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Monochromatic cycles in 2-coloured graphs

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Li, Nikiforov and Schelp [12] conjectured that any 2-edge coloured graph G with order n and minimum degree $\delta(G) > 3n/4$ contains a monochromatic cycle of length ℓ, for all $\ell \in [4, \lceil n/2 \rceil]$. We prove this conjecture for sufficiently large n and also find all 2-edge coloured graphs with $\delta(G) = 3n/4$ that do not contain all such cycles. Finally, we show that, for all $\delta > 0$ and $n > n_0(\delta)$, if G is a 2-edge coloured graph of order n with $\delta(G) \geq 3n/4$, then one colour class either contains a monochromatic cycle of length at least $(2/3 + \delta/2)\, n$, or contains monochromatic cycles of all lengths $\ell \in [3, (2/3 - \delta)\, n]$.

1. Introduction

A well-known theorem of Dirac [8] states that a graph with order $n \geq 3$ and minimum degree at least $n/2$ contains a cycle C_n on n vertices.

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Theorem 1.1 (Dirac [8]). *Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is hamiltonian.* □

In fact, as noted by Bondy [5], an immediate corollary of the following theorem is that such a graph will contain cycles of all lengths $\ell \in [3, n]$. We call such a graph *pancyclic*.

Theorem 1.2 (Bondy [5]). *If G is a hamiltonian graph of order n such that $|E(G)| \geq n^2/4$, then either G is pancyclic or n is even and G is isomorphic to $K_{n/2,n/2}$.* □

Corollary 1.3. *Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then either G is pancyclic or n is even and G is isomorphic to $K_{n/2,n/2}$. In particular, if $\delta(G) > n/2$, then G is pancyclic.* □

Given a graph G with edge set $E(G)$, a 2-*edge colouring* of G is a partition $E(G) = E(R) \cup E(B)$, where R and B are spanning subgraphs of G. We define a k-*edge colouring* of G similarly. In a recent paper [12], Li, Nikiforov and Schelp made the following conjecture, which would give an analogue of Corollary 1.3 for 2-edge coloured graphs.

Conjecture 1.4 (Li, Nikiforov and Schelp [12]). *Let $n \geq 4$ and let G be a graph of order n with $\delta(G) > 3n/4$. If $E(G) = E(R) \cup E(B)$ is a 2-edge colouring of G, then for each $\ell \in [4, \lceil n/2 \rceil]$, either $C_\ell \subseteq R$ or $C_\ell \subseteq B$.*

Note that we may only ask for ℓ in this range. For example, take the 2-colouring of K_5 consisting of a red and a blue C_5 and blow it up, that is, replace each vertex of K_5 with an independent set of p vertices and each edge with the complete monochromatic bipartite graph $K_{p,p}$ of the same colour. The resulting graph G has minimum degree $\delta(G) = 4|G|/5$ but no monochromatic C_3. Similarly letting R be the complete bipartite graph with vertex classes of order $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, and letting B be the complement, we obtain a 2-colouring of the complete graph K_n with no monochromatic odd cycle C_ℓ with $\ell > \lceil n/2 \rceil$. Li, Nikiforov and Schelp [12] proved the following partial result.

Theorem 1.5 (Li, Nikiforov and Schelp [12]). *Let $\epsilon > 0$, let G be a graph of sufficiently large order n, with $\delta(G) > 3n/4$. If $E(G) = E(R) \cup E(B)$ is a 2-edge colouring of G, then for all $\ell \in [4, \lfloor (1/8 - \epsilon)\, n \rfloor]$, either $C_\ell \subseteq R$ or $C_\ell \subseteq B$.* □

We will prove Conjecture 1.4 for sufficiently large n, but first we will define a set of 2-edge coloured graphs showing that the degree bound $3n/4$ is tight.

Definition. Let $n = 4p$ and let G be a graph isomorphic to $K_{p,p,p,p}$. A 2-*bipartite 2-edge colouring* of G is a 2-edge colouring $E(G) = E(R) \cup E(B)$ such that both R and B are bipartite.

If $G \cong K_{p,p,p,p}$ and $E(G) = E(R) \cup E(B)$ is a 2-bipartite 2-edge colouring of G, let $V_1 \cup V_2$ be the bipartition of R and $W_1 \cup W_2$ be the bipartition of B. Let $U_{i,j} = V_i \cap W_j$

for all $i, j \in \{1, 2\}$. Then the $U_{i,j}$ are four independent sets of G covering all vertices, and so must be the four independent sets of order p. So, a 2-bipartite 2-edge colouring of $K_{p,p,p,p}$ forces a labelling of the independent sets $\{U_{i,j} : i, j \in \{1, 2\}\}$ such that:

- all edges between $U_{1,1}$ and $U_{1,2}$ and between $U_{2,1}$ and $U_{2,2}$ are blue;
- all edges between $U_{1,1}$ and $U_{2,1}$ and between $U_{1,2}$ and $U_{2,2}$ are red;
- edges between $U_{1,1}$ and $U_{2,2}$ and between $U_{2,1}$ and $U_{1,2}$ can be either colour.

If 4 divides n, then the graph $K_{n/4,n/4,n/4,n/4}$ with a 2-bipartite 2-edge colouring has minimal degree $3n/4$ and no monochromatic odd cycles. Note that for a fixed labelling of this graph, there are $2^{n^2/8}$ 2-bipartite 2-edge colourings of $K_{n/4,n/4,n/4,n/4}$. However, $K_{n/4,n/4,n/4,n/4}$ has $24\left((n/4)!\right)^4 = 2^{O(n \log n)}$ automorphisms and so there are $2^{n^2/8+O(n \log n)}$ distinct 2-bipartite 2-edge colourings of $K_{n/4,n/4,n/4,n/4}$. In fact we will prove that $K_{n/4,n/4,n/4,n/4}$ is the only extremal graph; although any 2-bipartite 2-edge colouring of $K_{n/4,n/4,n/4,n/4}$ is extremal. We state our main result now.

Theorem 1.6. *There exists a positive integer n_0 with the following property. Let G be a graph of order $n > n_0$ with $\delta(G) \geq 3n/4$. Suppose that $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring of G. Then either $C_\ell \subseteq R$ or $C_\ell \subseteq B$ for all $\ell \in [4, \lceil n/2 \rceil]$, or $n = 4p$, $G \cong K_{p,p,p,p}$ and the colouring is a 2-bipartite 2-edge colouring.*

We define the *monochromatic circumference* of a k-edge coloured graph to be the length of its longest monochromatic cycle. Li, Nikiforov and Schelp [12] also posed the following question.

Question 1.7. *Let $0 < c < 1$ and n be sufficiently large integer. If G is a 2-coloured graph of order n with $\delta(G) > cn$, what is the minimum possible monochromatic circumference of G ?*

For graphs G with $\delta(G) \geq 3n/4$ we show that the monochromatic circumference is at least $(2/3 + o(1))\, n$. In fact, we show the following result.

Theorem 1.8. *For every $0 < \delta \leq 1/180$, there exists an integer $n_0 = n_0(\delta)$ such that the following holds. Let G be a graph of order $n > n_0$ with $\delta(G) \geq 3n/4$. Suppose that $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring of G. Then either G has monochromatic circumference at least $(2/3 + \delta/2)\, n$, or one of R_G and B_G contains cycles of all lengths $\ell \in [3, (2/3 - \delta)\, n]$.*

Note that the last statement requires monochromatic cycles of all lengths in some prescribed set of integers, as in Theorem 1.6. However, here these cycles are required to be of the *same* colour. Also, the upper bound on δ is of a technical nature, and we are only interested in small δ. There are similar technical bounds throughout this paper.

For integers $s \leq t$, we define the following 2-edge coloured graph $F_{s,t}$, which with $t = 2s$ shows that Theorem 1.8 is asymptotically sharp.

Definition. For $s \leq t$, let $F_{s,t}$ be the 2-edge coloured complete graph on $s+t$ vertices in which the blue edges form the complete bipartite graph $K_{s,t}$ and all the other edges are red.

The longest cycle in the red subgraph of $F_{s,t}$ has length t. The blue subgraph is bipartite and has circumference $2s$. Thus the monochromatic circumference of $F_{s,t}$ is $\max\{t, 2s\}$.

Let $n = 3s$. Then $|F_{s,2s}| = n$, $\delta(F_{s,2s}) = n-1$, and $F_{s,2s}$ has monochromatic circumference $2s = 2n/3$. Hence Theorem 1.8 is asymptotically sharp.

We shall show that the linear dependence between the two occurrences of δ in Theorem 1.8 is correct. Fix $\delta > 0$. Suppose that $n > 1/2\delta$, and let $G \cong F_{n-\lceil(2/3-\delta)n\rceil, \lceil(2/3-\delta)n\rceil}$. Then the monochromatic circumference of G is $2\lfloor(1/3+\delta)\,n\rfloor \geq (2/3+\delta)\,n$. However, G contains no monochromatic cycle of odd length $\ell \in \{\lceil(2/3-\delta)\,n\rceil+1, \lceil(2/3-\delta)\,n\rceil+2\}$.

In Section 2, we will introduce some theorems that will be used in our proofs. We will then prove Theorem 1.6 in two parts. Section 3 will deal with short (up to constant length) cycles and Section 4 will deal with long cycles. This will rely on a number of lemmas, whose proofs are postponed to Sections 5 and 6. In Section 7 we will look at the length of the longest monochromatic cycle, and in particular prove Theorem 1.8. We conclude in Section 8 with some open problems.

2. Results used in the proof

In the proof of Theorem 1.6, we shall use well-known extremal graph theory results and the regularity method to find long cycles. Before introducing these, we make some preliminary definitions and notation.

For a graph G, we denote by $e(G)$ its number of edges. Let X and Y be disjoint subsets of $V(G)$. We denote by $G[X]$ the subgraph of G induced by the vertices of X. We also denote by $E(X,Y)$ the set of edges joining X and Y, set $e(X,Y) := |E(X,Y)|$, and let $G[X,Y]$ be the bipartite subgraph of G with partite sets X and Y and edge set $E(X,Y)$. For a set of vertices S, we denote by $\Gamma_G(S)$ the set of all vertices adjacent to some vertex in S. We drop the subscript when there is no danger of confusion. We also write $\Gamma_G(v)$ instead of $\Gamma_G(\{v\})$, and set $d_G(v) := |\Gamma_G(v)|$.

Definition. Let G be a graph and X and Y be disjoint subsets of $V(G)$. The *density* of the pair (X,Y) is the value

$$d(X,Y) := \frac{e(X,Y)}{|X||Y|}.$$

We define a regular pair to be one where the density between not-too-small subsets of X and Y is close to the density between X and Y.

Definition. Let $\epsilon > 0$. Let G be a graph and X and Y be disjoint subsets of $V(G)$. We call (X,Y) an *ϵ-regular pair for G* if, for all $X' \subseteq X$ and $Y' \subseteq Y$ satisfying $|X'| \geq \epsilon|X|$ and $|Y'| \geq \epsilon|Y|$, we have

$$|d(X,Y) - d(X',Y')| < \epsilon.$$

The next fact shows that almost all vertices in any regular pair have large degrees.

Fact 2.1. *Let G be a graph and let (V_1, V_2) be an ϵ-regular pair for G with density $d := d(V_1, V_2)$. Then all but at most $\epsilon|V_1|$ vertices $v \in V_1$ satisfy $|\Gamma(v) \cap V_2| \geq (d-\epsilon)|V_2|$.* $\square$

This is particularly useful when we want to find paths of prescribed length in a regular pair, as shown by the next lemma from [1, Lemma 10].

