all splits with size m. By 3.3, the relation "is before" is a linear order on $\mathcal{S}$, and we need to show it is integral. For $S, S' \in \mathcal{S}$, we write $S \leq S'$ if S is before S'. Suppose that there exist $S_1, S_2 \in \mathcal{I}$ such that $S_1 \leq S_2$, and there are infinitely many $S \in \mathcal{S}$ such that $S_1 \leq S \leq S_2$. Choose a subset $X \subseteq V(G)$, maximal such that there are infinitely many $S \in \mathcal{S}$ with $S_1 \leq S \leq S_2$ and with $X \subseteq S$. (This is possible since every such X has size at most the width.) A path of G *avoids* X if it has no vertex in X.

(1) *There is no path of $G \setminus X$ between S_1, S_2 avoiding X.*

Suppose that there is a path P of G between S_1, S_2 avoiding X, and choose such a path P minimal. Thus, none of its internal vertices belong to $S_1 \cup S_2$. But for each $S \in \mathcal{S}$ with $S_1 \subseteq S \subseteq S_2$, some vertex of P belongs to S; and since P has finite length, there is a vertex $v \in V(P)$ such that v belongs to S for infinitely many S with $S_1 \subseteq S \subseteq S_2$, contradicting the maximality of X. This proves (1).

For $i = 1, 2$, choose an initial interval I_i such that S_i is the I_i-split, and let I be the set of all $t \in I_2$ such that either $t \in I_1$ or there is a path of G between S_1, W_t avoiding X.

(2) *I is an initial interval.*

Let $s, t \in L$ with $s \leq_L t$ and $t \in I$; we need to show that $s \in I$. If $s \in I_1$ the claim is true, so we assume that $s \notin I_1$; and so $s, t \in I \setminus I_1$. Hence there is a path of G between S_1, W_t avoiding X; but this path intersects W_s, and so $s \in I$. This proves (2).

Let S be the I-split.

(3) *$S \subseteq X$.*

Suppose that there exists $v \in S \setminus X$. Since $v \in W(L \setminus I)$, there exists $t \in L \setminus I$ with $v \in W_t$; and hence $t \notin I_1$ and there is no path of G between S_1, W_t avoiding X. In particular, $v \notin S_1$. Since $v \in W(I)$, there exists $s \in I$ with $v \in W_s$; and therefore either $s \in I_1$ or there is a path of G between S_1, W_s avoiding X. Since $v \in W_s \cap W_t$ and $v \notin S_1$, it follows that $s \notin I_1$. Hence there is a path of G between S_1, W_s avoiding X, and so there is a path of G between S_1, v avoiding X, since $v \in W_s \setminus X$ and W_s is a clique. But there is no path of G between S_1, W_t avoiding X, a contradiction. This proves (3).

Since S is a split, it follows from the hypothesis that $|S| \geq m$. But X is a subset of infinitely many distinct splits all of size m, and so $|X| < m$, contrary to (3). This proves 3.4. ∎

If $(L, (W_t : t \in L))$ is a line-decomposition of G, we say a vertex v of G is a *left-limit vertex* of the line-decomposition if for all $s, t \in L$ with $s \leq_L t$, if $v \in W_t$ then $v \in W_s$. (There may be no such vertices.) Similarly, v is a *right-limit vertex* if for all $s, t \in L$ with $s \leq_L t$, if $v \in W_s$ then $v \in W_t$.

3.5 *Let $(L, (W_t : t \in L))$ be a tidy line-decomposition of G, such that for some m, every split has size at least m. Let S be a split of size m, and let $\mathcal{I}$ be the set of all initial intervals I such that S is the I-split. Then either $\bigcap_{I \in \mathcal{I}} I = \emptyset$ and S is the set of all left-limit vertices of $(L, (W_t : t \in L))$, or $\bigcap_{I \in \mathcal{I}} I$ is the minimal member of $\mathcal{I}$. Similarly, either $\bigcup_{I \in \mathcal{I}} I = L$ and S is the set of all right-limit vertices, or $\bigcup_{I \in \mathcal{I}} I$ is the maximal member of $\mathcal{I}$.*

Proof. Let $I_1 = \bigcap_{I \in \mathcal{I}} I$. Suppose first that $I_1 = \emptyset$. We must show that S is the set of all left-limit vertices. We show first that every vertex in S is a left-limit vertex. Let $v \in S$ and let $s, t \in L$ with $s \leq_L t$ and $v \in W_t$. Since $\bigcap_{I \in \mathcal{I}} I = \emptyset$, there exists $I \in \mathcal{I}$ with $s \notin I$ and hence $t \notin I$. Since S is the I-split, $v \in W(I)$, and so $v \in W_r$ for some $r \in I$; but $r \leq_L s \leq_L t$ and $v \in W_r \cap W_t$, and therefore $v \in W_s$. This proves that every vertex in S is a left-limit vertex. For the converse inclusion, we must show that if $I_1 = \emptyset$ then every left-limit vertex belongs to S. Thus, let v be a left-limit vertex, and choose $t \in L$ with $v \in W_t$. Since $\bigcap_{I \in \mathcal{I}} I = \emptyset$, there exists $I \in \mathcal{I}$ with $t \in L \setminus I$. Choose $s \in I$; then

$s \leq_L t$, and so $v \in W_s$ since v is a left-limit vertex; and consequently $v \in W(I) \cap W(L \setminus I) = S$. This proves that if $\bigcap_{I \in \mathcal{I}} I = \emptyset$ then S is the set of all left-limit vertices.

