Monochromatic components in edge-coloured graphs with large minimum degree

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Abstract

For every $n \in \mathbb{N}$ and $k \geq 2$, it is known that every k-edge-colouring of the complete graph on n vertices contains a monochromatic connected component of order at least $\frac{n}{k-1}$. For $k \geq 3$, it is known that the complete graph can be replaced by a graph G with $\delta(G) \geq (1-\varepsilon_k)n$ for some constant ε_k. In this paper, we show that the maximum possible value of ε_3 is $\frac{1}{6}$. This disproves a conjecture of Gyárfas and Sárközy.

1 Introduction

Erdős and Rado noted that, for any graph G, either G or its complement is connected. This is equivalent to the statement that every 2-edge-colouring of a complete graph contains a monochromatic spanning tree. Gyárfás [2] extended this result to $k \geq 3$ colours. He proved the following theorem.

Theorem 1.1 (Gyárfás [2]). *Fix $k \geq 2$. In every k-edge-colouring of the complete graph on n vertices, there exists a monochromatic component of order at least $\frac{n}{k-1}$.*

The bound in this theorem is sharp if $k-1$ is a prime power and n is divisible by $(k-1)^2$. Consider the affine plane of order $k-1$ and colour the edges in the i^{th} parallel class with colour i for each $i \in [k]$. Every monochromatic component contains exactly $\frac{n}{k-1}$ vertices.

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For $k = 2$, it is easy to see that the conclusion of Theorem 1.1 does not hold for 2-colourings of the edges of a non-complete graph: if xy is not an edge, then colour edges red if they are incident with x and blue otherwise. However, Gyárfás and Sárközy [3] proved the following theorem.

Theorem 1.2 (Gyárfás and Sárközy [3]). *Let G be a graph of order n with $\delta(G) \geq \frac{3}{4}n$. If the edges of G are 2-coloured, then there exists a monochromatic component of order at least $\delta(G) + 1$.*

Gyárfás and Sárközy also showed that the bounds in this theorem are best possible in the following two senses.

Firstly, we cannot reduce $\delta(G)$ below $\frac{3}{4}n$. Suppose that n is divisible by 4 and partition the vertices into 4 sets V_1, V_2, V_3 and V_4, each of order $\frac{n}{4}$. Colour the following edges red: the edges within each V_i, those between V_1 and V_2 and those between V_3 and V_4. Colour the edges between V_1 and V_3 blue and the edges between V_2 and V_4 blue. All monochromatic components have order $\frac{n}{2}$ and $\delta(G) = \frac{3}{4}n - 1$.

Secondly, the largest monochromatic component we can guarantee has order $\delta(G) + 1$. Take the complete graph K_n and let $X, Y \subset V(K_n)$ be disjoint vertex sets with $|X| = |Y| = z(\delta)$ where $0 \leq z \leq \frac{n}{2}$. Form the graph G by removing all edges between X and Y. Colour the edges incident to X red, the edges incident to Y blue and all other edges arbitrarily. The largest monochromatic component in G in either colour has order $n - z = \delta(G) + 1$.

For $k \geq 3$, the situation is different. In this case, it is possible to remove some edges from a complete graph and still obtain a monochromatic component of order $\frac{n}{k-1}$ in every k-edge-colouring. Indeed, Gyárfás and Sárközy [4] showed that the complete graph can be replaced by any graph G with $\delta(G) \geq (1 - \varepsilon_k)n$ for some constant $\varepsilon_k > 0$. They made the following conjecture.

Conjecture 1.3 (Gyárfás and Sárközy [4]). *Fix $k \geq 3$. Let G be any graph with n vertices and $\delta(G) \geq \left(1 - \frac{k-1}{k^2}\right) n$. If the edges of G are k-coloured, then there exists a monochromatic component of order at least $\frac{n}{k-1}$.*

Recently, there has been some progress towards Conjecture 1.3. The best current general result was proved by DeBiasio, Krueger and Sárközy [1].

Theorem 1.4 (DeBiasio, Krueger and Sárközy [1]). *Fix an integer $k \geq 3$ and let G be any graph of order n with $\delta(G) \geq \left(1 - \frac{1}{3072(k-1)^5}\right) n$. If the edges of G are k-coloured, then there exists a monochromatic component of order at least $\frac{n}{k-1}$.*

For 3 colours, DeBiasio, Krueger and Sárközy [1] proved a stronger result.

Theorem 1.5 (DeBiasio, Krueger and Sárközy [1])**.** *Let G be a graph of order n with $\delta(G) \geq \frac{7}{8}n$. In every 3-colouring of the edges of G, there exists a monochromatic component of order at least $\frac{n}{2}$.*

This is not far from the constant $\frac{7}{9}$ predicted by Conjecture 1.3. However, in this paper, we show that the conjecture is in fact false by proving the following theorem.

Theorem 1.6. *For every $n \in \mathbb{N}$, there exists a graph G of order n with $\delta(G) \geq \left\lfloor \frac{5}{6}n \right\rfloor - 2$ and a 3-colouring of the edges of G such that every monochromatic component has order strictly less than $\frac{n}{2}$.*

We further show that $\frac{1}{6}$ is the largest possible value for ε_3 by proving the following theorem.