Lemma 2.2. *For every $0 < \beta < 1$ there is an $m_0(\beta)$ such that for every $m > m_0(\beta)$ the following holds: Let G be a graph, and let V_1, V_2 be disjoint subsets of $V(G)$ such that $|V_1|, |V_2| \geq m$. Furthermore, suppose that, for some ϵ satisfying $0 < \epsilon < \beta/100$, the pair (V_1, V_2) is ϵ-regular for G, with density at least $\beta/4$.*

Then, for every pair of vertices $v_1 \in V_1$, $v_2 \in V_2$ satisfying $|\Gamma(v_1) \cap V_2|$, $|\Gamma(v_2) \cap V_1| \geq \beta m/5$, and for every i, $1 \leq i \leq m - 5\epsilon m/\beta$, G contains a v_1-v_2 path of length $2i + 1$. $\square$

Combining Lemma 2.2 with Fact 2.1 yields the following straightforward consequence.

Corollary 2.3. *For every $0 < \epsilon < 10^{-5}$ there is an $m_0(\epsilon)$ such that for every $m > m_0(\epsilon)$ the following holds: Let G be a graph, and let $V_1, V_2, \ldots, V_\ell$ be disjoint subsets of $V(G)$ such that $|V_1|, |V_2|, \ldots, |V_\ell| \geq m$. Suppose that all pairs $(V_1, V_2), (V_2, V_3), \ldots, (V_{\ell-1}, V_\ell)$, (V_ℓ, V_1) are ϵ-regular for G, with density at least $\sqrt{\epsilon}$.*

Then, $C_i \subset G$ for every i, $\ell \leq i \leq (1 - 5\sqrt{\epsilon})(\ell - 1)m$, of the same parity as ℓ. $\square$

We will use the following 2-coloured version of the Szemerédi Regularity Lemma (see, for example, the survey paper of Komlós and Simonovits [11]) that is not hard to deduce from the standard form of the regularity lemma [13].

Theorem 2.4 (Degree form of 2-coloured Regularity Lemma). *For every $\epsilon > 0$ and positive integer k_0, there is an $M = M(\epsilon, k_0)$ such that if $G = (V, E)$ is any 2-edge coloured graph and $d \in [0, 1]$ is any real number, then there is $k_0 \leq k \leq M$, a partition $(V_i)_{i=0}^k$ of the vertex set V and a subgraph $G' \subseteq G$ with the following properties:*

(R1) $|V_0| \leq \epsilon|V|$,

(R2) all clusters V_i, $i \in [k] := \{1, 2, \ldots, k\}$, are of the same size $m \leq \lceil \epsilon|V| \rceil$,

(R3) $d_{G'}(v) > d_G(v) - (2d + \epsilon)|V|$ *for all $v \in V$,*

(R4) $e(G'[V_i]) = 0$ *for all $i \in [k]$,*

(R5) for all $1 \leq i < j \leq k$, the pair (V_i, V_j) is ϵ-regular for $R_{G'}$ with a density either 0 or greater than d and ϵ-regular for $B_{G'}$ with a density either 0 or greater than d, where $E(G') = E(R_{G'}) \cup E(B_{G'})$ is the induced 2-edge colouring of G'. $\square$

Having applied the above form of the Regularity Lemma to a 2-edge coloured graph G, we make the following definition, based on the clusters $\{V_i : 1 \leq i \leq k\}$. Note that this definition depends on the parameters ϵ and d.

Definition. Given a graph $G = (V, E)$ and a partition $(V_i)_{i=0}^k$ of V satisfying conditions

(R1)–(R5) above, we define the (ϵ, d)-*reduced* 2-*edge coloured graph* H on vertex set $\{v_i : 1 \le i \le k\}$ as follows: For $1 \le i < j \le k$,

- let $\{v_i, v_j\}$ be a blue edge of H when $B_{G'}[V_i, V_j]$ has density at least d;
- let $\{v_i, v_j\}$ be a red edge of H when it is not a blue edge and $R_{G'}[V_i, V_j]$ has density at least d.

Remark. Notice that the definition of the (ϵ, d)-reduced 2-edge coloured graph H is non-symmetric in the following sense: On one hand, if $\{v_i, v_j\}$ is a red edge of H, then we know that $R_{G'}[V_i, V_j]$ has density at least d and $B_{G'}[V_i, V_j]$ has density less than d. On the other hand, if $\{v_i, v_j\}$ is a blue edge of H, then we know that $R_{B'}[V_i, V_j]$ has density at least d, but we have no information about the density of $R_{G'}[V_i, V_j]$.

This asymmetry will never cause a problem in our arguments because we only use the facts that a red edge $\{v_i, v_j\}$ in H corresponds to $R_{G'}[V_i, V_j]$ with density at least d and a blue edge $\{v_i, v_j\}$ in H corresponds to $B_{G'}[V_i, V_j]$ with density at least d.

We aim to use matchings in the reduced graph H to find long cycles in G. To do so we will use the following lemma.

Lemma 2.5. *Given $0 < c < 1$ and $0 < \delta < 1 - c$, let $\epsilon > 0$ and $d > 0$ be sufficiently small real numbers such that $d \ge \epsilon^{1/3}$. Let G be a 2-edge coloured graph of sufficiently large order n with a vertex partition $(V_i)_{i=0}^k$ satisfying conditions (R1)–(R5), and let H be the corresponding (ϵ, d)-reduced 2-edge coloured graph.*

Suppose that H has a monochromatic component C that contains a matching on $(c+\delta)k$ vertices. Then

(a) G contains monochromatic cycles of length ℓ for all even $4k \le \ell \le (c + \delta/2)n$.
(b) If C also contains any odd cycle, then G contains monochromatic cycles of length ℓ for all $4k \le \ell \le (c + \delta/2)n$.

Furthermore, all cycles in (a) and (b) have the same colour as C.

Lemma 2.5 has by now a very standard proof using Lemma 2.2, which we omit here. The interested reader can easily modify the proof given in [1, Section 5, p.696].

In the proof of Theorem 1.6, we shall frequently show that there is a subset S of V on which one of R_G or B_G is hamiltonian, and apply Theorem 1.2. To prove hamiltonicity, it will normally be sufficient to use Dirac's Theorem (Theorem 1.1). However, we will also need the following generalisation and one of its consequences.

Theorem 2.6 (Chvátal [7]). *Let G be a graph of order $n \ge 3$ with degree sequence $d_1 \le d_2 \le \cdots \le d_n$ such that*

$$d_k \le k < \frac{n}{2} \Rightarrow d_{n-k} \ge n - k.$$

Then G contains a Hamilton cycle. □

Corollary 2.7. *Let G be a graph of order $n \geq 3$ with minimum degree $\delta(G) \geq n/2+2$. Then, for every two vertices u, v, there is a u-v path containing all vertices of G.* $\square$

In Section 5, we will need to use the following defect version of Tutte's 1-factor Theorem [14], due to Berge [2]. Here $q(G)$ denotes the number of components of a graph G of odd order.

Theorem 2.8 (Berge [2]). *A graph G contains a set of independent edges covering all but at most d of the vertices if, and only if,*

$$q(G-S) \leq |S| + d$$

for all $S \subseteq V$. $\square$

The next result of Bollobás [3, p.150] will be used in Section 3, as will be the three following results. Note that by the length of a path or a cycle we mean the number of its edges.

Theorem 2.9 (Bollobás [3]). *If G is a graph of order n, with $e(G) > n^2/4$, then G contains C_k for all $k \in [3, \lceil n/2 \rceil]$.* $\square$

Theorem 2.10 (Bondy and Simonovits [6]). *Let G be a graph of order n and let k be an integer. If $e(G) > 100kn^{1+1/k}$, then G contains a cycle of length $2k$.* $\square$

Theorem 2.11 (Erdős and Gallai [9]). *If G is a graph with order n and circumference at most L, then $e(G) \leq (n-1)L/2$. If G is a graph with order n and with no paths of length at least $L+1$, then $e(G) \leq nL/2$.* $\square$

Theorem 2.12 (Györi, Nikiforov and Schelp [10]). *Let k, m be positive integers. There exist $n_0 = n_0(k,m)$ and $c = c(k,m) > 0$ such that for every nonbipartite graph G on $n > n_0$ vertices with minimum degree*

$$\delta(G) > \frac{n}{2(2k+1)} + c,$$

if $C_{2s+1} \subseteq G$, for some $k \leq s \leq 4k+1$, then $C_{2s+2j+1} \subseteq G$ for every $j \in [m]$. $\square$

3. Existence of short cycles

In this section we shall prove that unless we are in the extremal case, we have monochromatic cycles of all lengths $\ell \in [4, K]$ for a given integer K. To prove this we shall use the following claim.

Claim 3.1. *Let L be an integer. Let n be sufficiently large and let G be a graph of order n with $\delta(G) \geq 3n/4$. Suppose that $E(G) = E(R) \cup E(B)$ is a 2-edge colouring of G. If there is a monochromatic C_3 or C_5, then one of R, B contains C_ℓ for all odd $\ell \in [5, 2L+1]$.*

The proof of Claim 3.1 follows exactly the method used in [12] to show the existence of short odd cycles. Note that we cannot appeal directly to Theorem 1.5, as the assumption there is that $\delta(G) > 3n/4$, whereas in Theorem 1.6 we assume only that $\delta(G) \geq 3n/4$.

Proof. Suppose first that $\Delta(B) > n/2 + 4L$. Let v be a vertex with $d_B(v) = \Delta(B)$, and $U = \Gamma_B(v)$. If $B[U]$ contains a path of length $2L$, then using the vertex v, there is a blue C_ℓ for all $\ell \in [3, 2L+1]$. Hence $B[U]$ does not contain a path of length $2L$, and by Theorem 2.11, we have $e(B[U]) \leq L|U|$. However, any vertex $u \in U$ has at most $n/4 - 1$ non-neighbours in U and so at least $|U| - n/4$ neighbours. Therefore,

$$\begin{aligned} e(G[U]) &= \frac{1}{2}\sum_{u\in U} d_{G[U]}(u) \\ &\geq \frac{1}{2}|U|\left(|U| - \frac{n}{4}\right) \\ &> \frac{1}{2}|U|\left(\frac{|U|}{2} + 2L\right). \end{aligned}$$

Hence $e(R[U]) = e(G[U]) - e(B[U]) > |U|^2/4$, and, consequently, by Theorem 2.9, $R[U]$ has cycles of all lengths from 3 to $|U|/2 \geq 2L+1$.

So we may assume that $\Delta(B) \leq n/2 + 4L$, and hence

$$\delta(R) \geq \frac{n}{4} - 4L > \frac{n}{6} + c(1, L),$$

where $c(1, L)$ is the constant from Theorem 2.12. Similarly we may assume that $\delta(B) > n/6 + c(1, L)$. Suppose that there is a monochromatic C_3 or C_5 and assume without loss of generality that it is red. Applying Theorem 2.12 to R with $k = 1$ and $m = L$, there is a red C_ℓ for all odd $\ell \in [5, 2L+1]$ as required. □

Now we prove the main result of this section.

