Induced subgraphs of graphs with large chromatic number. VIII. Long odd holes

Maria Chudnovsky[1]
Princeton University, Princeton, NJ 08544, USA

Alex Scott
Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour[2]
Princeton University, Princeton, NJ 08544, USA

Sophie Spirkl
Princeton University, Princeton, NJ 08544, USA

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Abstract

We prove a conjecture of András Gyárfás, that for all κ, ℓ, every graph with clique number at most κ and sufficiently large chromatic number has an odd hole of length at least ℓ.

1 Introduction

All graphs in this paper are finite and have no loops or parallel edges. We denote the chromatic number of a graph G by $\chi(G)$, and its clique number (the cardinality of its largest clique) by $\omega(G)$. A *hole* in G means an induced subgraph which is a cycle of length at least four, and an *odd* hole is one with odd length. In [4], András Gyárfás proposed three conjectures about the lengths of holes in graphs with large chromatic number and bounded clique number, the following:

1.1 *For all $\kappa \geq 0$ there exists $c \geq 0$ such that for every graph G, if $\omega(G) \leq \kappa$ and $\chi(G) > c$ then G has an odd hole.*

1.2 *For all $\kappa, \ell \geq 0$ there exists c such that for every graph G, if $\omega(G) \leq \kappa$ and $\chi(G) > c$ then G has a hole of length at least ℓ.*

1.3 *For all $\kappa, \ell \geq 0$ there exists c such that for every graph G, if $\omega(G) \leq \kappa$ and $\chi(G) > c$ then G has an odd hole of length at least ℓ.*

The third evidently contains the other two. The first was proved in [5], and the second in [2], but until now the third has remained open. In this paper we prove the third.

We remark that there have been some other partial results approaching 1.3, in [1, 6]. The second is the stronger, namely:

1.4 *For all $\ell \geq 0$, there exists $c \geq 0$ such that for every graph G, if $\omega(G) \leq 2$ and $\chi(G) > c$ then G has holes of ℓ consecutive lengths (and in particular has an odd hole of length at least ℓ).*

One might expect that an analogous result holds for all graphs G with $\omega(G) \leq \kappa$, for all fixed κ, as was conjectured in [6], but this remains open. We can prove that there are two holes both of length at least ℓ and with consecutive lengths, by a refinement of the methods of this paper, but we omit the details.

The proof of 1.3 will be by induction on κ, with ℓ fixed, so we may assume that $\kappa \geq 2$ and the result holds for all smaller κ. In particular there exists τ such that for every graph G, if $\omega(G) \leq \kappa - 1$ and G has no odd hole of length at least ℓ then $\chi(G) < \tau$. We might as well assume that ℓ is odd and $\ell \geq 5$. Let us say a graph G is a (κ, ℓ, τ)-*candidate* if $\omega(G) \leq \kappa$, and G has no odd hole of length at least ℓ, and every induced subgraph of G with clique number less than κ is τ-colourable. We will show that for all κ, ℓ, τ there exists c such that every (κ, ℓ, τ)-candidate has chromatic number at most c.

If H is a subgraph of G, and not necessarily induced, then adjacency in H may be different from in G, and we speak of being G-adjacent, a G-neighbour, and so on, to indicate in which graph we are using adjacency when there may be confusion. If $X \subseteq V(G)$, the subgraph of G induced on X is denoted by $G[X]$, and we often write $\chi(X)$ for $\chi(G[X])$. The *distance* or G-*distance* between two vertices u, v of G is the length of a shortest path between u, v, or ∞ if there is no such path. If $v \in V(G)$ and $\rho \geq 0$ is an integer, $N_G^\rho(v)$ or $N^\rho(v)$ denotes the set of all vertices u with distance exactly ρ from v, and $N_G^\rho[v]$ or $N^\rho[v]$ denotes the set of all v with distance at most ρ from v. If G is a nonnull graph and $\rho \geq 1$, we define $\chi^\rho(G)$ to be the maximum of $\chi(N^\rho[v])$ taken over all vertices v of G. (For the null graph G we define $\chi^\rho(G) = 0$.)

As in several other papers of this series, the proof of 1.3 examines whether there is an induced subgraph of large chromatic number such that every ball of small radius in it has bounded chromatic number. The proof breaks into three steps, as follows. We first show that for every (κ, ℓ, τ)-candidate G, $\chi(G)$ is bounded by a function of $\chi^3(G)$. Second, a similar argument shows the same for $\chi^2(G)$; that is, there is a function ϕ such that $\chi(G) \le \phi(\chi^2(G))$ for every (κ, ℓ, τ)-candidate G. For the third step, we apply a result of [2], which implies that in these circumstances there is an upper bound on the chromatic number of all (κ, ℓ, τ)-candidates.

The first two steps are both by application of lemmas proved in the next section, and are relatively simple. The remaining step would have been more difficult, but is already done and published. Except not quite; the result we need is not proved explicitly in [2] (we did it explicitly for graphs with no hole of length at least ℓ, and just sketched how to modify the argument to make it work for graphs with no odd hole of length at least ℓ). In the final section we fill in the details of this modification.

2 Gradings

Let G be a graph. We say a *grading* of G is a sequence $(W_1, \dots, W_n)$ of subsets of $V(G)$, pairwise disjoint and with union $V(G)$. If $w \ge 0$ is such that $\chi(G[W_i]) \le w$ for $1 \le i \le n$ we say the grading is *w-colourable.* We say that $u \in V(G)$ is *earlier* than $v \in V(G)$ (with respect to some grading $(W_1, \dots, W_n)$) if $u \in W_i$ and $v \in W_j$ where $i < j$. We need some lemmas about gradings. The first is:

2.1 *Let $w, c \ge 0$ and let $(W_1, \dots, W_n)$ be a w-colourable grading of a graph G with $\chi(G) > w + 2c$. Then there exist subsets X, Y of $V(G)$ with the following properties:*

- *$G[X], G[Y]$ are both connected;*
- *every vertex in Y is earlier than every vertex in X;*
- *some vertex in X has a neighbour in Y; and*
- *$\chi(X), \chi(Y) > c$.*

Proof. Let us say $v \in V(G)$ is *right-active* if there exists $X \subseteq V(G)$ such that

- $G[X]$ is connected;
- v is earlier than every vertex in X;
- v has a neighbour in X; and
- $\chi(X) > c$.