Theorem 1.7. *Let G be a graph of order n with $\delta(G) \geq \frac{5}{6}n$ where the edges of G have been* 3*-coloured. Then G has a monochromatic component with order at least $\frac{n}{2}$.*

The paper is organised in the following way. In Section 2, we will prove Theorem 1.6. Then, in Sections 3 and 4, we will prove Theorem 1.7. First we will reduce the problem to two specific cases in Section 3. We will then explain how these two cases can be formulated as collections of linear programs and solved in Section 4. Together these results show that $\varepsilon_3 = \frac{1}{6}$.

2 Proof of lower bound

In this section, we give constructions to prove the lower bound given in Theorem 1.6. We will give a separate construction for each residue class modulo 6.

Proof of Theorem 1.6. First, take $n = 6q + 4$ for some $q \in \mathbb{N}$. We will consider other residue classes later.

We will construct a graph G of order n with $\delta(G) = 5q + 2$. We will also show that there is a 3-colouring of the edges of G such that every monochromatic component has order strictly less than $\frac{n}{2} = 3q + 2$. The colours will be red, blue and green.

Partition the vertices into 8 sets $V_1, \ldots, V_8$ with the following sizes:

- $|V_1| = |V_2| = 2$
- $|V_3| = q - 3$
- $|V_4| = q - 1$
- $|V_5| = |V_6| = |V_7| = |V_8| = q+1$.

Observe that $|V_1| + \cdots + |V_8| = 6q + 4 = n$. There are no edges between:

- V_1 and V_2
- V_4 and $V_1 \cup V_2 \cup V_3$
- V_5 and V_8
- V_6 and V_7.

All other edges are present (including all edges inside vertex classes). This means that each vertex in V_3 has degree $5q + 4$ and every other vertex has degree $5q + 2$. Therefore $\delta(G) = 5q + 2 = \left\lfloor \frac{5}{6}n \right\rfloor - 1$ as required.

It remains to construct a 3-colouring of the edges in which every monochromatic component has order strictly less than $\frac{n}{2}$. We colour edges between vertex classes as shown in Figure 1; edges inside classes are coloured arbitrarily with the exception of the edge within V_1 which is red.

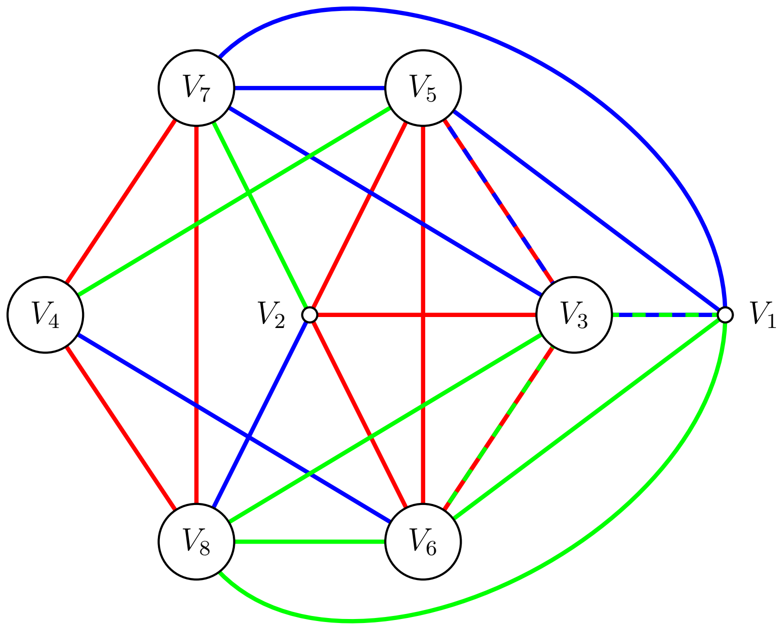

Figure 1: The graph G for $n = 6q + 4$.

In this colouring, the three red components have orders 2, $3q + 1$ and $3q + 1$, the three blue components have orders $q + 3$, $2q$ and $3q + 1$ and the three green components have orders $q + 3$, $2q$ and $3q + 1$. As $\frac{n}{2} = 3q + 2$, all monochromatic components have order strictly less than $\frac{n}{2}$ as required.

For the remaining residue classes modulo 6, we construct similar graphs with $\delta(G) = \left\lfloor \frac{5}{6}n \right\rfloor - c$ for some constant $c \in \{1, 2\}$ that depends on the residue class.

Case: $n = 6q$. Partition the vertices into 6 sets.

- $|V_1| = |V_2| = |V_3| = q + 1$
- $|V_4| = |V_5| = |V_6| = q - 1$

We colour edges between vertex classes as in Figure 2 and edges inside classes arbitrarily to get the graph G. The largest monochromatic component has order $3q - 1 = \frac{n}{2} - 1$ and $\delta(G) = 5q - 2 = \frac{5}{6}n - 2$.