Lemma 3.2. *Let K be an integer. Let n be sufficiently large and let G be a graph of order n with $\delta(G) \geq 3n/4$. If $E(G) = E(R) \cup E(B)$ is a 2-edge colouring of G, then either $C_\ell \subseteq R$ or $C_\ell \subseteq B$ for all $\ell \in [4, K]$, or $n = 4p$, $G \cong K_{p,p,p,p}$ and the colouring of G is a 2-bipartite 2-edge colouring.*

Proof. Note that the existence of monochromatic C_ℓ for all even $\ell \in [4, K]$ is immediate from Theorem 2.10 because one colour class has at least $e(G)/2 \geq 3n^2/16$ edges. Hence, by Claim 3.1, it is sufficient to prove that either there is a monochromatic C_3 or C_5, or $n = 4p$, $G \cong K_{p,p,p,p}$ and the colouring is a 2-bipartite 2-edge colouring.

Suppose that, in fact, none of these occur. Any 2-edge colouring of K_5 contains a monochromatic C_3 or C_5. Hence we may assume that $K_5 \nsubseteq G$. Since $e(G) \geq 3n^2/8 = t_4(n)$, by Turán's Theorem, we must therefore have that $G \cong T_4(n)$. However, $\delta(G) \geq 3n/4$ implies that $n = 4p$, and hence $G \cong K_{p,p,p,p}$. Let U_i, $1 \leq i \leq 4$, be the independent sets of G of order p.

We may assume that R is not bipartite. Let $C = v_1 v_2 \dots v_r$ be a shortest odd cycle of

R; we may assume that $r \geq 7$. We may properly 4-vertex colour C by setting $c(v_i) = j$ when $v_i \in U_j$. As C is an odd cycle, there must be three consecutive vertices with different colours under c. Without loss of generality, assume that $c(v_3) = 1$, $c(v_4) = 2$ and $c(v_5) = 3$.

We will aim to show that $G[V(C)]$ contains a triangle or 5-cycle which is edge-disjoint from C. Then we may assume that an edge of the triangle or 5-cycle is red, else we have a monochromatic C_3 or C_5. But this red edge, together with C, will create a shorter red odd cycle than C, contradicting our assumption that C was minimal. We shall find such a triangle or 5-cycle by case analysis.

If $c(v_1)$ is 2 or 4, then G contains the triangle $v_1v_3v_5$, as these vertices lie in different U_j. Hence $c(v_1) \in \{1, 3\}$, and similarly $c(v_7) \in \{1, 3\}$.

If $c(v_6) = 4$, then G contains the triangle $v_1v_4v_6$. So we may assume that $c(v_6) \neq 4$ and similarly $c(v_2) \neq 4$. Hence $c(v_2) \in \{2, 3\}$ and $c(v_6) \in \{1, 2\}$. If $c(v_2) = 3$ and $c(v_6) = 1$, then G contains the triangle $v_2v_4v_6$. Hence, by symmetry, we may assume that $c(v_2) = 2$ and $c(v_6) \in \{1, 2\}$.

If $c(v_7) = 1$, then G contains the triangle $v_2v_5v_7$. Hence $c(v_7) = 3$.

If $|C| = 7$, then $v_1v_7 \in E(C)$, and $c(v_1) = 1$ because c is a proper colouring of G. But then $v_1v_5v_2v_7v_4$ is a 5-cycle in G, not containing any edges of C.

So we may assume that $|C| > 7$. If $c(v_1) = 1$, then $v_1v_4v_7$ is a triangle in G which is edge-disjoint from C. Hence $c(v_1) = c(v_7) = 3$. But now, if $c(v_6) = 1$, then G contains the triangle $v_1v_4v_6$, while if $c(v_6) = 2$, then G contains the triangle $v_1v_3v_6$, giving a contradiction.

Hence, in fact, our assumption was false, and one of the cases of the lemma holds. □

4. Existence of long cycles

In order to find long monochromatic cycles, we will use the regularity method. Recall from Section 2 that having applied the Regularity Lemma to the graph G on n vertices, we define a reduced graph H on k vertices. Note that the Regularity Lemma implies that the minimum degree of the reduced graph H is not too much smaller than $\frac{k}{n}$ times the minimum degree of G.

At this point we use the following lemma, proved in Section 5 using extremal arguments, that shows that either there is a monochromatic component of H containing a large matching, or the reduced graph H has one of two particular forms.

Lemma 4.1. *Let $0 < \delta < 1/36$ and let G be a graph of sufficiently large order n with $\delta(G) \geq (3/4 - \delta)\, n$. Suppose that we are given a 2-edge colouring $E(G) = E(R) \cup E(B)$. Then one of the following holds.*

(i) There is a component of R or B which contains a matching on at least $(2/3 + \delta)\, n$ vertices.

(ii) There is a set S of order at least $(2/3 - \delta/2)\, n$ such that either $\Delta(R[S]) \leq 10\delta n$ or $\Delta(B[S]) \leq 10\delta n$.

(iii) There is a partition $V(G) = U_1 \cup \cdots \cup U_4$ with $\min_i |U_i| \geq (1/4 - 3\delta)\, n$ such that there are no red edges from $U_1 \cup U_2$ to $U_3 \cup U_4$ and no blue edges from $U_1 \cup U_3$ to $U_2 \cup U_4$.

In the first case, we will need the following lemma, which is also proved in Section 5.

Lemma 4.2. *Let $0 < \delta < 1/36$ and let G be a graph of sufficiently large order n with $\delta(G) \geq (3/4 - \delta)\, n$. Suppose that we are given a 2-edge colouring $E(G) = E(R) \cup E(B)$. Suppose that there is a monochromatic component containing a matching on at least $(2/3 + \delta)\, n$ vertices. Then there is a monochromatic component C containing a matching on at least $(1/2 + \delta)\, n$ vertices such that either C contains an odd cycle, or $|V(C)| \geq (1 - 5\delta)\, n$.*

To finish our argument in this case we shall use Lemma 2.5 and the next lemma.

Lemma 4.3. *Let n be sufficiently large, and let G be a graph of order n with $\delta(G) \geq 3n/4$. Suppose that $E(G) = E(R) \cup E(B)$ is a 2-edge colouring of G.*

(a) If B has an independent set S with $|S| > n/2$, then $C_\ell \subseteq R$ for all $\ell \in [3, |S|]$.

(b) If B is bipartite, then either $C_\ell \subseteq R$ for all $\ell \in [4, \lceil n/2 \rceil]$, or n is divisible by four, $G \cong K_{n/4,n/4,n/4,n/4}$ and the colouring is a 2-bipartite 2-edge colouring.

Both statements remain valid when we interchange B and R.

By analysing the original graph G, we will use the following two lemmas to show that, in cases (ii) and (iii) of Lemma 4.1, we will have the desired monochromatic cycles.

Lemma 4.4. *Given $0 < \delta < 1/144$, let ϵ and d be real numbers such that $0 < \epsilon \ll d \ll \delta$, where, as usual, $\ll$ means 'sufficiently smaller than'. Let G be a graph of sufficiently large order n with $\delta(G) \geq 3n/4$. Suppose that $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring of G, $(V_i)_{i=0}^k$ is a partition of $V(G)$ satisfying conditions (R1)–(R5), and let H be the corresponding (ϵ, d)-reduced 2-edge coloured graph.*

(a) If there is a set $S \subseteq V(H)$ of order at least $(2/3 - \delta/2)\, k$ such that $\Delta(R_H[S]) \leq 10\delta k$, then G contains a blue cycle of length ℓ for all $\ell \in [3, (2/3 - \delta)\, n]$.

(b) If there is a set $S \subseteq V(H)$ of order at least $(2/3 - \delta/2)\, k$ such that $\Delta(B_H[S]) \leq 10\delta k$, then G contains a red cycle of length ℓ for all $\ell \in [3, (2/3 - \delta)\, n]$.

Lemma 4.5. *Given $0 < \delta < 1/144$, let ϵ and d be real numbers such that $0 < \epsilon \ll d \ll \delta$. Let G be a graph of sufficiently large order n with $\delta(G) \geq 3n/4$. Suppose that $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring of G, $(V_i)_{i=0}^k$ is a partition of $V(G)$ satisfying conditions (R1)–(R5), and let H be the corresponding (ϵ, d)-reduced 2-edge coloured graph.*

Suppose that there is a partition $V(H) = U_1 \cup \cdots \cup U_4$ with $\min_i |U_i| \geq (1/4 - 3\delta)\, k$ such that there are no red edges from $U_1 \cup U_2$ to $U_3 \cup U_4$ and no blue edges from $U_1 \cup U_3$ to $U_2 \cup U_4$. Then G contains a monochromatic cycle of length at least $(1 - 38\delta)\, n$ and monochromatic cycles of length ℓ for all $\ell \in [4, \lceil n/2 \rceil]$.

Lemmas 4.3–4.5 will be proved in Section 6. We now prove Theorem 1.6 as outlined above.

4.1. Proof of Theorem 1.6

Choose $0 < \delta < 1/144$ and $0 < \epsilon \ll d \ll \delta$. In particular, we may assume that $d \geq \epsilon^{1/3}$. Choose $k_0 > 1/\epsilon^2$ so that one can apply Lemmas 4.1 and 4.2 on any graph with at least k_0 vertices. From Theorem 2.4, we obtain $M = M(\epsilon, k_0)$ and set $K = 6M$. We take n sufficiently large and let G be a graph of order n with $\delta(G) \geq 3n/4$, with 2-edge colouring $E(G) = E(R_G) \cup E(B_G)$.

By Lemma 3.2, we have that either $G \cong K_{n/4,n/4,n/4,n/4}$ and the colouring is a 2-bipartite 2-edge colouring, or G contains a monochromatic C_ℓ for all $\ell \in [4, K]$. Hence it is sufficient to prove that either $G \cong K_{n/4,n/4,n/4,n/4}$ and the colouring is a 2-bipartite 2-edge colouring, or G contains a monochromatic C_ℓ for all $\ell \in [K, \lceil n/2 \rceil]$.

We apply the degree form of the 2-colour Regularity Lemma to G, with parameters d and ϵ. Let $V_0, V_1, \ldots, V_k$ be the partition of $V(G)$ satisfying conditions (R1)–(R5), and G' be the subgraph of G defined by Theorem 2.4. Finally, let H be the (ϵ, d)-reduced graph defined from G' earlier, with 2-edge colouring $E(H) = E(R_H) \cup E(B_H)$.

We first observe that

$$\delta(H) \geq \left(\frac{3}{4} - \delta\right) k. \tag{4.1}$$

Indeed, by (R3), we have $\delta(G') \geq (3/4 - 2d - \epsilon)\, n$. Suppose that $\delta(H) < (3/4 - \delta)\, k$. Then there exists some $i \geq 1$ with $d_H(v_i) < (3/4 - \delta)\, k$. For a vertex $v \in V_i$, its neighbours in G' are only in V_0, or in V_j for those j such that $v_i v_j$ is an edge of H. Hence

$$d_{G'}(v) < \left(\frac{3}{4} - \delta\right) km + |V_0| \leq \left(\frac{3}{4} - \delta + \epsilon\right) n,$$

which is a contradiction, as $\delta \gg 2d + 2\epsilon$.

Applying Lemma 4.1 to H, we have one of the following possibilities.

(i) There is a component of R_H or B_H which contains a matching on at least $(2/3 + \delta)\, k$ vertices.

(ii) There is a set S of order at least $(2/3 - \delta/2)\, k$ such that either $\Delta(R_H[S]) \leq 10\delta k$ or $\Delta(B_H[S]) \leq 10\delta k$.