Let A be the set of all vertices of G that are not right-active, and $B = V(G) \setminus A$. Since $\chi(C) > w+2c$, either $\chi(A) > w + c$ or $\chi(B) > c$. Suppose that $\chi(A) > w + c$, and let C be a component of $G[A]$ with maximum chromatic number. Choose i with $1 \le i \le n$ minimum such that $W_i \cap V(C) \ne \emptyset$. Since $\chi(W_i) \le w$, it follows that $\chi(C \setminus W_i) > c$. Let C' be a component of $C \setminus W_i$ with maximum chromatic number; consequently $\chi(C') > c$. Choose $v \in W_i \cap V(C)$ with a neighbour in C' (this exists since C is connected). But then v is right-active, a contradiction.

This proves that $\chi(A) \le w + c$, and so $\chi(B) > c$. Let C be a component of $G[B]$ with maximum chromatic number. Choose i with $1 \le i \le n$ maximum such that $W_i \cap V(C) \ne \emptyset$, and choose $v \in W_i \cap V(C)$. Since v is right-active, there exists X as in the definition of "right-active" above; and then setting $Y = V(C)$ satisfies the theorem. This proves 2.1. ∎

Next, we need:

2.2 *Let $w, c \geq 0$ and let $(W_1, \ldots, W_n)$ be a w-colourable grading of a graph G. Let H be a subgraph of G (not necessarily induced) with $\chi(H) > w + 2(c + \chi^1(G))$. Then there is an edge uv of H, and a subset X of $V(G)$, such that*

- *$G[X]$ is connected;*
- *u, v are both earlier than every vertex in X;*
- *exactly one of u, v has a G-neighbour in X; and*
- *$\chi(X) > c$.*

Proof. We claim:

(1) *There exist $X, Y \subseteq V(G)$ with the following properties:*

- *$G[X]$ and $H[Y]$ are connected;*
- *every vertex of Y is earlier than every vertex of X;*
- *some vertex of Y has no G-neighbour in X, and some vertex of Y has a G-neighbour in X; and*
- *$\chi(G[X]) > c$.*

By 2.1 applied to H, there exist $X, Y \subseteq V(G)$ such that

- $H[X], H[Y]$ are both connected;
- every vertex in Y is earlier than every vertex in X;
- some vertex in X has an H-neighbour in Y; and
- $\chi(H[X]), \chi(H[Y]) > c + \chi^1(G)$.

If some vertex of Y has no G-neighbour in X, then (1) holds, so we may assume that every vertex of Y has a G-neighbour in X. Choose $y \in Y$ and let N be the set of vertices in X that are G-adjacent to y. Let C be a component of $G[X \setminus N]$ with maximum chromatic number. Since $\chi(G[N]) \leq \chi^1(G)$, it follows that $\chi(C) > c$. If some vertex of Y has a G-neighbour in $V(C)$ then (1) holds, so we assume not. Choose $x \in N$ with a G-neighbour in $V(C)$. Then $G[V(C) \cup \{x\}]$ is connected, and some vertex in Y has a G-neighbour in it, namely y. We may therefore assume that every vertex of Y has a G-neighbour in $V(C) \cup \{x\}$, and hence is G-adjacent to x, since no vertex of Y has a G-neighbour in $V(C)$. But this is impossible since $\chi(G[Y]) \geq \chi(H[Y]) > \chi^1(G)$. This proves (1).

Let X, Y be as in (1). Since some vertex of Y has a G-neighbour in X and some vertex of Y has no G-neighbour in X, and $H[Y]$ is connected, it follows that there is an edge uv of H with $u, v \in Y$ such that exactly one of u, v has a G-neighbour in X; and so the theorem holds. This proves 2.2. ∎

We also need the following:

2.3 *Let G be a graph, let $k \geq 0$, let $C \subseteq V(G)$, and let $x_0 \in V(G) \setminus C$, such that $G[C]$ is connected, x_0 has a neighbour in C, and $\chi(C) > k\chi^1(G)$. Then there is an induced path $x_0\text{-}\cdots\text{-}x_k$ of G where $x_1, \ldots, x_k \in C$, and a subset C' of C, with the following properties:*

- *$x_0, \ldots, x_k \notin C'$;*
- *$G[C']$ is connected;*
- *x_k has a neighbour in C', and $x_0, \ldots, x_{k-1}$ have no neighbours in C'; and*
- *$\chi(C') \geq \chi(C) - k\chi^1(G)$.*

Proof. We proceed by induction on k; the result holds if $k = 0$, so we assume that $k > 0$ and the result holds for $k-1$. Consequently there is an induced path $x_0\text{-}\cdots\text{-}x_{k-1}$ of G where $x_1, \ldots, x_{k-1} \in C$, and a subset C'' of C, such that

- $x_0, \ldots, x_{k-1} \notin C''$;
- $G[C'']$ is connected;
- x_{k-1} has a neighbour in C'', and $x_0, \ldots, x_{k-2}$ have no neighbours in C''; and
- $\chi(C'') \geq \chi(C) - (k-1)\chi^1(G)$.

Let N be the set of neighbours of x_{k-1}, and let C' be the vertex set of a component of $G[C'' \setminus N]$, chosen with $\chi(C')$ maximum (there is such a component since $\chi(C'') > \chi^1(G) \geq \chi(N)$). Let x_k be a neighbour of x_{k-1} with a neighbour in C''. Then $x_0\text{-}\cdots\text{-}x_k$ and C' satisfy the theorem. This proves 2.3. ∎

If G is a graph and $B, C \subseteq V(G)$, we say that B *covers* C if $B \cap C = \emptyset$ and every vertex in C has a neighbour in B. Let G be a graph, and let $B, C \subseteq V(G)$, where B covers C. Let $B = \{b_1, \ldots, b_m\}$. For $1 \leq i < j \leq m$ we say that b_i is *earlier* than b_j (with respect to the enumeration $(b_1, \ldots, b_m)$). For $v \in C$, let $i \in \{1, \ldots, m\}$ be minimum such that b_i, v are adjacent; we call b_i the *earliest parent* of v. An edge uv of $G[C]$ is said to be *square* (with respect to the enumeration $(b_1, \ldots, b_m)$) if the earliest parent of u is nonadjacent to v, and the earliest parent of v is nonadjacent to u. Let $B = \{b_1, \ldots, b_m\}$, and let $(W_1, \ldots, W_n)$ be a grading of $G[C]$. We say the enumeration $(b_1, \ldots, b_m)$ of B and the grading $(W_1, \ldots, W_n)$ are *compatible* if for all $u, v \in C$ with u earlier than v, the earliest parent of u is earlier than the earliest parent of v.