Next we assume that $I_1 \neq \emptyset$. Since $\mathcal{I} \neq \emptyset$, it follows that I_1 is an initial interval; let S_1 be the I_1-split. If $S_1 \subseteq S$, then since S_1 is a split, and S is a split of minimum size, it follows that $S_1 = S$ and therefore I_1 is the minimal member of $\mathcal{I}$; so we assume for a contradiction that there exists $v \in S_1 \setminus S$. Since $v \in S_1$, there exists $s \in I_1$ and $t \in L \setminus I_1$ such that $v \in W_s \cap W_t$. Since $t \notin I_1$, there exists $I \in \mathcal{I}$ with $t \notin I$ and therefore $t \in L \setminus I$. But $s \in I_1 \subseteq I$, and

$$v \in W_s \cap W_t \subseteq W(I) \cap W(L \setminus I) = S,$$

a contradiction. This proves the first statement, and the second follows similarly.

This proves 3.5. ∎

We say $(L, (W_t : t \in L))$ is a line-decomposition of G *from Z_1 to Z_2* if Z_1 is a set of left-limit vertices and Z_2 is a set of right-limit vertices. (We emphasize that Z_1 need not be the entire set of left-limit vertices, just a subset, and similarly for Z_2.) We will need the following lemma:

3.6 *Let $(L, (W_t : t \in L))$ be a line-decomposition of G from Z_1 to Z_2, of width at most k, and let I be an initial interval. Let S be the I-split, and let $G_1 = W(I)$ and $G_2 = W(L \setminus I)$. Then: $(I, (W_t : t \in W_i))$ is a line-decomposition of G_1 from Z_1 to S, and $(L \setminus I : (W_t : t \in L \setminus I))$ is a line-decomposition of G_2 from S to Z_2, both of width at most k.*

Proof. Certainly $(I, (W_t : t \in W_i))$ is a line-decomposition of G_1, and we need to check it is from Z_1 to S. Thus, we need to check that every member of Z_1 is a left-limit vertex of $(I, (W_t : t \in W_i))$, and every vertex in S is a right-limit vertex of $(I, (W_t : t \in W_i))$.

Let $v \in Z_1$. Then $v \in W_t$ for some $t \in L$, and $v \in W_s$ for all $s \in L$ with $s \leq_L t$. Since $I \neq \emptyset$, there exists $s \in I$, and hence either $t \in I$ or $v \in W_s$, and in the second case we may replace t by s. Thus, we may assume that $t \in I$. Since $v \in W_r$ for all $r \in L$ with $r \leq_L t$, it follows that v is a left-limit vertex of $(I, (W_t : t \in W_i))$. So every member of Z_1 is a left-limit vertex of $(I, (W_t : t \in W_i))$.

Now let $u \in S$. Since S is the I-split, there exists $s \in I$ and $t \in L \setminus I$ such that $u \in W_s \cap W_t$; but then $u \in W_{s'}$ for all $s' \in I$ with $s \leq_L s'$, since $s \leq_L s' \leq_L t$. Consequently u is a right-limit vertex of $(I, (W_t : t \in W_i))$, and so every vertex in S is a right-limit vertex of $(I, (W_t : t \in W_i))$.

This proves that $(I, (W_t : t \in W_i))$ is a line-decomposition of G_1 from Z_1 to S. Similarly, $(L \setminus I : (W_t : t \in L \setminus I))$ is a line-decomposition of G_2 from S to Z_2, and clearly both these decompositions have width at most k. This proves 3.6. ∎

3.7 *Let $Z_1, Z_2, S \subseteq V(G)$, and let G_1, G_2 be induced subgraphs of G with $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = S$, and $Z_i \subseteq V(G_i)$ for $i = 1, 2$. If G_1 admits a wo-decomposition from Z_1 to S, and G_2 admits a wo-decomposition from S to Z_2, both of width at most some k', then G admits a wo-decomposition from Z_1 to Z_2 of width at most k'.*

Proof. For $i = 1, 2$, let $(L_i, (W_t : t \in L_i))$ be a wo-decomposition of G_i of width at most k', from Z_1 to S if $i = 1$, and from S to Z_2 if $i = 2$. We may assume that $L_1 \cap L_2 = \emptyset$. Let L be the linear order on $L_1 \cup L_2$ defined by $x \leq_L y$ if and only if either $x \in L_1$ and $y \in L_2$, or for some $i \in \{1, 2\}$, $x, y \in L_i$ and $x \leq_{L_i} y$. To see that $(L, (W_t : t \in L))$ is a line-decomposition of G, observe first that

$$\bigcup_{t \in L} W_t = \bigcup_{t \in L_1} W_t \cup \bigcup_{t \in L_2} W_t = G_1 \cup G_2 = G.$$