(iii) There is a partition $V(H) = U_1 \cup \cdots \cup U_4$ with $\min_i |U_i| \geq (1/4 - 3\delta)\, k$ such that there are no blue edges from $U_1 \cup U_2$ to $U_3 \cup U_4$ and no red edges from $U_1 \cup U_3$ to $U_2 \cup U_4$.

If (ii) or (iii) occurs, then we are done immediately by Lemma 4.4 and Lemma 4.5 respectively. Hence we assume that there is a component of R_H or B_H which contains a matching on at least $(2/3 + \delta)\, k$ vertices. By Lemma 4.2, we may assume that there is a monochromatic component C which contains a matching on at least $(1/2 + \delta)\, k$ vertices, and that either C contains an odd cycle or $|V(C)| \geq (1 - 5\delta)\, k$.

Assume first that

$$C = R'_H \text{ is a component of } R_H. \tag{4.2}$$

If the component R'_H of R_H contains an odd cycle, then we are done because, by Lemma 2.5(b), G contains red cycles of any length between $4k$ and $(1/2 + \delta/2)n$ and $4k < K$.

Suppose now that R'_H contains no odd cycles and hence $|V(R'_H)| \geq (1 - 5\delta)\, k$. Then R'_H is bipartite, with classes H_1 and H_2. Applying Lemma 2.5(a), we deduce that $C_\ell \subseteq$

R_G for all even $\ell \in [4, (1/2 + \delta/2)\, n]$. Hence we are done if we can show that G contains a monochromatic C_ℓ for all odd $\ell \in [K, \lceil n/2 \rceil]$. We start with the following claim.

Claim 4.6. *There are disjoint sets of vertices X and Y with $|X \cup Y| \geq (1 - 6\delta)n$ with the following properties.*

(a) Between any vertices $u, v \in X$ (or $u, v \in Y$) there are, in $R_G[X \cup Y]$, paths of length ℓ for all even $\ell \in [3k, (1/2 + \delta/2)\, n]$.

(b) Between any $u \in X$ and any $v \in Y$ there are, in $R_G[X \cup Y]$, paths of length ℓ for all odd $\ell \in [3k - 1, (1/2 + \delta/2)\, n]$.

Proof. Take a matching M in R'_H with a maximal number of vertices. Let $r \geq (1/2 + \delta)k/2$ be the number of edges in the matching. Since R'_H is connected, there exists a spanning tree T that contains all edges of M.

For every vertex $v_i \in V(R'_H)$, fix an edge $D_i = v_i v_{\ell_i}$ in T. Let V'_i be the set of all vertices in V_i with at least $(d - \epsilon)m$ red neighbours in V_{ℓ_i}. By Fact 2.1, we have that $|V'_i| \geq (1 - \epsilon)|V_i|$.

Let X be the union of all V'_i such that $v_i \in H_1$, and Y be the union of all V'_i such that $v_i \in H_2$, so that X and Y are subsets of $V(G)$. Notice that

$$|X \cup Y| \geq |H_1|(1 - \epsilon)m + |H_2|(1 - \epsilon)m \geq (1 - 5\delta)k(1 - \epsilon)^2 \frac{n}{k} > (1 - 6\delta)n.$$

Now we show that (a) holds. Let $u, v \in X$ and suppose that $u \in V'_i$ and $v \in V'_j$. We claim there is a walk in R'_H from v_i to v_j, starting with edge D_i, ending with edge D_j, containing all edges of M, and whose number of edges is even and at most $3k$.

Indeed, one starts at v_i, continues to v_{ℓ_i} (using D_i), then walks around T using each edge of T twice (including those in M) until coming back to v_{ℓ_i}, then continues to v_{ℓ_j} via the unique v_{ℓ_i}-v_{ℓ_j} path in T, and finally ends in v_j (using D_j). This walk has at most $1 + 2(k - 1) + (k - 1) + 1 < 3k$ edges. Furthermore, the number of edges in this walk is even, because R'_H is bipartite and both endpoints of the walk are from the same partition class.

For convenience, denote the vertices of this walk $v_{t_1}, v_{t_2}, \ldots, v_{t_\ell}$, where $v_{t_1} = v_i$, $v_{t_2} = v_{\ell_i}$, $v_{t_{\ell-1}} = v_{\ell_j}$, and $v_{t_\ell} = v_j$. Using Fact 2.1 repeatedly, we construct a path $P = w_1, w_2 \ldots, w_\ell$ in R_G such that $w_1 = u$, $w_\ell = v$, and, for $1 < j < \ell$, $w_j \in V_{t_j}$ has at least $(d - \epsilon)m$ red neighbours in both $V_{t_{j-1}}$ and $V_{t_{j+1}}$.

Consider the edge $w_{t_j} w_{t_{j+1}}$. The bipartite graph

$$R_j = R[(V_{t_j} \setminus P) \cup \{w_{t_j}\}, (V_{t_{j+1}} \setminus P) \cup \{w_{t_{j+1}}\}]$$

is (2ϵ)-regular with density at least $d - \epsilon > \epsilon^{1/3}/2$, and w_{t_j}, $w_{t_{j+1}}$ have both large degree in R_j. Using Lemma 2.2 with $\beta = \epsilon^{1/3}$, we obtain that R_j contains a w_{t_j}-$w_{t_{j+1}}$ path of any odd length between 3 and $2(1 - 10\epsilon^{2/3})(m - 3k)$.

Hence, by replacing edge $w_{t_j} w_{t_{j+1}}$ in P by a w_{t_j}-$w_{t_{j+1}}$ path of appropriate odd length in B_j for one occurrence $v_{t_j} v_{t_{j+1}}$ of each edge in M, we obtain a red u-v path of any even length between $3k$ and $2(1 - 10\epsilon^{2/3})(m - 3k)r$. Using $\epsilon \ll d \ll \delta$, a short calculation shows that $2(1 - 10\epsilon^{2/3})(m - 3k)r > (1/2 + \delta/2)n$.

The proof of (b) is straightforward: if $u \in X$ and $v \in Y$, take any neighbour $w \in X$ of v and apply part (a) to u, w. □

If either X or Y contains an internal red edge uv, then, by Claim 4.6(a), we have red cycles of all odd lengths between $3k+1$ and $(1/2+\delta/2)\,n$. Hence we may assume that X and Y are independent sets in R_G. However, if $|X| > n/2$ or $|Y| > n/2$, then we are done by Lemma 4.3. Hence we may assume that $|X|, |Y| \leq n/2$ and, consequently, $\min\{|X|, |Y|\} \geq (1/2 - 6\delta)\,n$.

If any vertex w of $V(G) \setminus (X \cup Y)$ has at least one red neighbour in both X and Y, then using the paths between a red neighbour u of w in X and a red neighbour v of w in Y (see Claim 4.6(b)), we have cycles of all odd lengths between $3k+1$ and $(1/2+\delta/2)\,n$. Hence all vertices of $V(G) \setminus (X \cup Y)$ have no red neighbours in at least one of X or Y.

Define disjoint sets X' and Y' by letting X' be the set of vertices of $V(G) \setminus (X \cup Y)$ with at least two red neighbours in Y, and Y' be the set of vertices of $V(G) \setminus (X \cup Y)$ with at least two red neighbours in X. Then there are no red edges between X' and X or between Y' and Y. If there is a red edge $u'v'$ within X', let u and v be distinct vertices of Y with uu' and vv' both red edges. Then $uu'v'v$ is a red path of length three between vertices of Y, with internal vertices in $V \setminus (X \cup Y)$. Using the u-v paths obtained from Claim 4.6(a), we have red cycles of all odd lengths between $3k+3$ and $(1/2+\delta/2)\,n$.

So we may assume that $R_G[X \cup X' \cup Y \cup Y']$ is bipartite with partite sets $X \cup X'$ and $Y \cup Y'$. We may assume that both $X \cup X'$ and $Y \cup Y'$ have order at most $n/2$, otherwise, we are done by Lemma 4.3.

A vertex not in $X \cup X' \cup Y \cup Y'$ has at least $(3/4 - 6\delta)\,n$ neighbours in $X \cup Y$. Let X'' be the set of vertices not in $X \cup X' \cup Y \cup Y'$ with at least $(3/8 - 3\delta)\,n$ neighbours in X, and Y'' be the set of vertices not in $X \cup X' \cup Y \cup Y'$ with at least $(3/8 - 3\delta)\,n$ neighbours in Y. Letting $X_0 = X \cup X' \cup X''$ and $Y_0 = Y \cup Y' \cup Y''$, we see that $V(G)$ is the (not necessarily disjoint) union of X_0 and Y_0.

Without loss of generality, we may assume that $|X_0| \geq n/2$. By definition, all vertices in X'' have at least $(3/8 - 3\delta)\,n$ neighbours in X. However vertices in X'' have at most one red neighbour in X, else they would have been in Y'. All vertices in $X \cup X'$ have at most $n/4$ non-neighbours in G and so at least $|X_0| - n/4$ neighbours in X_0. Since there at most $|X''|$ red edges between X'' and X, the set X''' of vertices in X with a red neighbour in X'' has order at most $|X''|$. Vertices in $X \setminus X'''$ have no red neighbours in X_0, while vertices in $X' \cup X'''$ have no red neighbours in $X_0 \setminus X''$. Hence

$$d_{B[X_0]}(v) \geq \begin{cases} |X_0| - \frac{n}{4} & v \in X \setminus X''' \\ |X_0| - \frac{n}{4} - |X''| & v \in X''' \cup X' \\ \left(\frac{3}{8} - 3\delta\right) n - 1 & v \in X''. \end{cases} \tag{4.3}$$

Since $|X' \cup X'' \cup X'''| \leq |X' \cup X''| + |X''| \leq 12\delta n$, the conditions of Theorem 2.6 are satisfied on the graph $B_G[X_0]$, and so $B_G[X_0]$ is hamiltonian.

Furthermore, using (4.3), we have

$$\begin{aligned}
e\left(B_G[X_0]\right) &\geq \frac{1}{2}\left(|X_0| - \frac{n}{4}\right)|X \setminus X'''| + \left(\left(\frac{3}{16} - \frac{3\delta}{2}\right)n - \frac{1}{2}\right)|X''| \\
&\quad + \frac{1}{2}\left(|X_0| - \frac{n}{4} - |X''|\right)(|X'''| + |X'|) \\
&= \frac{1}{2}|X_0|\left(|X_0| - \frac{n}{4}\right) \\
&\quad + \left(\left(\frac{3}{16} - \frac{3\delta}{2}\right)n - \frac{1}{2} - \frac{1}{2}\left(|X_0| - \frac{n}{4}\right) - \frac{1}{2}(|X'''| + |X'|)\right)|X''| \\
&\geq \frac{1}{4}|X_0|^2 + \left(\left(\frac{1}{16} - \frac{15\delta}{2}\right)n - \frac{1}{2}\right)|X''|.
\end{aligned}$$

Here we have used

$$\begin{aligned}
\frac{1}{2}\left(|X_0| - \frac{n}{4}\right) + \frac{1}{2}(|X'''| + |X'|) &\leq \frac{1}{2}\left(|X| - \frac{n}{4}\right) + |X'| + |X''| \\
&\leq \frac{n}{8} + |V(G) \setminus (X \cup Y)| \\
&\leq \left(\frac{1}{8} + 6\delta\right)n.
\end{aligned}$$

From Theorem 1.2 we deduce that either $B_G[X_0]$ is pancyclic, in which case $C_\ell \subseteq B_G$ for all $\ell \in [3, |X_0|]$, or $B_G[X_0] \cong K_{|X_0|/2, |X_0|/2}$ and $e(B_G[X_0]) = |X_0|^2/4$. In the latter case, this means that $X'' = \emptyset$. Similarly, if $|Y_0| \geq n/2$, then either $B_G[Y_0]$ is pancyclic, or $Y'' = \emptyset$. Hence we may assume that $X'' = Y'' = \emptyset$ and, therefore, B_G is bipartite. Thus, by Lemma 4.3, we are done.