2.4 *Let G be a graph, and let $B, C \subseteq V(G)$, where B covers C. Let every induced subgraph J of G with $\omega(J) < \omega(G)$ have chromatic number at most τ. Let the enumeration $(b_1, \ldots, b_m)$ of B and the grading $(W_1, \ldots, W_n)$ of $G[C]$ be compatible. Let H be the subgraph of G with vertex set C and edge set the set of all square edges. Let $(W_1, \ldots, W_n)$ be w-colourable; then $\chi(G[C]) \leq w\tau\chi(H)$.*

Proof. Let $X \subseteq C$ be a stable set of H. We claim that $\chi(G[X]) \leq w\tau$. Suppose not. Since $(W_1, \ldots, W_n)$ is w-colourable, there is a partition $(A_1, \ldots, A_w)$ of C such that $W_i \cap A_j$ is stable in G for $1 \leq i \leq n$ and $1 \leq j \leq w$. Consequently there exists j such that $\chi(G[X \cap A_j]) > \tau$. From the choice of τ, it follows that $\omega(G[X \cap A_j]) = \omega(G)$. Let $Y \subseteq X \cap A_j$ be a clique with $|Y| = \omega(G)$. Choose $i \in \{1, \ldots, n\}$ maximum such that $W_i \cap Y \neq \emptyset$, and let $y \in W_i \cap Y$. Let b be the earliest parent of y. Since $|Y| = \omega(G)$, there exists $y' \in Y$ nonadjacent to b. Since $y, y' \in Y \subseteq A_j$, and y, y' are adjacent, it follows that $y' \notin W_i$, and so y' is earlier than y, from the choice of y. Since the enumeration and the grading are compatible, the earliest parent b' of y' is earlier than the earliest parent of y, and in particular is nonadjacent to y; and so yy' is a square edge, contradicting that $y, y' \in X$. This proves that $\chi(G[X]) \leq w\tau$. Since C can be partitioned into $\chi(H)$ sets that are stable in H, it follows that $\chi(G[C]) \leq w\tau\chi(H)$. This proves 2.4. ▮

Combining these lemmas yields the main result of this section:

2.5 *Let G be a graph, and let $B, C \subseteq V(G)$, where B covers C. Let every induced subgraph J of G with $\omega(J) < \omega(G)$ have chromatic number at most τ. Let the enumeration $(b_1, \ldots, b_m)$ of B and the grading $(W_1, \ldots, W_n)$ of $G[C]$ be compatible. Let $\ell, \rho \geq 3$ be integers. Let*

$$\chi(C) > w\tau(4(\ell + 2)\chi^\rho(G) + w).$$

Then there is an induced path $p_1\text{-}\cdots\text{-}p_k$ of $G[C]$, such that

- *$k \geq \ell$;*
- *p_1p_2 is a square edge;*
- *p_1, p_2 are both earlier than all of $p_3, \ldots, p_k$; and*
- *let b, b' be the earliest parent of p_1, p_2 respectively; then for each $v \in \{b, b', p_1, \ldots, p_\ell\}$, the G-distance between p_k and v is at least $\rho + 1$.*

Proof. Let $c = \ell\chi^1(G) + (\ell + 2)\chi^\rho(G)$. Thus

$$\chi(C) > w\tau(2(c + \chi^1(G)) + w).$$

Let H be the subgraph of $G[C]$ with edge set the square edges; then by 2.4,

$$\chi(H) > 2(c + \chi^1(G)) + w.$$

By 2.2, applied to $G[C]$ and H, there is a square edge uv, and a subset X of C, such that

- $G[X]$ is connected;
- u, v are both earlier than every vertex in X;
- exactly one of u, v is adjacent in G to a member of X; and
- $\chi(G[X]) > c$.

Let $\{p_1, p_2\} = \{u, v\}$ where p_2 has a neighbour in X. Since $\chi(X) > c \geq \ell\chi^1(G)$, 2.3 implies that there is an induced path p_1-$\cdots$-p_ℓ of G where $p_3, \ldots, p_\ell \in X$, and a subset C' of X, with the following properties:

- $p_1, \ldots, p_\ell \notin C'$;
- $G[C']$ is connected;
- p_ℓ has a neighbour in C', and $p_1, \ldots, p_{\ell-1}$ have no neighbours in C'; and
- $\chi(C') \geq \chi(X) - \ell\chi^1(G) > (\ell + 2)\chi^\rho(G)$.

Let b, b' be the earliest parent of p_1, p_2 respectively, and let $Z = \{b, b', p_1, \ldots, p_\ell\}$. Thus $|Z| = \ell + 2$. Since $\chi(C') > (\ell+2)\chi^\rho(G)$, there exists $x \in C'$ such that the G-distance between x and each member of Z is at least $\rho + 1$. Let p_ℓ-$\cdots$-p_k be an induced path of $G[C' \cup \{p_\ell\}]$ between p_ℓ and x; then p_1-$\cdots$-p_k satisfies the theorem. This proves 2.5. ∎

3 Little bounding balls

In this section we carry out the first two steps of the proof, showing that the chromatic number of every (κ, ℓ, τ)-candidate G can be bounded in terms of $\chi^3(G)$, and then can be bounded in terms of $\chi^2(G)$. More precisely we first prove the following.