Now suppose that $r \leq_L s \leq_L t$; we must show that $W_r \cap W_t \subseteq W_s$. This is true if $t \in L_1$ (because then $r, s \in L_1$), or if $r \in L_2$; so we assume that $r \in L_1$ and $t \in L_2$. Now s may belong to either of L_1, L_2, but from the symmetry we may assume that $s \in L_1$. (This is a small cheat, since we are working with well-orders, and so there is no real symmetry between L_1, L_2, but the argument for the case when $s \in L_2$ is exactly the same, with L_1, L_2 exchanged.) Now

$$W_r \cap W_t \subseteq V(G_1) \cap V(G_2) = S,$$

so it suffices to show that $W_r \cap S \subseteq W_s$. Let $v \in W_r \cap S$. Thus, v is a right-limit vertex in $(L_1, (W_t : t \in L_1))$, and since $r \leq_{L_1} s$ it follows that $v \in W_s$. This proves that $W_r \cap W_t \subseteq W_s$, and so $(L, (W_t : t \in L))$ is a line-decomposition of G. Evidently its width is at most k', and it is a wo-decomposition. This proves 3.7. ▮

We will prove the following, which immediately implies 1.3 by taking $Z_1, Z_2 = \emptyset$:

3.8 *Let $(L, (W_t : t \in L))$ be a line-decomposition of G from Z_1 to Z_2, of width at most k, and let $m \leq k$ such that every split has size at least m, and $|Z_1| \leq m$. Then G admits a wo-decomposition from Z_1 to Z_2 of width at most $2k - |Z_1|$.*

Proof. We proceed by induction on k, and for fixed k, by induction on $k - m$. As we discussed in the previous section, we may assume that $(L, (W_t : t \in L))$ is tidy. We observe first:

(1) *We may assume that G is connected.*

Suppose that G is not connected, and let C_1 be a component of G. Let I_1 be the set of $t \in L$ such that $W_t \cap V(C_1) \neq \emptyset$. Since each W_t is a clique, it follows that $W_t \subseteq V(C_1)$ for each $t \in I_1$. Moreover, we claim that I_1 is an interval of L. To see this, let $r, t \in I_1$ and suppose that $r \leq_L s \leq_L t$. There is a path of C_1 joining W_r, W_t, and every such path intersects W_s, and so $s \in I_1$. Thus, I_1 is an interval. Since $I_1 \neq L$ (because G is not connected, and so some W_t is disjoint from $V(C_1)$ and hence $t \notin I_1$), we may assume from the symmetry that there exists $s \in L$ such that $s \leq_L t$ for all $t \in I_1$, and $s \notin I_1$. Let J be the set of $s \in L$ with this property; then J is an initial interval. Let S be the J-split. We claim that $S = \emptyset$. Suppose not, and let $v \in S$. Choose $r, s \in L$ such that $v \in W_r \cap W_s$ and $r \in J$ and $s \notin J$. Since $r \in J$, $W_r \cap V(C_1) = \emptyset$; so $W_s \not\subseteq V(C_1)$, and therefore $s \notin I_1$. Since $s \notin J$, and $s \notin I_1$, the definition of J implies that there exists $t \in I_1$ such that $s \not\leq_L t$. Thus, $r \leq_L t \leq_L s$, and $v \in W_r \cap W_s$ and $v \notin W_t$, a contradiction. This proves that $S = \emptyset$. Since S is a split and all splits have size at least m, it follows that $m = 0$, and therefore $Z_1 = \emptyset$.

Since $(L, (W_t : t \in L))$ is tidy, it follows that Z_2 is a clique (because it is a subset of a bag of $(L, (W_t : t \in L))$); and therefore there is a component C^* of G with $Z_2 \subseteq C^*$. We can choose a well-ordering of the set of components of G different from C^*; choose an ordinal β such that the components of G can be numbered G_α $(\alpha < \beta)$, where α ranges over all ordinals $< \beta$. Define $C_\beta = C^*$.

Suppose that the result is true for all components of G; thus, for all $\alpha < \beta$, there is a wo-decomposition from $\emptyset$ to $\emptyset$ of width at most $2k$; and for C_β there is a wo-decomposition from $\emptyset$ to Z_2 of width at most $2k$. Then by concatenating these in the natural way, we obtain a wo-decomposition of G of width at most $2k - |I_1|$ (since $I_1 = \emptyset$), as required. This proves (1).

Next, suppose that there are no splits with size m. If $m \le k-1$, then the hypotheses all remain true with m replaced by $m+1$, and the result follows from the second inductive hypothesis. If $m = k$, then since all splits have size at most m (because $(L, (W_t : t \in L))$ is tidy and consequently all its bags are different), we deduce that there are no splits, and so $|L| = 1$; and since $|Z_1| \le k$, the result is true. Hence we may assume that there is a split of size m. Consequently, $m \ge 1$, since G is connected and therefore all splits are non-null.