Now we only need to discuss what changes if, instead of (4.2), we assume that the component C is a blue component B'_H of B_H.

If the component B'_H contains an odd cycle, then we are again done by Lemma 2.5(b). Otherwise, B'_H is bipartite and $|V(B'_H)| \geq (1 - 5\delta)\, k$. Applying Lemma 2.5(a), we again deduce that $C_\ell \subseteq B_G$ for all even $\ell \in [4, (1/2 + \delta/2)\, n]$. By following the proof of Claim 4.6 with colours red and blue swapped, we realize that the following holds.

Claim 4.7. *There are disjoint sets of vertices X and Y with $|X \cup Y| \geq (1 - 6\delta)n$ with the following properties.*

(a) Between any vertices $u, v \in X$ (or $u, v \in Y$) there are, in $B_G[X \cup Y]$, paths of length ℓ for all even $\ell \in [3k, (1/2 + \delta/2)\, n]$.

(b) Between any $u \in X$ and any $v \in Y$ there are, in $B_G[X \cup Y]$, paths of length ℓ for all odd $\ell \in [3k - 1, (1/2 + \delta/2)\, n]$. □

Notice that the asymmetry in the definition of the colouring of H is not an issue here: In the proof of Claim 4.6, we only use the fact that the red edges in H correspond to ϵ-regular pairs in G' whose density of red edges is at least d. Hence, in the proof of Claim 4.7, we only need the fact that the blue edges in H correspond to ϵ-regular pairs in G' whose density of blue edges is at least d.

Having Claim 4.7, we follow the lines of the proof after Claim 4.6 with colours red and blue interchanged. This completes the proof of Theorem 1.6.

5. Proof of Lemmas 4.1 and 4.2

In this section and in Section 6 we shall prove the lemmas used in the proof of Theorem 1.6. More precisely, we give the proofs of Lemma 4.1 and Lemma 4.2 here and leave the proofs of Lemmas 4.3–4.5 to the next section.

Throughout both proofs we shall assume that for a given 2-edge colouring $E(G) = E(R) \cup E(B)$ of a graph G, R' is a largest component of R, and that B' is a largest component of B.

We will need the following claim about the component structure.

Lemma 5.1. *Let $0 < \delta < 1/36$ and let G be a graph of sufficiently large order n with $\delta(G) \geq (3/4 - \delta)\, n$. Suppose that we are given a 2-edge colouring $E(G) = E(R) \cup E(B)$.*

Then one of the following holds.

(a) One of R or B is connected.

(b) For a largest component R' of R, and a largest component B' of B, we have that $V(G) = V(R') \cup V(B')$ and both R' and B' have order at least $(3/4 - \delta)\, n$.

(c) There is a partition $V(G) = U_1 \cup \cdots \cup U_4$ with $\min_i |U_i| \geq (1/4 - 3\delta)\, n$ such that there are no red edges from $U_1 \cup U_2$ to $U_3 \cup U_4$ and no blue edges from $U_1 \cup U_3$ to $U_2 \cup U_4$.

Proof. Assume that (a) does not hold, that is, both R and B are disconnected. Suppose first that $|V(R')| \leq (5/12 - \delta)\, n$. Then $\Delta(R) < (5/12 - \delta)\, n$ and hence $\delta(B) \geq \delta(G) - \Delta(R) > n/3$. Since B is disconnected, we see that B has exactly two components $B_1 = B'$ and B_2 with $n/3 < |V(B_2)| \leq |V(B_1)| < 2n/3$. Set $W_i = V(B_i) \cap V(R')$, for $i \in \{1, 2\}$.

Suppose that $W_i \neq \emptyset$, for some $i \in \{1, 2\}$. Let $v \in W_i$. Then v has no neighbours outside $R' \cup B_i$, and so $\Gamma_G(v) \subseteq R' \cup B_i$. Hence, as $W_{3-i} = V(R') \setminus V(B_i)$,

$$|W_{3-i}| \geq |\Gamma_G(v)| - |V(B_i)| > \left(\frac{1}{12} - \delta\right) n.$$

In particular $W_{3-i} \neq \emptyset$, and so W_1 is non-empty if and only if W_2 is non-empty. As $V(R') = W_1 \cup W_2$, we see that both W_1 and W_2 are therefore non-empty. But then

$$\begin{aligned} |V(R')| = |W_1| + |W_2| &\geq \left(\frac{3}{4} - \delta\right) n - |V(B_1)| + \left(\frac{3}{4} - \delta\right) n - |V(B_2)| \\ &= \left(\frac{1}{2} - 2\delta\right) n. \end{aligned}$$

This contradicts our assumption that $|V(R')| \leq (5/12 - \delta)\, n$.

We may thus assume that $|V(R')| > (5/12 - \delta)\, n$, and similarly $|V(B')| > (5/12 - \delta)\, n$. Let $W_1 = V(B') \cap V(R')$, $W_2 = V(R') \setminus V(B')$, $W_3 = V(B') \setminus V(R')$ and $W_4 = V(G) - (W_1 \cup W_2 \cup W_3)$. Note that there are no edges (of either colour) between W_1 and W_4 or between W_2 and W_3.

If $W_4 = \emptyset$, then $V(G) = V(R') \cup V(B')$. As neither R nor B is connected, we must have that both W_2 and W_3 are non-empty. Let $v \in W_2$. Since $\Gamma_G(v) \cap W_3 = \emptyset$, we see that $|W_3| \le (1/4+\delta)\,n$ and so $|V(R')| = |W_1| + |W_2| = n - |W_3| \ge (3/4-\delta)\,n$. We may similarly show that $|V(B')| \ge (3/4-\delta)\,n$. Hence, (b) holds.

If, however, $W_4 \neq \emptyset$, choose any $x \in W_4$. As $\Gamma_G(x) \cap W_1 = \emptyset$, we have $|W_1| \le (1/4+\delta)\,n$. However both R' and B' have order at least $(5/12-\delta)\,n$ and hence both W_2 and W_3 are non-empty. Thus, arguing as for W_4, we see that both W_2 and W_3 have order at most $(1/4+\delta)\,n$ and so W_1 is also non-empty. This in turn implies that W_4 has order at most $(1/4+\delta)\,n$. Hence each W_i has order at least $(1/4-3\delta)\,n$. Furthermore, there is no red edge between $W_1 \cup W_2 = V(R')$ and $W_3 \cup W_4 = V(G) \setminus V(R')$ and no blue edge between $W_1 \cup W_3 = V(B')$ and $W_2 \cup W_4 = V(G) \setminus V(B')$. Hence, (c) holds. □

5.1. Proof of Lemma 4.1

Let $0 < \delta < 1/36$. We assume throughout that n is sufficiently large. Suppose that G is a graph of order n with $\delta(G) \ge (3/4-\delta)\,n$ and that we are given a 2-edge colouring $E(G) = E(R) \cup E(B)$ such that the conclusions (i) and (iii) do not hold. Since (i) is not true, we assume that

$$\text{neither } R' \text{ nor } B' \text{ contains a matching on at least } \left(\tfrac{2}{3}+\delta\right) n \text{ vertices.} \tag{5.1}$$

We are aiming to show that (ii) holds, that is, that there is a large set on which one of the colours has a very low density. We first show that the orders of B' and R' cannot take certain values.

Claim 5.2. *Either* $|V(B')| < (1/3+\delta/2)\,n$ *or* $|V(B')| > (2/3-\delta/2)\,n$.

Proof. Suppose that $(1/3+\delta/2)\,n \le |V(B')| \le (2/3-\delta/2)\,n$. We apply Lemma 5.1: (c) cannot be true because (iii) does not hold, (b) fails because $|V(B')| < (3/4-\delta)n$, and so (a) must be true. Since B is disconnected, we conclude that R is connected.

Let V_1 be the smaller of $V(B')$ and $V \setminus V(B')$. Let $V_2 = V \setminus V_1$ and $F = G[V_1, V_2]$ be the bipartite graph between V_1 and V_2. There are no blue edges between V_1 and V_2 and so all edges of F are red. For a subset S of V_1 we shall find a lower bound on $|\Gamma_F(S)|$ by splitting into the cases that $|S| > (1/4+\delta)\,n$ and $|S| \le (1/4+\delta)\,n$.

For $S \subseteq V_1$ with $|S| > (1/4+\delta)\,n$, consider a vertex $v \in V_2$. Then, as $d_G(v) \ge (3/4-\delta)\,n$, v must have a neighbour in S. Hence $\Gamma_F(S) = V_2$, and so $|\Gamma_F(S)| = |V_2| \ge |V_1| \ge |S|$.

If $S \subseteq V_1$ and $|S| \le (1/4+\delta)\,n$, then every vertex in S has at least $|V_2| - (1/4+\delta)\,n$ neighbours in V_2. Hence

$$\begin{aligned} |\Gamma_F(S)| &\ge |V_2| - \left(\frac{1}{4}+\delta\right) n \\ &= |S| - \left(\left(\frac{1}{4}+\delta\right) n + |S| - |V_2|\right) \\ &\ge |S| - \left(\left(\frac{1}{2}+2\delta\right) n - |V_2|\right). \end{aligned}$$

Thus by the defect form of Hall's Theorem, F contains a matching with at least

$$|V_1| - \max\left\{0, \left(\frac{1}{2} + 2\delta\right) n - |V_2|\right\} = \min\left\{|V_1|, |V_1| + |V_2| - \left(\frac{1}{2} + 2\delta\right) n\right\}$$

edges. As $|V_1| + |V_2| = n$ and $|V_1| \geq (1/3 + \delta/2)\, n$, this matching contains at least $(2/3 + \delta)\, n$ vertices. As R is connected, this contradicts (5.1). Hence Claim 5.2 holds. $\square$

Let $X_R = V(G) \setminus V(R')$ and $X_B = V(G) \setminus V(B')$. We define the following sets when R' or B' is large.

Definition. Suppose that $|V(R')| \geq (2/3 + \delta)\, n$.

- Let $S_R \subseteq V(R')$ be a set such that
$$q\left(R\left[V(R') - S_R\right]\right) > |S_R| + |V(R')| - \left(\frac{2}{3} + \delta\right) n.$$
Note that, in view of (5.1), such a set exists by Theorem 2.8 applied with $d = |V(R')| - (2/3 + \delta)\, n$.
- For $1 \leq i \leq n$, let $T_{R,i}$ be the set of vertices which lie in components of $R\left[V(R') - S_R\right]$ of order i.
- Let $T_R = \bigcup_{1 \leq i \leq t} T_{R,i}$, where $t = \lceil \delta^{-1} \rceil$.

If $|V(B')| \geq (2/3 + \delta)\, n$, we define S_B, $T_{B,i}$ and T_B similarly.

We shall use the following result throughout. Note that, as with Claim 5.2, we may exchange the roles of R and B to obtain a symmetrical version of this result.