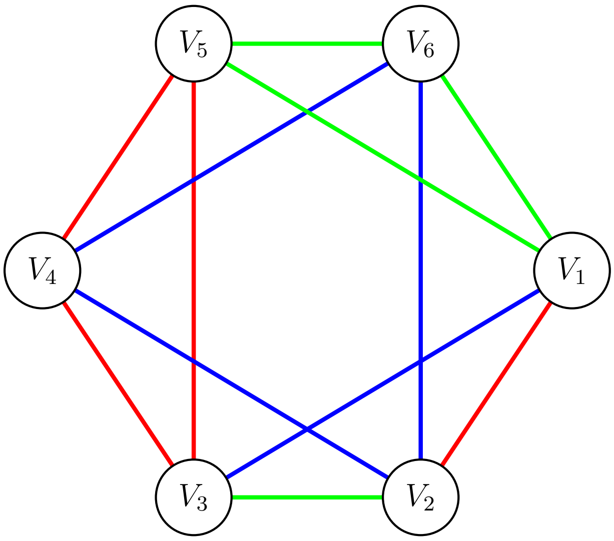

Figure 2: The graph G for $n = 6q$.

Case: $n = 6q + 1$. Partition the vertices into 8 sets.

- $|V_1| = |V_2| = 1$
- $|V_3| = q - 3$
- $|V_4| = q - 2$
- $|V_5| = |V_6| = |V_7| = |V_8| = q + 1$

We colour edges between the vertex classes the same as in Figure 1 and edges inside classes arbitrarily to get the graph G. (Note that the number of vertices in each set is different from the case where $n \equiv 4 \mod 6$.) The largest monochromatic component has order $3q < \frac{n}{2}$ and $\delta(G) = 5q - 1 = \left\lfloor \frac{5}{6}n \right\rfloor - 1$.

Case: $n = 6q + 2$. Partition the vertices into 8 sets.

- $|V_1| = |V_2| = 2$

- $|V_3| = q - 4$
- $|V_4| = q - 2$
- $|V_5| = |V_6| = |V_7| = |V_8| = q + 1$

We colour edges between the vertex classes the same as in Figure 1 and edges inside classes arbitrarily to get the graph G. We see that the largest monochromatic component has order $3q = \frac{n}{2} - 1$ and $\delta(G) = 5q = \lfloor \frac{5}{6}n \rfloor - 1$.

Case: $n = 6q + 3$. Partition the vertices into 6 sets.

- $|V_1| = |V_2| = |V_3| = q + 1$
- $|V_4| = |V_5| = |V_6| = q$

We colour edges between the vertex classes as in Figure 2 and edges inside classes arbitrarily to get the graph G. We see that the largest monochromatic component has order $3q + 1 < \frac{n}{2}$ and $\delta(G) = 5q + 1 = \lfloor \frac{5}{6}n \rfloor - 1$.

Case: $n = 6q + 5$. Partition the vertices into 9 sets.

- $|V_1| = |V_2| = |V_3| = 1$
- $|V_4| = |V_5| = |V_6| = |V_7| = q$
- $|V_8| = |V_9| = q + 1$

We colour edges between vertex classes as in Figure 3 and edges inside classes arbitrarily to get the graph G. The largest monochromatic component has order $3q + 2 < \frac{n}{2}$ and $\delta(G) = 5q + 2 = \lfloor \frac{5}{6}n \rfloor - 2$.

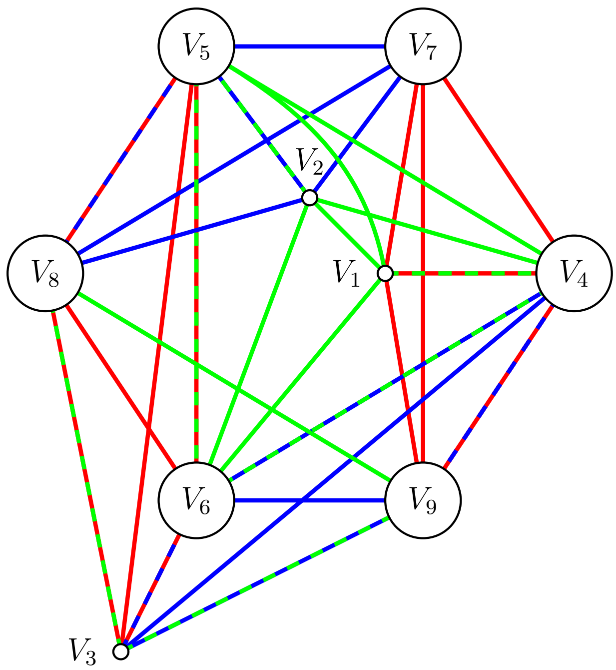

Figure 3: The graph G for $n \equiv 5 \mod 6$.

□

It is worth noting that, as $n \to \infty$, the graph shown in Figure 1 (corresponding to the cases where $n \equiv 1, 2, 4 \mod 6$) is close to the graph in Figure 4, which is one of the optimal cases found in Section 4. For the other residue classes, we similarly find that they are close to other optimal cases found in Section 4 as $n \to \infty$.

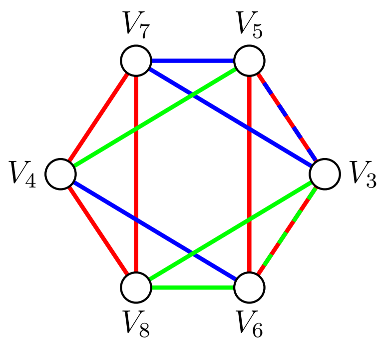

Figure 4: The limit graph of G for $n \equiv 1, 2, 4 \mod 6$.