3.1 *Let G be a (κ, ℓ, τ)-candidate; then $\chi(G) \leq 24(2\ell + 5)\tau\chi^3(G)^2$.*

Proof. We may assume that $\chi(G) > 4\chi^3(G)$. Since some component of G has the same chromatic number, we may assume that G is connected. Choose some vertex, and for $i \geq 0$ let L_i be the set of vertices with G-distance i from the vertex. There exists s such that $\chi(L_{s+1}) \geq \chi(G)/2$. Since $\chi(L_0 \cup \cdots \cup L_3) \leq \chi^3(G)$, it follows that $s \geq 3$. Let V_0 be the vertex set of a component of $G[L_{s+1}]$ with maximum chromatic number; and choose $z \in L_s$ with a neighbour in V_0. For $i \geq 0$ let M_i be the set of vertices in $V_0 \cup \{z\}$ with $G[V_0 \cup \{z\}]$-distance i from z. Choose t such that $\chi(M_{t+1}) \geq \chi(V_0)/2$. Again, since $\chi(V_0)/2 > \chi^3(G)$, it follows that $t \geq 3$.

The set of vertices in M_{t+1} with G-distance at most three from z has chromatic number at most $\chi^3(G)$; so there is a set $C \subseteq M_{t+1}$ with $\chi(C) \geq \chi(M_{t+1}) - \chi^3(G)$ such that every vertex in C has G-distance at least four from z. Let B be the set of vertices in L_s with a neighbour in C, and let D be the set of vertices in M_t with a neighbour in C. Thus every vertex in $B \cup D$ has G-distance at least three from z.

Let $V_1 = M_0 \cup \cdots \cup M_{t-1}$. Thus $G[V_1]$ is connected, and there are no edges between V_1 and C. Let B_0 be the set of vertices in B with no neighbour in V_1. Let B_1 be the set of vertices v in B with a neighbour in V_1 such that the $G[V_1 \cup \{v\}]$-distance between z, v is odd, and let B_2 be the set where this distance is even. Every vertex in C has a neighbour in at least one of B_0, B_1, B_2; let C_i be the set of vertices in C with a neighbour in B_i for $i = 0, 1, 2$.

(1) $\chi(C_0) \leq (4\ell + 9)\tau\chi^3(G)^2$.

Take an enumeration $(d_1, \ldots, d_n)$ of D, and for $1 \leq i \leq n$ let W_i be the set of vertices $v \in C_0$

such that v is adjacent to d_i and nonadjacent to $d_1, \dots, d_{i-1}$. Then $(W_1, \dots, W_n)$ is a grading of C_0, and is $\chi^3(G)$-colourable (indeed, τ-colourable, but we need the $\chi^3(G)$ bound), and the enumeration $(d_1, \dots, d_n)$ is compatible with it. Suppose that $\chi(C_0) > (4\ell + 9)\tau\chi^3(G)^2$. By 2.5 with $\rho = 3$ and $w = \chi^3(G)$, there is an induced path $p_1\text{-}\cdots\text{-}p_k$ of $G[C_0]$, such that

- $k \geq \ell$;
- p_1p_2 is a square edge;
- p_1, p_2 are both earlier than all of $p_3, \dots, p_k$; and
- let d, d' be the earliest parent of p_1, p_2 respectively; then for each $v \in \{d, d', p_1, \dots, p_\ell\}$, the G-distance between p_k and v is at least 4.

Since $d, d' \in D$, there are induced paths Q, Q' of $G[V_1 \cup D]$ between d, z and between d', z respectively, both of length t. Let $y \in B_0$ be adjacent to p_k. There is an induced path R between z, y with interior in $L_0 \cup \dots \cup L_{s-1}$, since $z, y \in L_s$. Now y is nonadjacent to d, d' since the G-distance between p_k and d, d' is at least four. It follows that $Q \cup R$ is an induced path between d, y, and $Q' \cup R$ is an induced path between d', y, of the same length. Also y is nonadjacent to $p_1, p_2, \dots, p_\ell$ since p_k has G-distance four from all these vertices. Choose $j \leq k$ minimum such that p_j is adjacent to y. Now d, d' are both nonadjacent to $p_3, \dots, p_j$ since p_1, p_2 are both earlier than $p_3, \dots, p_j$; and since p_1 is nonadjacent to d' and p_2 is nonadjacent to d (because p_1p_2 is a square edge) it follows that $d\text{-}p_1\text{-}p_2\text{-}\cdots\text{-}p_j\text{-}y$ and $d'\text{-}p_2\text{-}\cdots\text{-}p_j\text{-}y$ are both induced paths, joining d, y and d', y respectively. So the union of $d\text{-}p_1\text{-}p_2\text{-}\cdots\text{-}p_j\text{-}y$ with $Q \cup R$ is a hole, and the union of $d'\text{-}p_2\text{-}\cdots\text{-}p_j\text{-}y$ with $Q' \cup R$ is a hole; and these holes differ in length by one. Since they both have length more than ℓ, this is impossible. This proves (1).

(2) *For $h = 1, 2$, $\chi(C_h) \leq (4\ell + 9)\tau\chi^3(G)^2$.*

Enumerate the vertices of V_1 in increasing order of $G[V_1]$-distance from z, breaking ties arbitrarily; that is, take an enumeration $(a_1, \dots, a_n)$ of V_1 where for $0 \leq i < j \leq n$ the $G[V_1]$-distance between a_i, z in $G[V_1]$ is at most that between a_j, z. For each $v \in C_h$, choose $i \in \{1, \dots, n\}$ minimum such that some vertex in B_h is adjacent both to v and to a_i. (Such a value of i exists from the definition of B_h, C_h.) We call a_i the *earliest grandparent* of v. For $1 \leq i \leq n$, let W_i be the set of vertices in C_h with earliest grandparent a_i. Thus every vertex in W_i has G-distance two from a_i, and so $\chi(W_i) \leq \chi^3(G)$; and it follows that $(W_1, \dots, W_n)$ is a $\chi^3(G)$-colourable grading of $G[C_h]$ (indeed, it is $\chi^2(G)$-colourable, since $W_i \subseteq N^2[a_i]$). Take an enumeration $(b_1, \dots, b_m)$ of B_h such that vertices with earlier neighbours in V_1 come first; that is, such that for $1 \leq i < j \leq m$, there is a neighbour $a_h \in V_1$ of b_i such that $h \leq h'$ for every neighbour $a_{h'}$ of b_j in V_1. It follows that $(b_1, \dots, b_m)$ and $(W_1, \dots, W_n)$ are compatible.