Claim 5.3. *Suppose that $|V(R')| \geq (2/3 + \delta)\, n$. Then $|S_R| < (1/3 + \delta/2)\, n$ and*

$$|X_R \cup T_R| > |S_R| + \left(\frac{1}{3} - 2\delta\right) n.$$

Further, if C_B is a component of B with $|V(C_B)| \leq (5/12 - 2\delta)\, n$, then $V(C_B) \subseteq S_R$.

Proof. All vertices of $V(R')$ lie in S_R or some component of $R\left[V(R') - S_R\right]$. Hence

$$\begin{aligned} |V(R')| &\geq |S_R| + q\left(R\left[V(R') - S_R\right]\right) \\ &> 2|S_R| + |V(R')| - \left(\frac{2}{3} + \delta\right) n. \end{aligned}$$

This implies that $|S_R| < (1/3 + \delta/2)\, n$.

There are at most $|T_R|$ components of $R\left[V(R') - S_R\right]$ of order at most t. However, there are at most $n/t \leq \delta n$ components of $R\left[V(R') - S_R\right]$ of order at least t. Hence

$|T_R| \geq q\,(R\,[V(R') - S_R]) - \delta n$. As X_R and T_R are disjoint, we have

$$\begin{aligned}
|X_R \cup T_R| &\geq n - |V(R')| + q\,(R\,[V(R') - S_R]) - \delta n \\
&> (1-\delta)\,n - |V(R')| + |S_R| + |V(R')| - \left(\frac{2}{3} + \delta\right) n \\
&= |S_R| + \left(\frac{1}{3} - 2\delta\right) n.
\end{aligned}$$

Finally, suppose that C_B is a component of B with $|V(C_B)| \leq (5/12 - 2\delta)\,n$. A vertex in C_B has blue degree at most $|V(C_B)| - 1$. Hence any vertex in C_B must have red degree at least

$$\begin{aligned}
\delta(G) - |V(C_B)| + 1 &\geq \left(\frac{3}{4} - \delta\right) n - \left(\frac{5}{12} - 2\delta\right) n + 1 \\
&= \left(\frac{1}{3} + \delta\right) n + 1. \qquad (5.2)
\end{aligned}$$

A vertex in X_R has red degree at most

$$|X_R| - 1 = n - |V(R')| - 1 \leq \left(\frac{1}{3} - \delta\right) n - 1.$$

However, a vertex in T_R is in a component of $R\,[V(R') - S_R]$ of order at most t. Hence, for all $v \in T_R$,

$$\begin{aligned}
d_R(v) &\leq t + |S_R| \\
&< \left(\frac{1}{3} + \frac{1}{2}\delta\right) n + t.
\end{aligned}$$

Hence (5.2) and $n \gg 1/\delta$ imply that $V(C_B) \cap (X_R \cup T_R) = \emptyset$.

Suppose that there exists $v \in V(C_B) \setminus S_R$. All blue neighbours of v lie in C_B, and so v has no blue neighbours in $X_R \cup T_R$. However, $V(C_B) \subseteq V(R')$ because $V(C_B) \cap X_R = \emptyset$, and so v has no red neighbours in X_R. The only vertices with red neighbours in T_R are those in $S_R \cup T_R$, and so we see that v also has no red neighbours in T_R. Hence v has no neighbours in $X_R \cup T_R$, and so

$$d_G(v) \leq n - |X_R \cup T_R| < \left(\frac{2}{3} + 2\delta\right) n.$$

This contradicts $\delta(G) \geq (3/4 - \delta)\,n$, and so $V(C_B) \subseteq S_R$. □

We may thus assume that S_R is not much bigger than $n/3$. The following result shows that if R' is very large and S_R has order approaching $n/3$, then $S = V(G) \setminus S_R$ is the set we are looking for in (ii).

Claim 5.4. *Suppose that $|V(R')| \geq (1 - 5\delta/2)\,n$ and that $|S_R| \geq (1/3 - 2\delta)\,n$. Then $R[V(G) \setminus S_R]$ is a graph on at least $(2/3 - \delta/2)\,n$ vertices with maximum degree at most $10\delta n$.*

Proof. That $R[V(G)\backslash S_R]$ is a graph of order at least $(2/3-\delta/2)\,n$ follows immediately from Claim 5.3.

For all $1 \le i \le n$, there are exactly $|T_{R,i}|/i$ components of order i in $R\,[V(R')\setminus S_R]$. Hence

$$\begin{aligned}\sum_{i\ge 1}\frac{1}{2i-1}|T_{R,2i-1}| &= q\left(R\,[V(R')\setminus S_R]\right)\\ &> |S_R| + |V(R')| - \left(\frac{2}{3}+\delta\right)n.\end{aligned}$$

However,

$$\begin{aligned}\sum_{i\ge 1}\frac{1}{2i-1}|T_{R,2i-1}| &\le |T_{R,1}| + \frac{1}{3}\sum_{i\ge 2}|T_{R,2i-1}|\\ &\le |T_{R,1}| + \frac{1}{3}\big(|V(R')| - |S_R| - |T_{R,1}|\big).\end{aligned}$$

Combining these inequalities, and using the bounds on $|V(R')|$ and $|S_R|$, we have

$$\begin{aligned}\frac{2}{3}|T_{R,1}| &> \frac{4}{3}|S_R| + \frac{2}{3}|V(R')| - \left(\frac{2}{3}+\delta\right)n\\ &\ge \frac{4}{3}\left(\frac{1}{3}-2\delta\right)n + \frac{2}{3}\left(1-\frac{5\delta}{2}\right)n - \left(\frac{2}{3}+\delta\right)n\\ &= \left(\frac{4}{9}-\frac{16\delta}{3}\right)n.\end{aligned}$$

Hence $|T_{R,1}| > (2/3-8\delta)\,n$.

However, $T_{R,1}$ is a set of isolated vertices in $R[V(G)\backslash S_R]$. As $|V(G)\backslash S_R| \le (2/3+2\delta)\,n$, we see that $R[V(G)\setminus S_R]$ has maximum degree at most $10\delta n$. $\square$

We may now complete the proof of Lemma 4.1, using the preceding claims. Since (iii) does not hold, by Lemma 5.1, we may assume that

$$\text{either one of } R \text{ or } B \text{ is connected, or } V(G) = V(R') \cup V(B') \text{ and } \min\{|V(R')|, |V(B')|\} \ge (3/4-\delta)\,n. \tag{5.3}$$

In either case, there will be a monochromatic component of order at least $(3/4-\delta)\,n > (2/3+\delta)\,n$. We may without loss of generality assume that this component is R'. We consider several cases depending on the order of B'.

If $|V(B')| < (1/3+\delta/2)\,n$, then every component of B has order at most $(1/3+\delta/2)\,n$. By Claim 5.3, $|S_R| < (1/3+\delta/2)\,n$ and S_R contains all blue components of order at most $(5/12-2\delta)n$. Since $(5/12-2\delta)n > (1/3+\delta/2)n$, S_R contains all components of B, and hence has order n, a contradiction.

We cannot have $(1/3+\delta/2)n \le |V(B')| \le (2/3-\delta/2)\,n$ by Claim 5.2.

If $(2/3-\delta/2)\,n < |V(B')| < (2/3+\delta)\,n$, then, by (5.3), R is connected. Also, all components of B other than B' have order at most $(1/3+\delta/2)\,n$. Hence, by Claim 5.3, S_R contains $X_B = V(G)\setminus V(B')$ and so $|S_R| > (1/3-\delta)\,n$. Thus we are done by Claim 5.4.

Finally, suppose that $|V(B')| \geq (2/3+\delta)\,n$. In this case, sets $S_B \subseteq V(B')$ and T_B are defined, and, in particular, we have that

$$q\,(B\,[V(B') - S_B]) > |S_B| + |V(B')| - \left(\frac{2}{3}+\delta\right)n.$$

By Claim 5.3, we see that $X_R \subseteq S_B$ and $X_B \subseteq S_R$.

Suppose that there is a vertex $v \in T_R \cap T_B$. Then v has at most $|S_R| + t$ red neighbours and at most $|S_B| + t$ blue neighbours. As $|S_R|$ and $|S_B|$ both have order at most $(1/3+\delta/2)\,n$ and $n \gg 1/\delta$, this contradicts $d_G(v) \geq \delta(G) \geq (3/4-\delta)n$. Hence $T_R \cap T_B = \emptyset$.

Suppose that $T_B \setminus S_R$ is non-empty, and let $v \in T_B \setminus S_R$. As $X_R \subseteq S_B$, we have $T_B \subseteq V(R')$. Hence v has no red neighbours in X_R. Vertices in T_R only have red neighbours in $T_R \cup S_R$. However, $T_B \cap T_R = \emptyset$ and so $v \notin T_R \cup S_R$. In particular, v has no red neighbours in $X_R \cup T_R$.

Hence, v has at least $|X_R \cup T_R| - (1/4+\delta)\,n$ blue neighbours in $X_R \cup T_R$, as v has at most $(1/4+\delta)\,n$ non-neighbours. However $v \in T_B$ and so all but t of its blue neighbours are in S_B. Hence

$$|S_B| \geq |X_R \cup T_R| - \left(\frac{1}{4}+\delta\right)n - t > |S_R| + \left(\frac{1}{12} - 3\delta\right)n - t,$$

where the second inequality uses Claim 5.3.

Similarly, if $T_R \setminus S_B$ is non-empty, then

$$|S_R| > |S_B| + \left(\frac{1}{12} - 3\delta\right)n - t.$$

As these cannot both occur, one of $T_R \setminus S_B$ or $T_B \setminus S_R$ is empty.

We assume without loss of generality that $T_B \subseteq S_R$. Then S_R contains the disjoint sets T_B and X_B. Hence, using Claim 5.3, again

$$|S_R| \geq |T_B \cup X_B| > |S_B| + \left(\frac{1}{3} - 2\delta\right)n.$$

Thus $|S_R| \geq (1/3-2\delta)\,n$. As $|S_R| < (1/3+\delta/2)\,n$, we must have $|S_B| \leq 5\delta n/2$. Since $X_R \subseteq S_B$, we see that $|V(R')| \geq (1-5\delta/2)\,n$. Hence, by Claim 5.4, we are done.

This concludes the proof Lemma 4.1. We now prove Lemma 4.2, using similar methods to those used in the proof of Lemma 4.1.

5.2. Proof of Lemma 4.2

Let $0 < \delta < 1/36$. We assume throughout that n is sufficiently large. Suppose that G is a graph of order n with $\delta(G) \geq (3/4-\delta)\,n$ and that we are given a 2-edge colouring $E(G) = E(R) \cup E(B)$.

Suppose that R' contains a matching on at least $(2/3+\delta)\,n$ vertices. We may assume that $|V(R')| < (1-5\delta)\,n$ and R' is bipartite with classes Y_R and Z_R, otherwise we are done. Without loss of generality, we assume that $|Z_R| \geq |Y_R|$ and so $|Y_R| \leq |V(R')|/2 < (1/2-5\delta/2)\,n$. As each edge of the matching contains one vertex from Z_R and one from

Y_R, we have

$$\left(\frac{1}{3}+\frac{\delta}{2}\right)n \le |Y_R| \le |Z_R| < \left(\frac{2}{3}-\frac{11\delta}{2}\right)n. \tag{5.4}$$

As in the proof of Lemma 4.1, we let $X_R = V(G)\backslash V(R')$ and $X_B = V(G)\backslash V(B')$. Note that $5\delta n < |X_R| \le (1/3-\delta)\,n$. We apply Lemma 5.1. Since $|V(R')| \ge (2/3+\delta)\,n$, we are not in case (c) of this lemma because, in that case, each monochromatic component has at most $(1/2+6\delta)n$ vertices. Hence, either B is connected by (a), or, by (b), both R' and B' have order at least $(3/4-\delta)n$. Consequently, $|X_B| \le (1/4+\delta)\,n$ and $|V(R')\cap V(B')| \ge (1/2-2\delta)n$. Thus, every vertex in $X_B \cap X_R$ must have a neighbour in $V(R')\cap V(B')$, which is a contradiction. So, we must have $X_B \cap X_R = \emptyset$.