From 3.4, we can number the splits with size exactly m as $\{S_i : i \in K\}$, where K is an interval of $\mathbb{Z}$, in such a way that S_i is before S_j for all $i, j \in K$ with $i \le j$. For each $i \in K$, let $\mathcal{A}_i$ be the set of all initial intervals I of L such that S_i is the I-split. Let $I_i = \bigcap_{I \in \mathcal{A}_i} I$, and $J_i = \bigcup_{I \in \mathcal{A}_i} I$. By 3.5, either $I_i = \emptyset$ and S_i is the set of all left-limit vertices of $(L, (W_t : t \in L))$, or I_i is the minimal element of $\mathcal{A}_i$; and either $J_i = L$ and S_i is the set of all right-limit vertices, or J_i is the maximal element of $\mathcal{A}_i$.

(2) *If $I_i = \emptyset$ then i is the smallest member of K, and if $J_i = L$ then i is the largest member of K.*

Suppose that $I_i = \emptyset$. Then $I_i \notin \mathcal{A}_i$, since it is not an initial interval of L; and so S_i is the set of all left-limit vertices of $(L, (W_t : t \in L))$. Suppose that i is not the smallest member of K, and so $i-1 \in K$. Choose $H_i \in \mathcal{A}_i$ and $H_{i-1} \in \mathcal{A}_{i-1}$. For each $v \in S_i$, we have $v \in W(L \setminus H_i) \subseteq W(L \setminus H_{i-1})$. Moreover, $v \in W_t$ for some $t \in L$, and hence $v \in W_s$ for some $s \in H_{i-1}$ since v is a left-limit vertex. Consequently $v \in W(H_{i-1}) \cap W(L \setminus H_{i-1}) = S_{i-1}$. This proves that $v \in S_{i-1}$ for each $v \in S_i$, which is impossible since S_{i-1}, S_i are distinct and have the same size. Consequently, i is the smallest member of K. The second statement is proved similarly. This proves (2).

(3) *Either $Z_1 \subseteq S_i$ for some $i \in K$, or K has a smallest member. Similarly, either $Z_2 \subseteq S_i$ for some $i \in K$, or K has a largest member.*

Suppose that there is no $i \in K$ with $Z_1 \subseteq S_i$. Since Z_1 is finite, and $S_i \cap Z_1 \supseteq S_j \cap Z_1$ for all $i, j \in K$ with $i \le j$, it follows that there exists $z \in Z_1$ such that there is no $i \in K$ with $z \in S_i$. Let I be the intersection of all the initial intervals in $\bigcup_{i \in K} \mathcal{A}_i$. Since $z \in Z_1$ is a left-limit vertex, and z belongs to none of the splits S_i, it follows that I contains all $t \in L$ such that $z \in W_t$; and in particular, $I \neq \emptyset$. Consequently I is an initial interval; let S be the I-split. Suppose that there exists $v \in S$ that belongs to none of the splits S_i $(i \in K)$. Since v belongs to the I-split, there exist $s \in I$ and $t \in L \setminus I$ with $v \in W_s \cap W_t$. Since $t \in L \setminus I$, there exists $i \in I$ and $H \in \mathcal{A}_i$ such that $t \in L \setminus H$. But $s \in I \subseteq H$, and so

$$v \in W_s \cap W_t \subseteq W(H) \cap W(L \setminus H) = S_i,$$

a contradiction. Thus, for each $v \in S$ there exists $i \in K$ with $v \in S_i$. Since S is before each S_i, it follows that $S \cap S_i \supseteq S \cap S_j$ for all $i, j \in K$ with $i \le j$; and so there exists $j \in K$ with $S \subseteq S_j$. But all splits have size at least m, and S is a split, and S_j has size m; so $S_j = S$. Since S is before S_i for all $i \in K$, it follows that j is the smallest member of K. The second statement of (3) is proved similarly. This proves (3).

(4) *We may assume that K has a smallest element.*

Suppose that K has no smallest element. By (3), there exists $i \in K$ such that $Z_1 \subseteq S_i$, and so we may assume that $Z_1 \subseteq S_0$. Let $G_1 = W(I_0)$ and $G_2 = W(L \setminus I_0)$. Thus, $V(G_1) \cap V(G_2) = S_0$. Suppose that:

- Either $Z_1 \neq \emptyset$, or G_1 admits a wo-decomposition from S_0 to $\emptyset$ of width at most $2k - |S_0|$; and
- G_2 admits a wo-decomposition from S_0 to Z_2 of width at most $2k - |S_0|$.