The graphs given in the proof of Theorem 1.6 are by no means unique. For each residue class, we can find other graphs G' with $\delta(G') = \left\lfloor \frac{5}{6}n \right\rfloor - c$

for some small c that have no monochromatic component covering half of the vertices. Indeed, for each of the optimal graphs found in Section 4, it is possible to find some such G' which is close to it as $n \to \infty$.

It is also worth remarking that the constructions used in the proof of Theorem 1.6 are sharp in some cases. Suppose that there is a graph G on n vertices with no monochromatic component of order $\frac{n}{2}$ and $\delta(G) = \frac{5}{6}n - a$ for some a. Theorem 1.7 tells us that $a > 0$. If G' is a t-blow-up of G, then G' has no monochromatic component of order $\frac{tn}{2}$. By Lemma 3.2, we find

$$\begin{aligned} \delta(G') &= t\delta(G) + (t-1) \\ &= \frac{5}{6}nt - at + (t-1) \end{aligned}$$

and Theorem 1.7 tells us that $\delta(G') < \frac{5}{6}tn$. Combining these inequalities gives $a > 1 - \frac{1}{t}$ for every $t \in \mathbb{N}$ and so $a \geq 1$. Hence it follows that $\delta(G) \leq \left\lfloor \frac{5}{6}n - 1 \right\rfloor = \left\lfloor \frac{5}{6}n \right\rfloor - 1$. In the proof of Theorem 1.6, the graphs given for $n \equiv 1, 2, 3, 4 \mod 6$ each had minimum degree $\left\lfloor \frac{5}{6}n \right\rfloor - 1$.

3 Reducing the upper bound to special cases

To prove Theorem 1.7, we will reduce the problem to one which can be written as a series of linear programs. We solve these linear programs by computer to show that the theorem is true.

We begin by proving a series of lemmas. Throughout this section, we will assume that the 3 colours used to colour the edges are red, blue and green.

Lemma 3.1. *Let G be a graph of order n with $\delta(G) \geq \frac{5}{6}n$. Suppose that the edges of G are 3-coloured. If there exists a vertex v which is not incident to edges of all 3 colours, then there exists a monochromatic component of order at least $\frac{n}{2}$.*

Proof. Colour the edges of G red, blue and green. First consider the case where the vertex v is only incident to edges of one colour, say red. As $\delta(G) \geq \frac{5}{6}n$, the red component containing v covers at least $\frac{5}{6}n + 1 > \frac{n}{2}$ of the vertices of G.

Now consider the case where v is only incident to edges of two colours, say red and blue. Let $R \subseteq V(G)$ be the vertices of the red component containing v and $B \subseteq V(G)$ be the vertices of the blue component containing v. We may assume that $|R| < \frac{n}{2}$ and $|B| < \frac{n}{2}$.

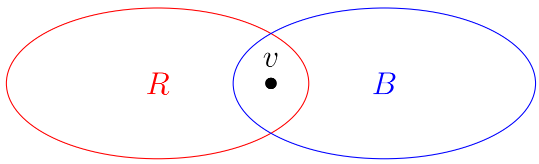

Figure 5: The vertices of the red and blue components containing v.

Let $x = |R \cap B|$, $r = |R \setminus B|$ and $b = |B \setminus R|$. Without loss of generality, we will assume that $r \geq b$. As $v \in R \cap B$ and $\delta(G) \geq \frac{5}{6}n$, we find that $x > 0$, $\frac{5}{12}n < r + x < \frac{n}{2}$ and $\frac{1}{3}n < b + x \leq r + x$.

Suppose that $x \leq \frac{1}{6}n$. As $r + x + b \geq \frac{5}{6}n$ and $r \geq b$, we must have $2r \geq \frac{5}{6}n - x$ giving $r > \frac{1}{3}n$. It follows from $b + x > \frac{1}{3}n$ that $b > \frac{1}{6}n$. Any edges that are incident to both $R \setminus B$ and $B \setminus R$ must be green. As $\delta(G) \geq \frac{5}{6}n$ and $r > \frac{1}{3}n$, every pair of vertices in $B \setminus R$ must have a green neighbour in common in $R \setminus B$. As $b > \frac{1}{6}n$, every vertex in $R \setminus B$ has a green neighbour in $B \setminus R$. Hence there is a green component covering all of $(R \cup B) \setminus (R \cap B)$. This green component has order at least $\frac{1}{6}n + \frac{1}{3}n \geq \frac{n}{2}$.

Now suppose that $x \geq \frac{1}{6}n$. As $r + x + b \geq \frac{5}{6}n$ and $r \geq b$, it follows that $r \geq \frac{5}{12}n - \frac{1}{2}x$ and hence $|R| = r + x \geq \frac{5}{12}n + \frac{1}{2}x \geq \frac{n}{2}$ giving a red component covering half of the vertices. $\square$

Given a graph G, the *t-blow-up* G' is the graph formed from G by replacing each vertex with a copy of K_t and each edge with a copy of $K_{t,t}$. If the edges of G have been coloured, then the edges of G' are coloured as follows:

- edges between two K_t are coloured according to the 3-edge-colouring on G
- edges within a K_t are coloured arbitrarily

The graph G' behaves like a larger version of G.