Claim 5.5. *B' contains a matching on at least $(1/2+\delta)\,n$ vertices.*

Proof. Suppose not; then by Theorem 2.8 there is a set $S \subseteq V(B')$ such that

$$q\,(B\,[V(B')-S]) > |S| + |V(B')| - \left(\frac{1}{2}+\delta\right)n.$$

We will apply the same arguments as used in Claim 5.3 to the set S. All vertices of $V(B')$ lie in S or some component of $B\,[V(B')-S]$. Hence

$$\begin{aligned}|V(B')| &\ge |S| + q\,(B\,[V(B')-S])\\ &> 2|S| + |V(B')| - \left(\frac{1}{2}+\delta\right)n.\end{aligned}$$

This implies that $|S| < (1/4+\delta/2)\,n$.

We let T_B be the set of vertices in components of $B\,[V(B')-S]$ with order at most $t = \lceil \delta^{-1} \rceil$. Then, as in the proof of Claim 5.3, $|T_B| \ge q\,(B\,[V(B')-S]) - \delta n$.

Any vertex in T has blue degree at most

$$|S| + t - 1 \le \left(\frac{1}{4}+\frac{\delta}{2}\right)n + t - 1,$$

and any vertex in X_B has blue degree at most

$$|X_B| - 1 \le \left(\frac{1}{4}+\delta\right)n - 1.$$

Also any vertex in X_R has red degree at most

$$|X_R| - 1 = n - |V(R')| - 1 \le \left(\frac{1}{3}-\delta\right)n - 1,$$

and any vertex in Z_R has red degree at most

$$|Y_R| < \left(\frac{1}{2}-\frac{5\delta}{2}\right)n.$$

Hence, any vertex in the intersection of $T_B \cup X_B$ and $Z_R \cup X_R$ has degree at most

$(3/4 - 3\delta/2)\,n - 1$. Since $\delta(G) \geq (3/4 - \delta)\,n$, we deduce that $T_B \cup X_B$ does not intersect $Z_R \cup X_R$.

Hence $T_B \cup X_B \subseteq Y_R$. However, T_B and X_B are disjoint sets, and so

$$\begin{aligned} |Y_R| &\geq |T_B \cup X_B| \\ &\geq q\left(B\left[V(B') - S\right]\right) - \delta n + n - |V(B')| \\ &> |S| + |V(B')| - \left(\frac{1}{2} + \delta\right) n + (1 - \delta)\,n - |V(B')| \\ &\geq \left(\frac{1}{2} - 2\delta\right) n, \end{aligned}$$

a contradiction with $|Y_R| \leq (1/2 - 5\delta/2)$. So B' contains a matching on at least $(1/2 + \delta)\,n$ vertices. □

We will show that B' contains all vertices in $X_R \cup Z_R$. All vertices of G have at most $(1/4 + \delta)\,n$ non-neighbours, and so any two vertices have at least $(1/2 - 2\delta)\,n$ common neighbours. As $|Y_R| \leq (1/2 - 5\delta/2)\,n$, every pair of vertices in Z_R have a common neighbour in $V(G) \setminus Y_R$. Since all vertices in Z_R have no red neighbours in $V(G) \setminus Y_R$, any two vertices in Z_R have a common blue neighbour. Hence all vertices of Z_R lie in the same blue component. Similarly, if $|Z_R| < (1/2 - 2\delta)\,n$ all vertices of Y_R lie in a single blue component.

Every vertex in X_R has at most $(1/4 + \delta)\,n$ non-neighbours in both Y_R and Z_R. Thus, by (5.4), every vertex in X_R has at least one neighbour in both Z_R and Y_R, which is necessarily blue. Hence $X_R \cup Z_R$ lies within one component of B. If $|Z_R| < (1/2 - 2\delta)\,n$, then B is connected and $B' = B$. If B is not connected, then the component of B containing $X_R \cup Z_R$ has order at least $n - |Y_R| \geq (1/2 + 5\delta/2)\,n$, and hence this component is B'.

If $|V(B')| \geq (1 - 5\delta)\,n$ or if B' is not bipartite, then, in view of Claim 5.5, we are done. So, suppose now that $|V(B')| < (1 - 5\delta)\,n$ and B' is bipartite, with classes Z_B and Y_B.

Both $Z_B \cap Z_R$ and $Y_B \cap Z_R$ are independent sets of G, and hence have order at most $(1/4 + \delta)\,n$. If $|Z_R| < (1/2 - 2\delta)\,n$, then, by the above argument, B is connected – a contradiction. So we may assume that $|Z_R| \geq (1/2 - 2\delta)\,n$. Hence, as $Z_R \subseteq V(B') = Y_B \cup Z_B$, both $Z_B \cap Z_R$ and $Y_B \cap Z_R$ have order at least $(1/4 - 3\delta)\,n$.

Let $v \in X_R$. Since $X_R \cap X_B = \emptyset$, we see that $v \in Z_B \cup Y_B$. We may assume without loss of generality that $v \in Z_B$. Then v has no blue neighbours in Z_B, and no red neighbours in Z_R. In particular, v has no neighbours of either colour in $Z_B \cap Z_R$, which is a set of order at least $(1/4 - 3\delta)\,n$. As v has at most $(1/4 + \delta)\,n$ non-neighbours, it thus has at most $4\delta n$ non-neighbours in $Y_R \subseteq V(G) \setminus (Z_R \cap Z_B)$. However, all edges from v to Y_R are blue. Thus all but at most $4\delta n$ vertices in Y_R lie in the same blue component as v. However, $v \in V(B')$, and $X_R \cup Z_R \subseteq V(B')$. Hence B' contains all but $4\delta n$ vertices, contradicting our assumption that $|V(B')| < (1 - 5\delta)\,n$, and concluding the proof of Lemma 4.2.

6. Proof of Lemmas 4.3–4.5

We shall now prove Lemmas 4.3–4.5, which deal with particular cases arising from the reduced graph. In the proof of Lemma 4.5, we shall be using the graph $G' \subseteq G$ defined by the Regularity Lemma.

Proof of Lemma 4.3. Suppose that B has an independent set S with $|S| \geq n/2$. All vertices in S have at most $n/4-1$ non-neighbours in G, and so

$$\delta(R[S]) \geq (|S|-1) - \left(\frac{n}{4}-1\right) \geq \frac{|S|}{2}.$$

By Corollary 1.3, either $R[S]$ is pancyclic, or $R[S] \cong K_{|S|/2,|S|/2}$. In the latter case, $\delta(R[S]) = |S|/2$, and so $|S| = n/2$. Hence, if $|S| > n/2$, then $C_\ell \subseteq R_G$ for all $\ell \in [3, |S|]$. This completes the proof of part (a).

In order to see (b), suppose that B is bipartite with classes S_1 and S_2, chosen so that $|S_1| \geq |S_2|$. If $|S_1| > n/2$, then $C_\ell \subseteq R$ for all $\ell \in [3, |S_1|]$ by part (a), and we are done. Hence we may assume that n is even and $|S_1| = |S_2| = n/2$. But by the proof of (a) above, we must have either that $C_\ell \subseteq R$ for all $\ell \in [3, n/2]$, or both $R[S_1]$ and $R[S_2]$ are isomorphic to $K_{n/4,n/4}$. This implies that n is divisible by four. Also, both $B[S_1]$ and $B[S_2]$ are isomorphic to the empty graph and so $G \cong K_{n/4,n/4,n/4,n/4}$.

For $i \in \{1,2\}$, let $S_{i,1}$ and $S_{i,2}$ be the independent sets of G partitioning S_i. Then if R is not bipartite, without loss of generality, there are red edges between $S_{1,1}$ and both $S_{2,1}$ and $S_{2,2}$. Hence there is a red path of length either two or four between a vertex of $S_{2,1}$ and a vertex of $S_{2,2}$, with all internal vertices in S_1. As R is complete between $S_{2,1}$ and $S_{2,2}$, R contains C_ℓ for all $\ell \in [4, \lceil n/2 \rceil]$. If, however, R is bipartite then the colouring is a 2-bipartite 2-edge colouring. □

Proof of Lemma 4.4. We shall prove part (a) first. Suppose that $S \subset V(H)$ is such that $|S| \geq (2/3 - \delta/2)\, k$ and $\Delta(R_H[S]) \leq 10\delta k$. We know that $\delta(H) \geq (3/4 - (2d+\epsilon))k$. Hence,

$$\delta(B_H[S]) > \left(\frac{3}{4} - (2d+\epsilon)\right) k - 10\delta k > \frac{k}{2} \geq \frac{|S|}{2}.$$

By Corollary 1.3, the graph $B_H[S]$ is pancyclic. We repeatedly apply Corollary 2.3 with $\ell = 3, \ldots, |S|$ and conclude that B_G contains a monochromatic cycle of every length between 3 and $(1-5\sqrt{\epsilon})(|S|-2)m$. As $m \geq (1-\epsilon)n/k$, $|S| \geq (2/3 - \delta/2)\, k$ and $\delta \gg \epsilon \gg 1/k$, we see that

$$(1-5\sqrt{\epsilon})(|S|-2)m > (1-5\sqrt{\epsilon})(1-\epsilon)^2 \left(\frac{2}{3} - \frac{\delta}{2}\right) n > \left(\frac{2}{3} - \delta\right) n.$$

The proof of (b) follows the same lines, with colours red and blue interchanged.

Notice that the asymmetry of H is not a problem in both proofs because we only use the fact that a red (blue, respectively) edge of H corresponds to an ϵ-regular pair of density at least d in $R_{G'}$ (in $B_{G'}$, respectively). □

6.1. Proof of Lemma 4.5

Our main tools to prove the lemma will be the following two claims. The first excludes a particular structure in a given graph.

Claim 6.1. *Let G be a graph on n vertices such that $\delta(G) \geq 3n/4$. Then there is no set S of order at most three such that $V(G) \setminus S$ can be partitioned into non-empty sets $X_1, \dots, X_4$ such that, for $i = 1, \dots, 4$ G has no edges between X_i and X_{5-i}.*

Proof. Suppose that there is such a set S. Then $\sum_{i=1}^{4} |X_i| \geq n - 3$, and so, for some $1 \leq i \leq 4$, we have $|X_i| \geq (n-3)/4$. As $X_{5-i} \neq \emptyset$, we may consider a vertex $v \in X_{5-i}$. Then v has no neighbours in X_{5-i} and is also not adjacent to itself. Hence $d_G(v) \leq n - (|X_i| + 1) < 3n/4$, contradicting the minimal degree of G. □

The second claim gives us long monochromatic paths in bipartite subgraphs with large minimum degree. We first need one definition.