Suppose that $\chi(C_h) > (4\ell + 9)\tau\chi^3(G)^2$. By 2.5 with $\rho = 3$ and $w = \chi^3(G)$, there is an induced path $p_1\text{-}\cdots\text{-}p_k$ of $G[C_h]$, such that

- $k \geq \ell$;
- p_1p_2 is a square edge;
- p_1, p_2 are both earlier than all of $p_3, \dots, p_k$; and

- let b, b' be the earliest parent of p_1, p_2 respectively; then for each $v \in \{b, b', p_1, \ldots, p_\ell\}$, the G-distance between p_k and v is at least 4.

Let b'' be the earliest parent of p_k, and let a, a', a'' be the earliest grandparents of p_1, p_2, p_k respectively. It follows that $ab, a'b', a''b''$ are edges. Since both a, a' occur in the enumeration $(a_1, \ldots, a_n)$ before a'', and a'' is the earliest grandparent of p_k, there are induced paths Q, Q' of $G[V_1]$ between a, z and between a', z respectively, with lengths of the same parity as $h+1$ (because $p_1, p_2 \in C_h$), such that b'' has no neighbours in Q, Q'. There is an induced path R between z, b'' with interior in $L_0 \cup \cdots \cup L_{s-1}$, using only two vertices of L_{s-1} (neighbours of z, b'' respectively). It follows that b, b' both have no neighbours in R; for b, b' both have G-distance at least three from z, and at least four from p_k and hence G-distance at least three from b'', and consequently they both have G-distance at least two from the vertices of R in L_{s-1} (it is to arrange this that we need to control the chromatic number of balls of radius three). Consequently $Q \cup R$ is an induced path between b, b'', and $Q' \cup R$ is an induced path between b', b'', and they have lengths of the same parity. Now b'' is nonadjacent to $p_1, \ldots, p_\ell$ since the G-distance between p_k and $p_1, \ldots, p_\ell$ is at least four. Choose j minimum such that p_j is adjacent to b''. Also, b, b' are nonadjacent to $p_3, \ldots, p_k$ since p_1, p_2 are earlier than $p_3, \ldots, p_k$; and since p_1p_2 is square, the union of $Q \cup R$ with b-p_1-p_2-$\cdots$-p_j-b'' and the union of $Q' \cup R$ with b'-p_2-$\cdots$-p_j-b'' are holes of opposite parity, both of length more than ℓ, which is impossible. This proves (2).

From (1) and (2), we deduce that $\chi(C) \le 3(4\ell+9)\tau\chi^3(G)^2$. But $\chi(C) \ge \chi(M_{t+1}) - \chi^3(G)$, so $\chi(M_{t+1}) \le 3(4\ell+9)\tau\chi^3(G)^2 + \chi^3(G)$. Since $\chi(M_{t+1}) \ge \chi(V_0)/2$, and $\chi(V_0) \ge \chi(G)/2$, we deduce that

$$\chi(G) \le 12(4\ell+9)\tau\chi^3(G)^2 + 4\chi^3(G) \le 24(2\ell+5)\tau\chi^3(G)^2.$$

This proves 3.1. ∎

Next we bound $\chi(G)$ in terms of $\chi^2(G)$, as follows.

3.2 *Let G be a (κ, ℓ, τ)-candidate; then $\chi(G) \le 96(2\ell+5)^3\tau^3\chi^2(G)^8$.*

Proof. Let z be a vertex such that $\chi(N^3[z]) = \chi^3(G)$, and let L_i be the set of vertices with G-distance i from z, for $0 \le i \le 3$. Fix a $\chi^2(G)$-colouring of $G[L_0 \cup L_1 \cup L_2]$, and for each $v \in L_3$ choose a path of length three from z to v, and let $\alpha(v), \beta(v)$ be the colours of its second and third vertex respectively. Choose colours α, β, and let C be the set of $v \in L_3$ such that $\alpha(v) = \alpha$ and $\beta(v) = \beta$. Let B be the set of vertices in L_2 with colour β and with a neighbour in L_1 with colour α. Consequently B covers C, and any two vertices in B are joined by an induced path of even length with interior in $L_0 \cup L_1$.

Let $(b_1, \ldots, b_n)$ be some enumeration of B, and for $1 \le i \le n$ let W_i be the set of $v \in C$ such that v is adjacent to b_i but not to $b_1, \ldots, b_{i-1}$. Thus $(W_1, \ldots, W_n)$ is a τ-colourable grading of $G[C]$ compatible with $(b_1, \ldots, b_n)$. Suppose that $\chi(C) > (4\ell+9)\tau\chi^2(G)^2$. Then by 2.5 with $\rho = 2$ and $w = \chi^2(G)$ there is an induced path p_1-$\cdots$-p_k of $G[C]$, such that

- $k \ge \ell$;
- p_1p_2 is a square edge;

- p_1, p_2 are both earlier than all of $p_3, \dots, p_k$; and
- let b, b' be the earliest parent of p_1, p_2 respectively; then for each $v \in \{b, b', p_1, \dots, p_\ell\}$, the G-distance between p_k and v is at least 3.

It follows that b, b' are nonadjacent to $p_3, \dots, p_k$; let $b'' \in B$ be adjacent to p_k, and choose j minimum such that p_j is adjacent to b''. Since p_k has G-distance at least three from each of $b, b', p_1, \dots, p_\ell$, it follows that b'' is nonadjacent to all these vertices. Choose j minimum such that p_j is adjacent to b''; then b-p_1-p_2-p_j-b'' and b'-p_2-$\cdots$-p_j-b'' are induced paths both of length more than ℓ and with lengths of opposite parity. Let Q be one of them with odd length. There is an induced path of even length joining the ends of Q with interior in $L_0 \cup L_1$; and its union with Q is an odd hole of length more than ℓ, which is impossible.