Lemma 3.2. *Fix $t > 1$. Let G be a graph of order n. Suppose the edges of G are 3-coloured so that every vertex is incident to all three colours and there is no monochromatic component covering half of the vertices. Let G' be the t-blow-up of G. Then G' has no monochromatic component covering half of its vertices and further $\delta(G') = t\delta(G) + t - 1$.*

Proof. Let $v_1, \dots, v_n$ be the vertices of G and let $V_1, \dots, V_n$ be the corresponding copies of K_t in G'. Fix a colour, say red. Since every vertex of G is incident with a red edge, each set V_i is contained in some red component of G' and furthermore V_i and V_j lie in the same red component if and only if v_i and v_j do. The remaining assertions are immediate. $\square$

Lemma 3.3. *Let G be a graph of order n with $\delta(G) \geq \frac{5}{6}n$. Suppose that the edges of G are 3-coloured and there are exactly 2 red components. Then there exists a monochromatic component of order at least $\frac{n}{2}$.*

Proof. By Lemma 3.1, we are done unless every vertex is incident to edges of all three colours. Therefore every vertex is in a red component. As there are only two red components, one of them must cover at least $\frac{n}{2}$ of the vertices. $\square$

Lemma 3.4. *Let G be a graph of order n where the edges are 3-coloured and there is no monochromatic component of order at least $\frac{n}{2}$. Suppose that G has r red, b blue and g green components and there exist red components R_1 and R_2 such that $|R_1| + |R_2| < \frac{n}{2}$. Then there is a graph G' together with a 3-edge-colouring such that there are $(r-1)$ red, b blue and g green components, $\frac{\delta(G')}{|G'|} \geq \frac{\delta(G)}{|G|}$ and there is no monochromatic component in G' covering at least half of the vertices.*

Proof. Note that, by Lemma 3.1, we may assume that every vertex is incident to all three colours.

Suppose first that there exists $v_1 \in R_1$ and $v_2 \in R_2$ such that $v_1v_2 \notin E(G)$. Let G' be a copy of G with the additional red edge v_1v_2. Then $\delta(G') \geq \delta(G)$ and all components in G' have the same number of vertices as they do in G with the exception of R_1 and R_2 which form a single component in G'. As $|R_1| + |R_2| < \frac{n}{2}$, G' contains no monochromatic component covering at least half of the vertices.

Now suppose that every vertex in R_1 is connected to every vertex in R_2. Fix $u \in R_1$ and $v \in R_2$ and, without loss of generality, assume that the edge uv is blue. Let G' be a 2-blow-up of G with same 3-edge-colouring. The vertex u in G corresponds to vertices u_1 and u_2 in G' and v corresponds to v_1 and v_2. Change the colour of the edge u_1v_1 from blue to red.

The red components corresponding to R_1 and R_2 in G' now form a single component of order $2(|R_1|+|R_2|) < n = \frac{1}{2}|G'|$. The vertices u_1 and v_1 still lie in the same blue component via the blue path $u_1v_2u_2v_1$ and so changing the colour of the edge u_1v_1 does not change the orders of the other components in G'. By Lemma 3.2, we have $\delta(G') = 2\delta(G) + 1$ and G' does not contain a monochromatic component of order at least $\frac{1}{2}|G'|$. As $|G'| = 2|G|$, it follows that $\frac{\delta(G')}{|G'|} > \frac{\delta(G)}{|G|}$. $\square$

Lemma 3.2 means we may make the following assumption: in each colour, G either has 3 components or 4 components each of order exactly $\frac{n}{4}$.

Lemma 3.5. *Let G be a graph of order n where the edges of G are 3-coloured. Suppose that G has no monochromatic component of order at least $\frac{n}{2}$. If, in* *two colours, there are* 4 *components of order exactly* $\frac{n}{4}$ *in that colour, then* $\delta(G) < \frac{5}{6}n$.

Proof. Without loss of generality, suppose G has 4 red components and 4 blue components, each with order exactly $\frac{n}{4}$. By Lemma 3.1, every vertex lies in components of all three colours. Lemma 3.3 tells us that there must be at least 3 green components. The smallest green component has order at most $\frac{n}{3}$. Choose a vertex v in the smallest green component. Then we find that $d(v) \leq (\frac{n}{3} - 1) + 2(\frac{n}{4} - 1) < \frac{5}{6}n$. □

The above lemmas allow us to make the following assumptions about G:

- Every vertex is incident to an edge of every colour (Lemma 3.1).
- There are either 3 components in each colour or there are 3 components in two colours and 4 components of order $\frac{n}{4}$ in the third colour (Lemmas 3.3, 3.4 and 3.5).

4 Linear programs for the upper bound

In Section 3, we reduced the proof of Theorem 1.7 to the following two cases:

1. Every vertex is incident to an edge of every colour. There are 3 components of each colour.
2. Every vertex is incident to an edge of every colour. There are 3 components in two of the colours (without loss of generality, red and blue) and 4 components of order exactly $\frac{n}{4}$ in the third colour (without loss of generality, green).

We now formulate these cases as collections of linear programs. More details about the code used to implement these linear programs may be found in Appendix A.

4.1 Three components in each colour

We begin by considering the first case where there are 3 components in each of the three colours. Let the vertex sets of the red components be R_i, the blue components be B_j and the green components be G_k for $i, j, k \in [3]$. We know that every vertex v of G lies in the intersection $R_i \cap B_j \cap G_k$ for some $i, j, k \in [3]$ and so $d(v) \leq |R_i \cup B_j \cup G_k| - 1$ with equality if v is adjacent to every vertex in $R_i \cup B_j \cup G_k$. Proving Theorem 1.7 for the first case is equivalent to showing that Question 4.1 has an answer of $\alpha < \frac{5}{6}$.