We claim that G_1 admits a wo-decomposition from Z_1 to S_0 of width at most $2k - |Z_1|$. To see this, there are two cases. If $Z_1 = \emptyset$, then adding S_0 to every bag of the decomposition given in the first bullet gives a wo-decomposition from S_0 to S_0, and hence from Z_1 to S_0, as required (since $2k - |Z_1| = 2k$). Now we assume that $Z_1 \neq \emptyset$. By 3.6, G_1 admits a line-decomposition from Z_1 to S_0 of width at most k, and all its bags include Z_1, since $Z_1 \subseteq S_0$. By removing Z_1 from all these bags, we obtain a line-decomposition of $G_1 \setminus Z_1$ from $\emptyset$ to $S_0 \setminus Z_1$ of width at most $k - |Z_1|$. By reversing its order, we obtain a line-decomposition of $G_1 \setminus Z_1$ from $S_0 \setminus Z_1$ to $\emptyset$ of width at most $k - |Z_1|$. Since $Z_1 \neq \emptyset$, the first inductive hypothesis implies that $G_1 \setminus Z_1$ admits a wo-decomposition from $S_0 \setminus Z_1$ to $\emptyset$ of width at most $2(k - |Z_1|) - |S_0 \setminus Z_1|$. By adding S_0 to all the bags of this wo-decomposition, we have proved that G_1 admits a wo-decomposition of width at most

$$2(k - |Z_1|) - |S_0 \setminus Z_1| + |S_0| = 2k - |Z_1|$$

from S_0 to S_0, and hence from Z_1 to S_0 since $Z_1 \subseteq S_0$, as claimed.

From the second bullet above, G_2 admits a wo-decomposition from S_0 to Z_2 of width at most $2k - |Z_1|$, since $|Z_1| \leq |S_0|$. From 3.7, we deduce that G admits a wo-decomposition from Z_1 to Z_2 of width at most $2k - |Z_1|$, so the statement of the theorem holds.

Consequently, to prove the theorem, it suffices to prove the two bullets above. But in both cases, we claim that the corresponding graph G_1 or G_2 admits a line-decomposition $\mathcal{W}$ of width at most k (from S_0 to $\emptyset$ in the first case, and from S_0 to Z_2 in the second) in which all splits have size at least m, and either $\mathcal{W}$ has no split of size m, or there is one that is before all other splits of $\mathcal{W}$ of size m. To see this, for G_1 we take as $\mathcal{W}$ the line-decomposition $(\overleftarrow{I} : (W_t : t \in \overleftarrow{I}))$, where $\overleftarrow{I}$ is the line obtained from I by reversing its order (then all splits of $\mathcal{W}$ have size at least m, and S_{-1} is a split of $\mathcal{W}$ of size m, and is before all other splits of size m; note that S_0 is not a split of $\mathcal{W}$). For G_2 we take as $\mathcal{W}$ the line-decomposition $(L \setminus I_0 : (W_t : t \in L \setminus I_0))$; then if $1 \in K$, S_1 is a split of $\mathcal{W}$ that is before all other splits of the minimum size m, and if $1 \notin K$ then $\mathcal{W}$ has no split of size at most m. This proves that $\mathcal{W}$ exists with the properties claimed. But if the theorem holds for $\mathcal{W}$, then the two bullets above hold, so it suffices to prove the theorem for $\mathcal{W}$. This proves (4).

For each $i \in K$, if $I_i = J_i$, let G_i be the complete graph with vertex set S_i. If $I_i \neq J_i$, let G_i be the union of the graphs $G[W_t]$ over all $t \in J_i \setminus I_i$.

(5) *For each $i \in K$, there is a wo-decomposition $\mathcal{W}_i$ of G_i from S_i to S_i with width at most $2k - m$.*

This is trivial if $I_i = J_i$, so we assume that $J_i \neq I_i$; let $M_i = J_i \setminus I_i$, with order inherited from L. It follows that M_i is a line, and therefore $(M_i : (W_t : t \in M_i))$ is a line-decomposition of G_i from S_i to S_i (because S_i is a subset of W_t for each $t \in M_i$). Thus, $(M_i, (W_t \setminus S_i : t \in M_i))$ is a line-decomposition of $G_i \setminus S_i$ from $\emptyset$ to $\emptyset$, of width at most $k - m$, since S_i is a subset of W_t for each $t \in M_i$ and $|S_i| = m$. Since $m \geq 1$, the first inductive hypothesis tells us that there is a wo-decomposition of $G_i \setminus S_i$ from $\emptyset$ to $\emptyset$, of width at most $2(k-m)$, and by adding S_i to each of its bags, we obtain a wo-decomposition of G_i from S_i to S_i, of width at most $2k-m$. This proves (5).

For each $i \in K$, we define G'_i as follows. If $J_i = L$ (and so i is the largest member of K, by (2)), let G'_i be the complete graph with vertex set S_i. If $J_i \neq L$, and $i+1 \notin K$, let $G'_i = W(L \setminus J_i)$. If $i+1 \in K$, let $G'_i = W(I_{i+1} \setminus J_i)$.