This proves that $\chi(C) \leq (4\ell + 9)\tau\chi^2(G)^2$. Since this holds for every choice of α, β, and there are only $\chi^2(G)^2$ such choices, it follows that $\chi(L_3) \leq (4\ell + 9)\tau\chi^2(G)^4$. But $\chi^3(G) \leq \chi(L_3) + \chi^2(G)$, so $\chi^3(G) \leq 2(2\ell + 5)\tau\chi^2(G)^4$. From 3.1, it follows that $\chi(G) \leq 96(2\ell + 5)^3\tau^3\chi^2(G)^8$. This proves 3.2. ∎

4 The good old stuff

That concludes the "new" part of the paper; the remainder is an attempt to persuade the reader that what we proved in [2] together with 3.2 implies 1.3. The main idea is "clique control", so let us define that. If X, Y are disjoint subsets of the vertex set of a graph G, we say

- X is *complete* to Y if every vertex in X is adjacent to every vertex in Y; and
- X is *anticomplete* to Y if every vertex in X nonadjacent to every vertex in Y.

(If $X = \{v\}$ we say v is complete to Y instead of $\{v\}$, and so on.) Let X be a clique in a graph G. If $|X| = h$ we call it an *h-clique*. We define $N_G^2(X)$ to be the set of vertices v of G that are not in X and are anticomplete to X, but are adjacent to some vertex u that is complete to X. Let $\mathbb{N}$ denote the set of nonnegative integers, let $\phi : \mathbb{N} \to \mathbb{N}$ be a nondecreasing function, and let $h \geq 1$ be an integer. We say a graph G is *(h, ϕ)-clique-controlled* if for every induced subgraph H of G and every integer $n \geq 0$, if $\chi(H) > \phi(n)$ then there is an h-clique X of H such that $\chi(N_H^2(X)) > n$. Consequently 3.2 implies that:

4.1 *Every (κ, ℓ, τ)-candidate G is $(1, \phi)$-clique-controlled where ϕ is the function*

$$\phi(x) = 96(2\ell + 5)^3\tau^3x^8.$$

(This will be a substitute for theorem 5.2 of [2].) To avoid having to go through all this again in later papers, let us try to maintain as much generality as possible. We assume that $\kappa, \tau \geq 0$ and $\mathcal{C}_0$ is a class of graphs, such that:

(C1) every induced subgraph of a member of $\mathcal{C}_0$ also belongs to $\mathcal{C}_0$;

(C2) $\omega(G) \leq \kappa$ for every $G \in \mathcal{C}_0$;

(C3) $\chi(G) \le \tau$ for every graph $G \in \mathcal{C}_0$ with $\omega(G) < \kappa$;

(C4) there is a nondecreasing function ϕ such that every member of $\mathcal{C}_0$ is $(1, \phi)$-clique-controlled; and

(C5) for all c there is a graph in $\mathcal{C}_0$ with chromatic number more than c.

We will guide the reader to what the results of [2] tell us about such a class $\mathcal{C}_0$; and at the end take $\mathcal{C}_0$ to be the class of all (κ, ℓ, τ)-candidates to deduce 1.3. But at the moment we prefer not to assume that the members of $\mathcal{C}_0$ have no odd hole of length at least ℓ.

Since every member of $\mathcal{C}_0$ has clique number at most κ, and there are members of $\mathcal{C}_0$ with arbitrarily large chromatic number, there is no function ϕ such that every member of $\mathcal{C}_0$ is (κ, ϕ)-clique-controlled. Consequently there exists $h < \kappa$, such that

- there is a function ϕ such that every member of $\mathcal{C}_0$ is (h, ϕ)-clique-controlled, and
- there is no function ϕ' such that every member of $\mathcal{C}_0$ is $(h + 1, \phi')$-clique-controlled.

The nonexistence of ϕ' means that for some value σ, there is no appropriate value of $\phi'(\sigma)$; that is, there exist graphs $G \in \mathcal{C}_0$ with arbitrarily large chromatic number such that $\chi(N_G^2(X)) \le \sigma$ for every $(h + 1)$-clique X in G. Let $\mathcal{C}$ be the class of graphs $G \in \mathcal{C}_0$ such that $\chi(N_G^2(X)) \le \sigma$ for every $(h + 1)$-clique X in G. Thus there are members of $\mathcal{C}$ with arbitrarily large chromatic number. In these circumstances we can apply theorem 4.1 of [2], which we explain next.

Let G be a graph and let $t \ge 0$ and $h \ge 1$ be integers. An *h-cable* of *length t* in G consists of:

- t h-cliques $X_1, \dots, X_t$, pairwise disjoint and anticomplete;
- for $1 \le i \le t$, a subset N_i of $V(G)$ disjoint from and complete to X_i, such that the sets $N_1, \dots, N_t$ are pairwise disjoint;
- for $1 \le i \le t$, disjoint subsets $Y_{i,t}$ and $Z_{i,i+1}, \dots, Z_{i,t}$ of N_i; and
- a subset $C \subseteq V(G)$ disjoint from $X_1 \cup \dots \cup X_t \cup N_1 \cup \dots \cup N_t$

satisfying the following conditions.

1. For $1 \le i \le t$, $Y_{i,t}$ covers C, and C is anticomplete to $Z_{i,j}$ for $i + 1 \le j \le t$, and C is anticomplete to X_i.
2. For $i < j \le t$, X_i is anticomplete to N_j.
3. For all $i < j \le t$, every vertex in $Z_{i,j}$ has a non-neighbour in X_j.
4. For $i < j < k \le t$, $Z_{i,j}$ is anticomplete to $X_k \cup N_k$.
5. For all $i < j \le t$, either
   - some vertex in X_j is anticomplete to $Y_{i,t}$, and $Z_{i,j} = \emptyset$, or
   - X_j is complete to $Y_{i,t}$, and $Z_{i,j}$ covers N_j.

We say $\chi(C)$ is the *chromatic number* of the h-cable. Theorem 4.1 of [2] says:

4.2 *Let $t, c, \tau, \sigma \geq 0$ and $h > 0$, and let $\phi : \mathbb{N} \to \mathbb{N}$ be nondecreasing. Then there exists c' with the following property. Let G be a graph such that*

- *$\chi(N^1(v)) \leq \tau$ for every $v \in V(G)$;*
- *G is (h, ϕ)-clique-controlled; and*
- *$\chi(N^2(X)) \leq \sigma$ for every $(h+1)$-clique X of G.*

If $\chi(G) > c'$ then G admits an h-cable of length t with chromatic number more than c.