Question 4.1. *What is the maximum value of α such that the following conditions can hold simultaneously?*

1. $|R_i|, |B_j|, |G_k| < \frac{n}{2} \qquad \forall i, j, k \in [3]$
2. $d(v) \geq \alpha n \qquad \forall v \in V(G)$

The addition of any missing edges between v and $R_i \cup B_j \cup G_k$ will not change the number of vertices in each monochromatic component but may increase $\delta(G)$ and hence α. Therefore we may assume that $d(v) = |R_i \cup B_j \cup G_k| - 1$ for every vertex $v \in R_i \cap B_j \cap G_k$ (although it is important to note that $R_i \cap B_j \cap G_k$ may be empty).

We can avoid dependence on n by rescaling. Let $x_{ijk} = \frac{1}{n}|R_i \cap B_j \cap G_k|$ for each $i, j, k \in [3]$. For fixed $i \in [3]$, we find that $|R_i| = n \sum_{j=1}^{3} \sum_{k=1}^{3} x_{ijk}$ (and similarly for $|B_j|$ and $|G_k|$) and, for fixed $i, j, k \in [3]$, we have

$$\frac{1}{n}|R_i \cup B_j \cup G_k| = \sum_{\substack{i',\ j',\ k' \\ i'=i \text{ or } j'=j \text{ or } k'=k}} x_{i'j'k'}.$$

Using this notation and dividing through by n, the first condition in Question 4.1 becomes:

$$\sum_{j=1}^{3} \sum_{k=1}^{3} x_{ijk} < \frac{1}{2} \qquad \forall i \in [3]$$
$$\sum_{i=1}^{3} \sum_{k=1}^{3} x_{ijk} < \frac{1}{2} \qquad \forall j \in [3]$$
$$\sum_{i=1}^{3} \sum_{j=1}^{3} x_{ijk} < \frac{1}{2} \qquad \forall k \in [3]$$

In linear programs, the condition statements must consist of weak, rather than strict, inequalities in order to guarantee that the space of feasible solutions is closed and, if this space is non-empty, that an optimal solution exists. Relaxing the above conditions to allow equality will increase the space of feasible solutions. As the maximum value of α for our original problem will be at most the maximum value of α for the relaxed problem, showing that $\alpha < \frac{5}{6}$ for the relaxed problem is sufficient to prove Theorem 1.7.

The second condition holds whenever $R_i \cap B_j \cap G_k$ is non-empty. We obtain:

$$\begin{cases} \displaystyle\sum_{\substack{i',\ j',\ k' \\ i'=i \text{ or } j'=j \text{ or } k'=k}} x_{i'j'k'} \geq \alpha + \frac{1}{n} & \text{if } R_i \cap B_j \cap G_k \neq \emptyset \\ x_{ijk} = 0 & \text{otherwise.} \end{cases}$$

We would like to remove the dependence on n completely. We therefore relax the second condition by removing the $\frac{1}{n}$; we will obtain an upper bound of $\alpha \leq \frac{5}{6}$ which is sufficient to prove Theorem 1.7.

We encode whether $R_i \cap B_j \cap G_k$ is empty with an additional variable y_{ijk} by setting:

$$y_{ijk} = \begin{cases} 1 & \text{if } R_i \cap B_j \cap G_k \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

The variables y_{ijk} represent the pattern of intersections. For a fixed intersection pattern, an upper bound on α in Question 4.1 can be found by solving Linear Program 1.

$$\begin{aligned}
&\textbf{maximise} && \alpha \\
&\textbf{subject to} && \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} x_{ijk} = 1 \\
&&& \sum_{j=1}^{3}\sum_{k=1}^{3} x_{ijk} \leq \frac{1}{2} \qquad \forall i \in [3] \\
&&& \sum_{i=1}^{3}\sum_{k=1}^{3} x_{ijk} \leq \frac{1}{2} \qquad \forall j \in [3] \\
&&& \sum_{i=1}^{3}\sum_{j=1}^{3} x_{ijk} \leq \frac{1}{2} \qquad \forall k \in [3] \\
&&& \sum_{\substack{i',\ j',\ k' \\ i'=i \text{ or } j'=j \text{ or } k'=k}} x_{i'j'k'} - \alpha \geq 0 \quad \forall i,j,k \in [3] \text{ such that } y_{ijk} = 1 \\
&&& x_{ijk} = 0 \quad \forall i,j,k \in [3] \text{ such that } y_{ijk} = 0 \\
&&& x_{ijk} \geq 0 \quad \forall i,j,k \in [3]
\end{aligned}$$

Linear Program 1: 3 red, 3 blue and 3 green components.

We have already assumed that there are 3 components of each colour or else we would be done by Lemma 3.3. Therefore, we only need to consider intersection patterns that satisfy the following condition: for each $i \in [3]$, there exists $j, k \in [3]$ such that $y_{ijk} \neq 0$ (and similarly for j and k).

Using a computer, we ran the linear program for all valid intersection patterns (roughly 2^{27} linear programs were run) and found the maximum overall value of α and the optimal solutions corresponding to this value of α. The maximum value was $\alpha = \frac{5}{6}$.