(6) *For each $i \in K$, there is a wo-decomposition $\mathcal{W}'_i$ of G'_i from S_i to S_{i+1} if $i+1 \in K$, and from S_i to Z_2 if $i+1 \notin K$, of width at most $2k-m$.*

First, we assume that $i+1 \in K$. Since S_i is before S_{i+1}, it follows that $J_i \subseteq I_{i+1}$. By (2), $J_i \in \mathcal{A}_i$ and $I_{i+1} \in \mathcal{A}_{i+1}$, and therefore $I_{i+1} \neq J_i$, since $S_i \neq S_{i+1}$. Consequently $I_{i+1} \setminus J_i \neq \emptyset$. Let $M'_i = I_{i+1} \setminus J_i$, with order inherited from L. It follows that $(M'_i, (W_t : t \in M'_i))$ is a line-decomposition from S_i to S_{i+1} of G'_i of width at most k; and all its splits have size at least $m+1$. If $m = k$, since all splits of $(M_i, (W_t : t \in M'_i))$ are splits of $(L, (W_t : t \in L))$ and hence have size at most m, it follows that $(M_i, (W_t : t \in M'_i))$ has no splits, and hence $|M'_i| = 1$. In that case $(M'_i, (W_t : t \in M'_i))$ satisfies the statement of (6). Thus, we may assume that $m < k$. From the second inductive hypothesis, applied to $(M_i, (W_t : t \in M'_i))$, since all its splits have size at least $m+1$, there is a wo-decomposition of G'_i from S_i to S_{i+1} of width at most $2k - |S_i| = 2k - m$.

Now we assume that $i+1 \notin K$, and so i is the largest member of K. If $J_i = L$, the claim is clear, using a line-decomposition with only one bag S_i; so we assume that $J_i \neq L$. Thus, $G'_i = W(L \setminus J_i)$. Since $J_i \neq L$, $(L \setminus J_i, (W_t : t \in L \setminus J_i))$ is a line-decomposition of G_i from S_i to Z_2 of width at most k, and all its splits have size at least $m+1$ (since i is the largest member of K). Again, the claim follows from the second inductive hypothesis if $m < k$, or trivially if $m = k$. This proves (6).

From (4), we may assume that 1 is the smallest member of K. Define $G'_0 = W(I_1)$ if $I_1 \neq \emptyset$, and let G'_0 be the complete graph with vertex set S_1 if $I_1 = \emptyset$. Then, similarly, there is a wo-decomposition $\mathcal{W}'_0$ of G_0 from Z_1 to S_1 of width at most $2k - |Z_1|$. By concatenating

$$\mathcal{W}'_0, \mathcal{W}_1, \mathcal{W}'_1, \ldots, \mathcal{W}'_{i-1}, \mathcal{W}_i$$

if K has a largest member i, or by concatenating

$$\mathcal{W}'_0, \mathcal{W}_1, \mathcal{W}'_1, \mathcal{W}_2, \ldots$$

otherwise, we deduce that G admits a wo-decomposition of width at most $2k - |Z_1|$ from Z_1 to Z_2. (Note that in the case when K has no largest member, Z_2 is included in S_i for all sufficiently large i, by (3), and so Z_2 is a set of right-limit vertices of the wo-decomposition we just constructed.) This proves 3.8. ∎

4 Prime decompositions

There is another route we could take to making line-decompositions as palatable as possible. In the proof of step (1) of 3.8, we used ordinals of arbitrary magnitude, but all the remainder of the proof of 3.8 was just concatenating sequences of wo-decompositions to make longer wo-decompositions, and this process was only iterated k times. If we could find a way to avoid step (1), we might get something nicer.

That motivates looking at "prime" line-decompositions. Let us say a line-decomposition $(L, (W_t : t \in L))$ is *prime* if it is tidy, and G is connected, and the I-split is different from the J-split, for all distinct initial intervals I, J. If we run the proof of 3.8 starting with a prime line-decomposition, three nice things happen:

- G is already connected, so we don't need step (1) of the proof;
- the graphs G_i used in step (5) of the proof are all trivial, because $I_i = J_i$ in the notation of that proof, so we could skip step (5);
- the line-decompositions $(M_i, (W_t : t \in M_i'))$ of the graphs G_i' used in step (6) are also prime, so we can apply the inductive hypothesis to them.

We deduce:

4.1 *If $(L, (W_t : t \in L))$ is a prime line-decomposition of G, of width at most k, then L has order-type that of a subset of $\mathbb{Z}^k$, and G admits a wo-decomposition $(L', (W_t' : t \in L'))$ of width at most $2k$ where L' is an ordinal and $L' \leq \omega^k$.*

Of course, not every connected graph that admits a tidy line-decomposition admits a prime line-decomposition of the same width. For instance, a *caterpillar* is a tree in which some path passes through every vertex of degree more than two. A connected graph G admits a tidy line-decomposition of width one if and only if G is a caterpillar, but G admits a prime line-decomposition of width one if and only if G is a path. Indeed, from the first statement of 4.1, if G admits a prime line-decomposition of width at most k, then $V(G)$ is countable.

On the other hand, we will show that a general tidy line-decomposition can be built by substituting prime line-decompositions in one another (iterating only k times, where k is the width). Let us say this more exactly. Let L be a linear order, and let $\mathcal{I}$ be a set of some of its initial intervals. For each $I \in \mathcal{I}$, let L_I be a linear order, such that L and all the orders L_I are pairwise disjoint. Let M be the linear order with element set $L \cup \bigcup_{I \in \mathcal{I}} L_I$, where $s \leq t$ if either:

- for some $I \in \mathcal{I}$, $s, t \in L_i$, and $s \leq t$ in the order of L_i; or
- $s, t \in L$, and $s \leq t$ in the order of L; or
- $s \in L$, and $t \in L_I$ for some $I \in \mathcal{I}$, and $s \in I$; or
- $s \in L_I$ for $I \in \mathcal{I}$, and $t \in L$, and $t \notin I$.