By applying this to our class $\mathcal{C}$ we deduce that for all $c, t \geq 0$, there exists $G \in \mathcal{C}$ with an h-cable of length t with chromatic number more than c.

Given an h-cable in the notation above, let $I \subseteq \{1, \ldots, t\}$; then the cliques X_i $(i \in I)$, the sets N_i $(i \in I)$, the sets $Z_{i,j}$ $(i, j \in I)$, the sets $Y_{i,t}$ $(i \in I)$ and C (after appropriate renumbering) define an h-cable of length $|I|$. We call this a *subcable*. In an h-cable there are two types of pair (i, j) with $i < j \leq t$, and we can apply Ramsey's theorem on these pairs to get a large subcable where all the pairs have the same type. Consequently, two special kinds of h-cables are of interest (t is the length in both cases):

- h-cables of *type 1*, where for all $i < j \leq t$, some vertex in X_j has no neighbours in $Y_{i,t}$, and $Z_{i,j} = \emptyset$; and
- h-cables of *type 2*, where for all $i < j \leq t$, X_j is complete to $Y_{i,t}$, and $Z_{i,j}$ covers N_j.

Ramsey's theorem, applied to the h-cable given by 4.2, implies:

4.3 *For all $t, c \geq 0$ there is a graph $G \in \mathcal{C}$ that admits an h-cable of length t of type 1 or 2 with chromatic number more than c.*

Theorem 3.2 of [2] says:

4.4 *Let $\sigma \geq 0$, $\ell \geq 5$ and $h \geq 1$, and let G be a graph with no ℓ-hole, such that $\chi(N^2(X)) \leq \sigma$ for every $(h+1)$-clique X of G. Then G admits no h-cable of type 2 and length $\ell' - 3$ with chromatic number more than $(\ell - 3)\sigma$.*

If we knew that for some fixed ℓ, the members of our class $\mathcal{C}_0$ have no hole of length exactly ℓ, we could eliminate cables of type 2 from 4.3. And we *do* know this, of course, for the application we have in mind, when $\mathcal{C}_0$ is the class of all (κ, ℓ, τ)-candidates, since $\ell \geq 5$ and is odd. But still we prefer not to make this assumption, because it may be that in later papers we will be excluding something more complicated than just an odd hole. Let us just assume instead that

(C6) there exist $t, c \geq 0$ such that no member of $\mathcal{C}_0$ admits an h-cable of type 2 and length t with chromatic number more than c.

Under the assumption **(C6)**, 4.3 becomes:

4.5 *For all $t, c \geq 0$ there is a graph $G \in \mathcal{C}$ that admits an h-cable of length t of type 1 with chromatic number more than c.*

The next step involves "multicovers", so let us define these. (A warning: in some papers of this series we used the word "multicover" with a different meaning, but here we revert to the meaning in [2].) Let $x \in V(G)$, let N be some set of neighbours of x, and let $C \subseteq V(G)$ be disjoint from $N \cup \{x\}$, such that x is anticomplete to C and N covers C. In this situation we call (x, N) a *cover* of C in G. For $C, X \subseteq V(G)$, a *multicover of* C in G is a family $(N_x : x \in X)$ such that

- X is stable;
- for each $x \in X, (x, N_x)$ is a cover of C;
- for all distinct $x, x' \in X$, x' is anticomplete to N_x (and in particular all the sets $\{x\} \cup N_x$ are pairwise disjoint).

The multicover $(N_x : x \in X)$ is *stable* if each of the sets N_x $(x \in X)$ is stable.

Now we need two theorems, implicit in the proofs of theorems 3.1 and 2.3 of [2] respectively, but let us prove them explicitly since the proofs are short.

4.6 *For all $m, c \geq 0$, there is a graph $G \in \mathcal{C}$ that admits a multicover $(N_x : x \in X)$ of a set C, such that $|X| = m$ and $\chi(C) > c$.*

Proof. By Ramsey's theorem there exists t such that for every partition of the edges of the complete graph K_t into h sets, there is an m-clique of K_t for which all edges joining its vertices are in the same set. Choose $G \in \mathcal{C}$ that admits an h-cable of length t of type 1 with chromatic number more than c. In the usual notation for h-cables, fix an ordering of the members of X_i for each i; thus we may speak of the rth member of X_i for $1 \leq r \leq h$. For each pair (i, j) with $i < j \leq t$, let $f(i, j) = r$ where the rth member of X_j has no neighbours in $Y_{i,t}$ (such a value exists from the definition of "type 1"). From the choice of t, there exist $I \subseteq \{1, \ldots, t\}$ with $|I| = m$ and $r \in \{1, \ldots, h\}$ such that $f(i, j) = r$ for all $i, j \in I$ with $i < j$. For each $j \in I$, let x_j be the rth member of X_j. Then the sets (x_j, N_j) $(j \in I)$ form a multicover of C. This proves 4.6. ∎

4.7 *For all $m, c \geq 0$, there is a graph $G \in \mathcal{C}$ that admits a stable multicover $(N_x : x \in X)$ of a set C, such that $|X| \geq m$ and $\chi(C) > c$.*

Proof. Let $c' = c\tau^m$. By 4.6 there exists $G \in \mathcal{C}$ that admits a multicover $(N'_x : x \in X)$ of a set C', such that $|X| = m$ and $\chi(C') > c'$. For each $x \in X$, the subgraph induced on N'_x is τ-colourable; choose some such colouring, with colours $1, \ldots, \tau$, for each x. For each $v \in C'$, let $f_v : X \to \{1, \ldots, \tau\}$ such that for each $x \in X$, some neighbour of v in N'_x has colour $f_v(x)$. There are only $\tau^{|X|}$ possibilities for f_v, so there is a function $f : X \to \{1, \ldots, \tau\}$ and a subset $C \subseteq C'$ with $\chi(C) \geq \chi(C')\tau^{-|X|} > c$, such that $f_v = f$ for all $v \in C$. For each $x \in X$, let N_x be the set of vertices in N'_x with colour $f(x)$; then $(N_x : x \in X)$ is a stable multicover of C. This proves 4.7. ∎