There were five different optimal solutions and these are shown below in graph format in Figure 6. Each circle represents a set of vertices and circles of the same size contain the same number of vertices. The large circles contain twice as many vertices as the small circles. Where two circles are connected by striped lines, the edges between these vertex sets may be either of the two colours indicated.

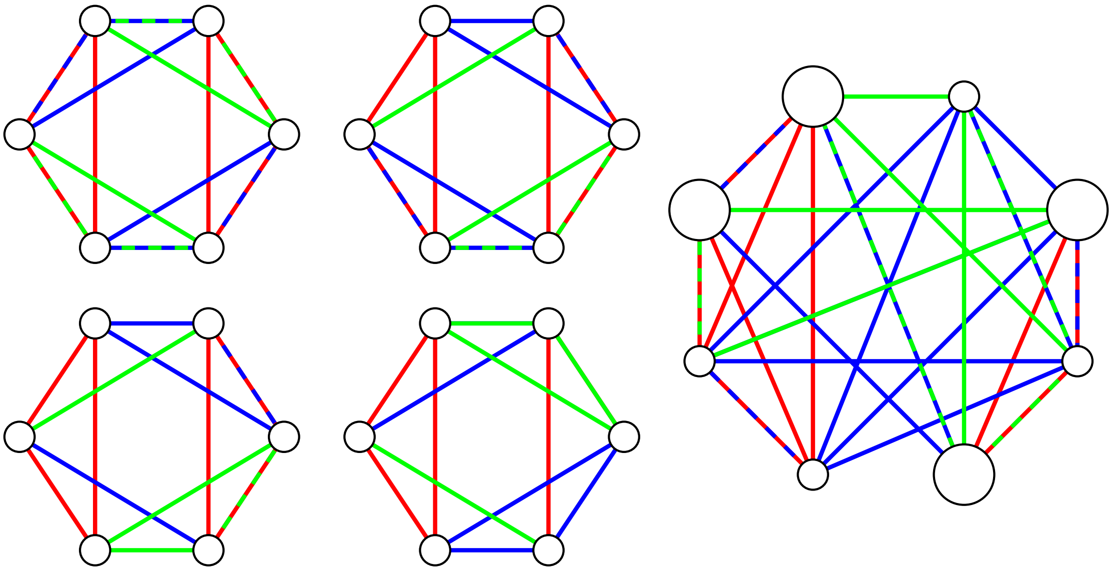

Figure 6: Optimal solutions of Linear Program 1.

4.2 Three components in two colours; four in the third

Now we consider the case where there are 3 red, 3 blue and 4 green components and each green component has order exactly $\frac{n}{4}$. The set-up is very similar to the case above. For a given intersection pattern y_{ijk}, the only difference in the linear program is the condition that each green component has size exactly $\frac{1}{4}$ (rather than just being at most $\frac{1}{2}$).

$$
\begin{array}{llll}
\textbf{maximise} & & \alpha & \\
\textbf{subject to} & & \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{4} x_{ijk} = 1 & \\
& & \displaystyle\sum_{j=1}^{3}\sum_{k=1}^{4} x_{ijk} \leq \frac{1}{2} & \forall i \in [3] \\
& & \displaystyle\sum_{i=1}^{3}\sum_{k=1}^{4} x_{ijk} \leq \frac{1}{2} & \forall j \in [3] \\
& & \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{3} x_{ijk} = \frac{1}{4} & \forall k \in [4] \\
& \displaystyle\sum_{\substack{i',\ j',\ k' \\ i'=i \text{ or } j'=j \text{ or } k'=k}} & x_{i'j'k'} - \alpha \geq 0 & \forall i,j \in [3], k \in [4] \text{ such that } y_{ijk} = 1 \\
& & x_{ijk} = 0 & \forall i,j \in [3], k \in [4] \text{ such that } y_{ijk} = 0 \\
& & x_{ijk} \geq 0 & \forall i,j \in [3], k \in [4]
\end{array}
$$

Linear Program 2: 3 red, 3 blue and 4 green components.

As above, we only considered intersection patterns where, for each $i \in [3]$, there exists $j \in [3], k \in [4]$ such that $y_{ijk} \neq 0$ (and similarly for j and k). Using a computer, we ran the linear program for all valid intersection patterns ($O(2^{30})$ linear programs were run) and found that the maximum overall value of α was $\frac{3}{4}$. As $\frac{3}{4}$ is strictly smaller than $\frac{5}{6}$, we may conclude that Theorem 1.7 holds in the case where there are four components in one colour and three components in the other two colours.

5 Conclusion

For any $n \in \mathbb{N}$ and $k \geq 3$, let $f_k(n)$ be the maximum value such that there exists a graph G on n vertices with $\delta(G) = f_k(n)$ and a k-edge-colouring of G where every monochromatic component has order strictly less than $\frac{n}{k-1}$.

In the proof of Theorem 1.6, we found that

$$f_3(n) = \left\lfloor \frac{5}{6}n \right\rfloor - 1 \qquad \text{if } n \equiv 1,2,3,4 \mod 6$$
$$f_3(n) \in \left\{ \left\lfloor \frac{5}{6}n \right\rfloor - 2, \left\lfloor \frac{5}{6}n \right\rfloor - 1 \right\} \qquad \text{if } n \equiv 0,5 \mod 6.$$

We believe that $f_3(n) = \lfloor \frac{5}{6}n \rfloor - 1$ for all residue classes but we were unable to find examples when $n \equiv 0, 5 \mod 6$.