It is easy to check that M is indeed a linear order. Now let $(L, (W_t : t \in L))$ be a line-decomposition of some graph G, and for each $I \in \mathcal{I}$ let $(L_I, (W_t^I : t \in L_I))$ be a line-decomposition of some non-null graph G_I, where the graphs G and G_I $(I \in \mathcal{I})$ are all pairwise vertex-disjoint. Let H be the graph obtained from the disjoint union of G and the graphs G_I $(I \in \mathcal{I})$ by adding an edge between u, v for each $I \in \mathcal{I}$, each $u \in V(G_I)$ and each $v \in W(I) \cap W(L \setminus I)$. Define M as before, and for each $t \in M$, define $X_t = W_t$ if $t \in L$, and $X_t = W_t \cup (W(I) \cap W(L \setminus I))$ if $t \in L_I$ for some $I \in \mathcal{I}$. Then $(M, (X_t : t \in M))$ is a line-decomposition of H, tidy if $(L, (W_t : t \in L))$ and each $(L_I, (W_t^I : t \in L_I))$ are tidy; and we say that $(M, (X_t : t \in M))$ is obtained by *substituting* $((L_I, (W_t^I : t \in L_I))\ I \in \mathcal{I})$ *into* $(L, (W_t : t \in L))$. We will prove:

4.2 *Let $(M,(X_t : t \in M))$ be a tidy line-decomposition of some connected graph H. Then there exist $G, (L,(W_t : t \in L))$, $\mathcal{I}$, G_I $(I \in \mathcal{I})$, and $(L_I,(W_t^I : t \in L_I))$ for each $I \in \mathcal{I}$, as above, such that*

- *$(M,(X_t : t \in M))$ is obtained by substituting $((L_I,(W_t^I : t \in L_I))\ I \in \mathcal{I})$ into $(L,(W_t : t \in L))$;*
- *$(L,(W_t : t \in L))$ is prime, with width at most that of $(M,(X_t : t \in M))$; and*
- *for each $I \in \mathcal{I}$, $(L_I,(W_t^I : t \in L_I))$ is tidy and has width strictly smaller than that of $(M,(X_t : t \in M))$.*

Proof. For each split C of $(M,(X_t : t \in M))$, let F_C be the set of all initial intervals I of M such that the I-split of $(M,(X_t : t \in M))$ equals C. If $|F_C| > 1$, we say C is a *repeated split.* If C is a repeated split, the set of all $t \in M$ such that t belongs to some but not all members of F_C is an interval of M; let us call this interval the *repeat interval of* C and denote it by L_C.

(1) *If C is a repeated split of $(M,(X_t : t \in M))$ and $t \in L_C$, then $C \subseteq W_t$.*

Since $t \in L_C$, there exist initial intervals $I, J \in F_C$ such that $t \in J \setminus I$. Let $v \in C$. Thus, v belongs to the I-split, and therefore to $X(I)$; and so there exists $s \in I$ with $v \in X_s$. Similarly there exists $s' \in M \setminus J$ such that $v \in X_{s'}$. Since $s \le t \le s'$, it follows that $v \in X_t$. This proves (1).

(2) *If C is a repeated split of $(M,(X_t : t \in M))$, and C' is a split with $C \not\subseteq C'$, and $J \in F_{C'}$, then either $L_C \cap J = \emptyset$ or $L_C \subseteq J$.*

Choose $v \in C \setminus C'$. Since $v \notin C' = X(J) \cap X(M \setminus J)$, either $v \notin X(J)$ or $v \notin X(M \setminus J)$. But $v \in X_t$ for all $t \in I$, by (1); and so either $L_C \cap J = \emptyset$ or $L_C \cap (M \setminus J) = \emptyset$. This proves (2).

(3) *If C, C' are repeated splits of $(M,(X_t : t \in M))$, then either $L_C \subseteq L_{C'}$ and C is a proper subset of C', or $L_{C'} \subseteq L_C$ and C' is a proper subset of C, or $L_C \cap L_{C'} = \emptyset$.*

We may assume that $C \neq C'$, and so without loss of generality, we assume that $C \not\subseteq C'$. By (2), for each initial interval $J \in F_{C'}$, either $L_C \cap J = \emptyset$ or $L_C \subseteq J$. If $L_C \cap J = \emptyset$ for every $J \in F_{C'}$, then $L_C \cap L_{C'} = \emptyset$; and if $L_C \subseteq J$ for all $J \in F_C$ then again $L_C \cap L_{C'} = \emptyset$. Finally, we assume that there exist $J_1, J_2 \in F_{C'}$ such that $L_C \cap J_1 = \emptyset$ and $L_C \subseteq J_2$. It follows that $L_C \subseteq L_{C'}$. If $C' \subseteq C$, then C' is a proper subset of C and the claim holds;, so we may assume that $C' \not\subseteq C$. By the same argument with C, C' exchanged, we deduce that either

- $L_C \cap L_{C'} = \emptyset$ (and this is false since $\emptyset \neq L_C \subseteq L_{C'}$); or
- $L_{C'} \subseteq L_C$, and so $L_{C'} = L_C$.