Let $(N_x : x \in X)$ be a multicover of C in G. It is said to be *k-crested* if there are vertices $a_1, \ldots, a_k$ and vertices $a_{ix} (1 \leq i \leq k, x \in X)$ of G, all distinct, with the following properties:

- $a_1, \ldots, a_k$ and the vertices $a_{ix} (1 \leq i \leq k, x \in X)$ do not belong to $X \cup C \cup \bigcup_{x \in X} N_x$;
- for $1 \leq i \leq k$ and each $x \in X$, a_{ix} is adjacent to x, and there are no other edges between the sets $\{a_1, \ldots, a_k\} \cup \{a_{ix} : 1 \leq i \leq k, x \in X\}$ and $X \cup C \cup \bigcup_{x \in X} N_x$;

- for $1 \leq i \leq k$ and each $x \in X$, a_{ix} is adjacent to a_i, and there are no other edges between $\{a_1, \ldots, a_k\}$ and $\{a_{ix} : 1 \leq i \leq k, x \in X\}$
- $a_1, \ldots, a_k$ are pairwise nonadjacent;
- for all distinct $i, j \in \{1, \ldots, k\}$ and all distinct $x, y \in X$, a_{ix} is nonadjacent to a_{jy}.

Note that a_{ix} may be adjacent to a_{iy}. We say the multicover is *stably k-crested* if for $1 \leq i \leq k$ and all distinct $x, y \in X$, a_{ix}, a_{iy} are nonadjacent. Theorem 2.1 of [2] (setting j of the theorem to be κ) implies:

4.8 *For all $m, c, \kappa, \tau \geq 0$ there exist $m', c' \geq 0$ with the following property. Let G be a graph with $\omega(G) \leq \kappa$, such that $\chi(H) \leq \tau$ for every induced subgraph H of G with $\omega(H) < \kappa$. Let $(N'_x : x \in X')$ be a stable multicover in G of some set C', such that $|X'| \geq m'$ and $\chi(C') > c'$. Then there exist $X \subseteq X'$ with $|X| \geq m$, and $C \subseteq C'$ with $\chi(C) > c$, and a stable multicover $(N_x : x \in X)$ of C contained in $(N'_x : x \in X')$ that is 1-crested.*

Ramsey's theorem applied to the vertices $\{a_{1x} : x \in X\}$ together with 4.8 yields (under the same hypotheses) that such a stable multicover exists which is stably 1-crested. And repeated application of this yields:

4.9 *For all $m, c, k, \kappa, \tau \geq 0$ there exist $m', c' \geq 0$ with the following property. Let G be a graph with $\omega(G) \leq \kappa$, such that $\chi(H) \leq \tau$ for every induced subgraph H of G with $\omega(H) < \kappa$. Let $(N'_x : x \in X')$ be a stable multicover in G of some set C', such that $|X'| \geq m'$ and $\chi(C') > c'$. Then there exist $X \subseteq X'$ with $|X| \geq m$, and $C \subseteq C'$ with $\chi(C) > c$, and a stable multicover $(N_x : x \in X)$ of C contained in $(N'_x : x \in X')$ that is stably k-crested.*

Applied to our class $\mathcal{C}$, we deduce:

4.10 *For all $m, c, k \geq 0$, there is a graph $G \in \mathcal{C}$ that admits a stably k-crested stable multicover $(N_x : x \in X)$ of a set C, where $|X| = m$ and $\chi(C) > c$.*

In summary, we have proved so far that:

4.11 *Let $\mathcal{C}_0$ be a class of graphs satisfying* **(C1)–(C6)**. *Then for all $m, c, k \geq 0$ there is a graph $G \in \mathcal{C}_0$ that admits a stably k-crested stable multicover $(N_x : x \in X)$ of a set C, where $|X| = m$ and $\chi(C) > c$.*

Proof of 1.3. We saw in section 1 that it suffices to prove an upper bound on the chromatic number of (κ, ℓ, τ)-candidates. Let $\mathcal{C}_0$ be the class of all (κ, ℓ, τ)-candidates. It satisfies **(C1)–(C4)** by 4.1, and **(C6)** by 4.4; suppose for a contradiction that it also satisfies **(C5)**. By 4.11, taking $k = m = \ell$ and $c = \tau$, there is a graph $G \in \mathcal{C}_0$ that admits a stably ℓ-crested stable multicover $(N_x : x \in X)$ of a set C, where $|X| = \ell$ and $\chi(C) > c$. Let $a_1, \ldots, a_\ell$ and the vertices $a_{ix}(1 \leq i \leq \ell, x \in X)$ be as in the definition of stable k-crested. Choose distinct $x_1, x_2 \in X$. Let $\ell_1 \in \{\ell+1, \ell+3\}$ be a multiple of four. There is an induced path P between x_1, x_2 of length ℓ_1 such that $V(P) \subseteq \{a_1, \ldots, a_\ell\} \cup X \cup \{a_{ix} : 1 \leq i \leq \ell, x \in X\}$. Since $\chi(C) > \tau$, there is an κ-clique $Y \subseteq C$. Choose $b_1 \in N_{x_1} \cup N_{x_2}$ with as many neighbours in Y as possible. By exchanging x_1, x_2 if necessary, we may assume that $b_1 \in N_{x_1}$. Now b_1 is not complete to Y since $\omega(G) \leq \kappa$; so there exists $y_2 \in Y$

nonadjacent to b_1. Choose $b_2 \in N_{x_2}$ adjacent to y_2. From the choice of b_1, there exists $y_1 \in Y$ adjacent to b_1 and not to b_2. If b_1, b_2 are adjacent let Q be the path x_1-b_1-b_2-x_2, and otherwise let Q be the path x_1-b_1-y_1-y_2-b_2-x_2. Thus Q is induced, and has length three or five, and the union of P and Q is an odd hole of length more than ℓ, which is impossible. This contradiction shows that $\mathcal{C}_0$ does not satisfy **(C5)**, and this proves 1.3.

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