Theorems 1.6 and 1.7 prove that Conjecture 1.3 is false for $k = 3$ and that the correct constant is $\frac{5}{6}$. It is natural to ask what the correct bound is for other values of k. Unfortunately, although our methods extend in principle to 4 or more colours, the computational time needed to run all of the required linear programs makes it infeasible to do so. For example, when $k = 4$, a naive implementation of our approach would entail solving around 2^{240} linear programs. It would be nice to resolve Conjecture 1.3 for more values of k.

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A Implementing the linear programs

The main obstacle in implementing Linear Programs 1 and 2 was the large number of possible intersection patterns (2^{27} and 2^{36} respectively) that needed to be checked. We therefore used the implicit symmetry of the problem to reduce the number of linear programs which needed to be run.

Recall that the intersection pattern is given by the variables (y_{ijk}) where

$$y_{ijk} = \begin{cases} 1 & \text{if } R_i \cap B_j \cap G_k \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

We may assume that there are at least 3 components of each colour (Lemma 3.3) and that each component intersects at least one component in each of the other colours (Lemma 3.1).

Given an intersection pattern (y_{ijk}), if there exists i such that $y_{ijk} = 0$ for all j and all k, then this corresponds to the red component R_i being empty (i.e there are at most two red components). It is therefore not necessary to run the linear program for this intersection pattern. (Indeed, the linear program has a value of $\alpha = \frac{7}{8}$ which is optimal if we do not specify that there must be at least three components of each colour.)

We therefore excluded intersection patterns which corresponded to one of the components being empty. As this check only requires knowledge of the current intersection pattern, it is straightforward to do it when the intersection pattern has been generated.

We also used the symmetry of the problem. There are two sources of symmetry: between components of the same colour and between components of different colours. Let us consider both.

Firstly, suppose we are given two intersection patterns, (y_{ijk}) and (y'_{ijk}). Suppose that, for some $t \in [2]$ and every j and k, we have $y_{ijk} = y'_{(i+t)jk}$ where $i + t$ is calculated modulo 3. Any optimal solution for (y_{ijk}) will also be an optimal solution for (y'_{ijk}) but with the red components relabelled. Therefore we only need to run the linear program for one of these intersection patterns to obtain the optimal solution for both. We can extend this idea to any intersection patterns which are the same up to relabelling of the components.

Secondly, if we have two intersection patterns (y_{ijk}) and (y'_{ijk}) such that, for every k, we have $y_{ijk} = y'_{jik}$, then any optimal solution for (y_{ijk}) will also be an optimal solution for (y'_{ijk}) but with the red and blue components swapped. Again we would only need to run the linear program for one of these intersection patterns to obtain the optimal solution for both. In the case where all three colours have exactly three components, all three colours are interchangeable; in the case where there are three red, three blue and four green components, only the red and blue components may be switched.

Unlike checking whether an intersection pattern corresponds to one of the components being empty, finding intersection patterns which are the same up to symmetry requires knowledge of both the current intersection pattern and other possible intersection patterns. Memory constraints make it impractical to generate all "non-symmetric" intersection patterns before running the linear programs. Instead, we consider only a subset of symmetries that we can handle efficiently using a version of lexicographic ordering.

First, for simplicity, suppose that we only have two colours, red and blue, and that the intersection matrix is given by $Z = (z_{ij})$ where z_{ij} represents whether or not $R_i \cap B_j$ is empty. Swapping two rows in Z corresponds to swapping the labels of two red components and similarly for columns and blue components. We define the *lex value* of Z to be

$$\text{lex}(Z) = \sum_{i=0}^{2} \sum_{j=0}^{2} z_{ij} 100^{-i-j}.$$

Swapping pairs of rows and/or pairs of columns in Z can change its lex value and configurations with more 1 entries in the top left corner of the matrix will give higher lex values.

By only swapping pairs of rows or pairs of columns that strictly increase the value of $\text{lex}(Z)$, we will eventually reach Z', a configuration of Z where the lex value is $\text{maxlex}(Z)$, the unique maximum possible lex value of Z. Both Z and Z' are possible intersection patterns and, because we obtained Z' from Z through a series of row and column swaps, Z and Z' are "symmetric" intersection patterns.

We only run a linear program on Z if $\text{lex}(Z) = \text{maxlex}(Z)$. This significantly reduces the number of linear programs that need to be run whilst still ensuring that at least one linear program is run for each class of "symmetric" intersection patterns.

Now consider the situation we actually have where the intersection pattern is given by $Y = (y_{ijk})$. Whilst we could extend the definition of lex value to a three-dimensional matrix, it proved cumbersome to calculate the maximum lex. Instead, we calculated $\text{lex}(Y^{(k)})$ where $Y^{(k)}$ is the 3×3 matrix obtained by restricting to a fixed value of k. We ran the linear program on Y if $\text{lex}(Y^{(k)}) \geq \text{lex}(Y^{(k+1)})$ for every k and $\text{lex}(Y^{(1)}) = \text{maxlex}(Y^{(1)})$. This method eliminated sufficiently many intersection patterns to make the computation tractable.