Thus, it remains to handle the case when $L_{C'} = L_C$ and neither of C, C' is a subset of the other. Let $J \in F_C$, and suppose that $L_C \cap J \neq \emptyset$ and $L_C \not\subseteq J$. Then J includes some but not all of L_C, and so there exist $s, t \in L_C$ with $s \in J$ and $t \notin J$. Hence $X_s \cap X_t$ is a subset of the J-split, that is, of C; but by (1), $X_s \cap X_t$ includes C' since $s, t \in L_{C'} = L_C$, so $C' \subseteq C$, a contradiction. Thus, if $J \in F_C$ then either $L_C \cap J = \emptyset$ or $L_C \subseteq J$. Let I_1 be the set of all $s \in M$ such that $s \notin L_C$ and $s < t$ for all $t \in I$. Thus, I_1 is either an initial interval or empty. Similarly, let I_2 be the set of all $s \in M$ such that $s \le t$ for some $t \in L_C$; then I_2 is either an initial interval or equals L.

We claim that if $J \in F_C$ then $J = I_1$ or $J = I_2$. To see this, choose $s \in L_C$. Suppose first that $L_C \cap J = \emptyset$, so $J \subseteq I_1$. If there exists $t \in I_1 \setminus J$, then t belongs to some but not all of the members of F_C, because it does not belong to J, and belongs to any member of F_C that contains s (and there is such a member, by definition of L_C). But then $t \in L_C$, a contradiction. So if $L_C \cap J = \emptyset$ then $J = I_1$. Next suppose that $L_C \subseteq J$, If there exists $t \in J \setminus I_2$, then t belongs to some but not all of the members of F_C, because it belongs to J, and does not belong to any member of F_C that does not contains s (and there is such a member, by definition of L_C). But then again $t \in L_C$, a contradiction. This proves the claim that if $J \in F_C$ then either $J = I_1$ or $J = I_2$. Since $|F_C| \geq 2$, it follows that $F_C = \{I_1, I_2\}$. In particular, I_1, I_2 are both initial intervals. Similarly $F_{C'} = \{I_1, I_2\}$; but then the I_1-split equals both C and C', a contradiction. This proves (3).

Let $\mathcal{C}$ be the set of all repeated splits C such that L_C is maximal. By (3), the sets L_C ($C \in \mathcal{C}$) are disjoint. Moreover:

(4) *For every repeated split C, there exists $C' \in \mathcal{C}$ such that $L_C \subseteq L_{C'}$.*

Choose a repeated split C' with C' maximal such that $L_C \subseteq L_{C'}$ (this is possible since all splits have size at most the width of $(M, (X_t : t \in M))$). But then by (3), $C' \in \mathcal{C}$. This proves (4).

Let L be the set of all $t \in L$ that do not belong to $\bigcup_{C \in \mathcal{C}} L_C$, with the linear order inherited from that of M. For each $t \in L$, let $W_t = X_t$; then $(L, (W_t : t \in L))$ is a line-decomposition of some subgraph G of H. For each $C \in \mathcal{C}$, and each $t \in L_C$, define $W_t^C = X_t \setminus C$; then $(L_C, (W_t^C : t \in L_C))$ is a tidy line-decomposition of some subgraph G_C of H, of width at most the width of $(M, (X_t : t \in M))$ minus $|C|$. Since each $C \neq \emptyset$ (because H is connected), it follows that each $(L_C, (W_t^C : t \in L_C))$ has width less than that of $(M, (X_t : t \in M))$. To complete the proof, we need to check that $(L, (W_t : t \in L))$ is prime. If I is an initial interval of L, then there is an initial interval J of M such that the split of I in $(L, (W_t : t \in L))$ equals the split of J in $(M, (X_t : t \in M))$; and if $I \subseteq I'$ are distinct initial intervals of L, the corresponding initial intervals $J \subseteq J'$ of M satisfy that $J' \setminus J$ is not a subset of any member of $\mathcal{C}$. By (4), the splits of J and of J' in $(M, (X_t : t \in M))$ are distinct, and so the splits of I and of I' in $(L, (W_t : t \in L))$ are distinct. Since H is connected, and is obtained from the disjoint union of the graphs G and the graphs G_C ($C \in \mathcal{C}$) by adding edges from each G_C to the clique C of G, it follows that G is connected. Thus, $(L, (W_t : t \in L))$ is prime. This proves 4.2. ∎

This result shows that we can build a tidy line-decomposition of width k of a *connected* graph by starting with a prime decomposition and substituting tidy line-decompositions of width $< k$ into it. But the tidy line-decomposition we are substituting are decompositions of subgraphs that might not be connected. So to complete a recursive structure theorem, we need another step, a proof that we can build any tidy line-decomposition of width $\leq k$ of a graph G from tidy line-decompositions of width $\leq k$ of the components of G, by concatenating the latter in some order. This is easy and we omit it.

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