Definition. Let G be a graph and U and W be two disjoint subsets of vertices. We say that the bipartite graph $G[U, W]$ is *t-complete* if every vertex in U has at least $|W| - t$ neighbours in W and every vertex in W has at least $|U| - t$ neighbours in U.

Claim 6.2. *Let G be a graph and U and W be two disjoint subsets of vertices. If $G[U, W]$ is t-complete, then the following holds.*

(a) For any two vertices $u, w \in U$, the graph $G[U, W]$ contains a u-w path of length ℓ for all even $2 \leq \ell \leq 2\min\{|U|, |W| - 2t\}$. If $G[U]$ or $G[W]$ contains an edge other than uw, then G also contains a u-w path of length ℓ for every odd $7 \leq \ell \leq 2\min\{|U|, |W| - 2t\} - 1$.

(b) For any two vertices $u \in U$ and $w \in W$, the graph $G[U, W]$ contains a u-w path of length ℓ for all odd $3 \leq \ell \leq 2\min\{|U|, |W| - 2t\} - 1$. If $G[U]$ or $G[W]$ contains an edge, then G also contains a u-w path of length ℓ for every even $6 \leq \ell \leq 2\min\{|U|, |W| - 2t\} - 2$.

(c) The graph G contains cycles of all even lengths between 4 and $2\min\{|U|, |W| - 2t\}$. If $G[U]$ is non-empty, then G also contains cycles of all odd lengths between 3 and $2\min\{|U|, |W| - 2t\} - 1$.

Proof. Suppose that u, w are two vertices in U, and let $1 \leq r \leq \min\{|U|, |W| - 2t\}$ be given. Consider any sequence $v_1, v_2, \dots v_r, v_{r+1}$ of distinct vertices in U such that $v_1 = u$ and $v_{r+1} = w$. Clearly, any two vertices in U have at least $|W| - 2t$ common neigbours. Hence, there are distinct vertices $w_1, \dots, w_r \in W$ such that, for all $1 \leq i \leq r$, w_i is a common neighbour of v_i and v_{i+1}. Hence,

$$v_1 w_1 v_2 w_2 \dots v_r w_r v_{r+1}$$

is a u-w path of length $2r$.

Let $xy \neq uw$ be an edge in $G[U]$ such that $\{x, y\} \cap \{u, w\} = \emptyset$. Then in the proof above for $r \geq 3$ we take $v_1 = u$, $v_2 = x$, $v_3 = y$, and $v_{r+1} = w$, and find distinct common neighbours w_i of v_i and v_{i+1} for all $1 \leq i \leq r$, $i \neq 2$. Hence,

$$v_1 w_1 v_2 v_3 w_3 \dots v_r w_r v_{r+1}$$

is a u-w path of length $2r - 1$.

Let $xy \neq uw$ be an edge in $G[U]$ such that $x = u$. In this case, we take $v_1 = u$, $v_2 = y$, and $v_{r+1} = w$, and find distinct common neighbours w_i of v_i and v_{i+1} for all $2 \leq i \leq r$. Hence,

$$v_1 v_2 w_2 v_3 w_3 \dots v_r w_r v_{r+1}$$

is a u-w path of length $2r - 1$, $r \geq 3$.

Finally, let xy be an edge in $G[W]$. Then in the proof above we take $v_1 = u$, v_2 to be any neighbour of $w_2 = x$, v_4 to be any neighbour of $w_3 = y$, and $v_{r+1} = w$. We again find distinct common neighbours w_i of v_i and v_{i+1} for all $1 \leq i \leq r$, $i \neq 2, 3$. Hence,

$$v_1 w_1 v_2 w_2 w_3 v_4 \dots v_r w_r v_{r+1}$$

is a u-w path of length $2r - 1$, $r \geq 4$.

To see (b), take any $u \in U$ and $w \in W$, and let $2 \leq r \leq \min\{|U|, |W| - 2t\}$. We again consider any sequence $v_1, v_2, \dots v_r$ of distinct vertices in U such that $v_1 = u$ and $v_r \neq u$ is any neighbour of w. For all $1 \leq i \leq r$, v_i and v_{i+1} have $|W| - 2t - 1$ common neigbours other than w. Hence, there are distinct vertices $w_1, \dots, w_{r-1} \in W$ such that, for all $1 \leq i \leq r$, $w_i \neq w$ is a common neighbour of v_i and v_{i+1}. Hence,

$$v_1 w_1 v_2 w_2 \dots v_{r-1} w_{r-1} v_r w$$

is a u-w path of length $2r - 1$. The proof of the second part of (b) is similar to (a).

The first part of (c) follows from (b) by taking an edge uw in $G[U, W]$. For the second part, take an edge $uw \in G[U]$ and apply part (a). $\square$

Proof of Lemma 4.5. We shall first prove that there exists a partition $V(G) = W_0 \cup W_1 \cup W_2 \cup W_3 \cup W_4$ such that

(i) $|W_0| \leq \delta n$ and $\min_i |W_i| \geq (1/4 - 4\delta)\, n$.
(ii) The graphs $G'[W_1, W_4]$ and $G'[W_2, W_3]$ are empty.
(iii) There are no blue edges from $W_1 \cup W_3$ to $W_2 \cup W_4$ in G'.

For $1 \leq i \leq 4$, let $W_i \subseteq V(G)$ be the union of the clusters in U_i, so that $|W_i| \geq (1/4 - 3\delta)\,(1 - \epsilon)n$. Note that in G' there are no blue edges from $W_1 \cup W_3$ to $W_2 \cup W_4$. Since there are no edges between U_1 and U_4 and between U_2 and U_3 in H, it follows there are no edges in $G'[W_1, W_4]$ and $G'[W_2, W_3]$. Furthermore, as $\epsilon \ll \delta$, we have that

$$|W_i| \geq \left(\frac{1}{4} - 3\delta\right)(1 - \epsilon)n \geq \left(\frac{1}{4} - 4\delta\right).$$

The set W_0 contains the vertices from V_0, so $|W_0| \leq \epsilon n < \delta n$. Notice that if we later move a constant number of vertices to W_0, then (i)-(iii) will still hold.

Recall that $\delta(G') \geq (3/4-\delta)\,n$ and hence vertices in $W_1 \cup \cdots \cup W_4$ have at most $(1/4+\delta)\,n$ non-neighbours in G'. For a vertex in W_1, at least $(1/4-4\delta)\,n$ of these non-neighbours are in W_4. Hence vertices in W_1 are adjacent in G' (and hence in G) to all but at most $5\delta n$ vertices in $W_1 \cup W_2 \cup W_3$. Similar results hold for W_2, W_3 and W_4. Hence,

(iv) $\delta\,(G[W_i]) \geq |W_i| - 5\delta n - 1$ for all $1 \leq i \leq 4$.
(v) The graphs $R_G[W_1, W_2]$ and $R_G[W_3, W_4]$ are $5\delta n$-complete.

If there is a red edge uv in W_1 or W_3, then, by Claim 6.2, G contains red cycles of length ℓ for all $\ell \in [3, (1/2-19\delta)n]$. Otherwise, all edges in $G[W_1]$ and $G[W_3]$ are blue. Since $|W_1| > 10\delta n + 2$, we have that $\delta\,(G[W_1]) > |W_1|/2$. By Corollary 1.3, G contains blue cycles of length ℓ for all $\ell \in [3, |W_1|]$. But there are also two vertex disjoint blue edges between W_1 and W_3 because there was a blue edge between U_1 and U_3 in H. By (iv), we can join greedily their endpoints in W_i, $i \in \{1,3\}$, by a blue path of any length between 2 and $|W_i| - 10\delta n - 2$. By concatenating these two paths and two edges, we get a cycle of any length between $|W_1|$ and $|W_1| + |W_3| - 20\delta - 4 > (1/2-29\delta)n$.

To complete the proof, we need to show that G contains a monochromatic cycle of length ℓ for all $\ell \in [(1/2-29\delta)\,n, \lceil n/2 \rceil]$, and a monochromatic cycle of length at least $(1-38\delta)\,n$. We distinguish two cases.

Case 1: there are two red edges in $G[W_i]$ for some $i \in \{1,2,3,4\}$.

Without loss of generality assume that $G[W_1]$ contain two red edges. Suppose that there are two disjoint paths P_1 and P_2 from $W_1 \cup W_2$ to $W_3 \cup W_4$ in R_G. Let P_1 have endpoints u in $W_1 \cup W_2$ and u' in $W_3 \cup W_4$ and let P_2 have endpoints w in $W_1 \cup W_2$ and w' in $W_3 \cup W_4$. By restricting to a smaller path if necessary, we may assume that all internal vertices of P_1 and P_2 are in W_0. Note that this also includes the case when there are two vertex disjoint red edges between $W_1 \cup W_2$ and $W_3 \cup W_4$.

We now use Claim 6.2 to find u-w paths of length ℓ for all $\ell \in [6, (1/2-19\delta)\,n]$ in $R[W_1, W_2]$. However, Claim 6.2 also implies that $R_G[W_3, W_4]$ contains u'-w' paths of length ℓ for all $\ell \in [6, (1/2-19\delta)\,n]$ of a given parity. By concatenating these paths with P_1 and P_2 we see that in this case we have monochromatic cycles of length ℓ for all $\ell \in [14, (1-38\delta)\,n]$.

So, we may assume that there are no two vertex disjoint red paths from $W_1 \cup W_2$ to $W_3 \cup W_4$ in R_G. By a corollary of Menger's Theorem, there is a vertex v_R such that there are no red paths from $W_1 \cup W_2$ to $W_3 \cup W_4$ in $G - \{v_R\}$. Hence, the set W_0 spilts into sets W_{12}, W_{34} such that there are no red edges between $W_1 \cup W_2 \cup W_{12}$ and $W_3 \cup W_4 \cup W_{34}$ in $G - \{v_R\}$. If there is also a vertex v_B such that there are no blue paths from $W_1 \cup W_3$ to $W_2 \cup W_4$ in $G - \{v_B\}$, then we can split W_{12} into W_1', W_2' and W_{34} into W_3', W_4' so that there are no edges between $X_i := W_i \cup W_i'$ and $X_{5-i} = W_{5-i} \cup W_{5-i}'$. Taking $S = \{v_R, v_B\}$, we have a contradiction to Claim 6.1. Hence we may assume that there are two disjoint blue paths between $W_1 \cup W_3$ and $W_2 \cup W_4$. Similarly to the above, applying Claim 6.2 to the ends of these paths in $B_G[W_1, W_3]$ and $B_G[W_2, W_4]$, there is a blue cycle of length at least $(1-38\delta)\,n$. Moreover, if there are two blue edges in $G[W_i]$ for some i, then we get blue cycles of length ℓ for all $\ell \in [14, (1-38\delta)n]$.

So, by moving at most eight vertices to W_0, we may assume that all edges of each $G[W_i]$ are red. To complete the proof in Case 1, we need to show that G contains a monochromatic cycle of length ℓ for all $\ell \in [(1/2-29\delta)\,n, \lceil n/2 \rceil]$.

Suppose that some vertex $v \in W_0$ has at least $(1/2+8\delta)\,n+3$ blue neighbours. Then it must have at least two blue neighbours in at least three of the sets W_i. If there is no blue path P from $W_1 \cup W_3$ to $W_2 \cup W_4$ in $G-\{v\}$, then, as above, we would have a contradiction with Claim 6.1 applied with $S=\{v, v_R\}$. Hence, let P be a blue path from $W_1 \cup W_3$ to $W_2 \cup W_4$ in $G-\{v\}$. Without loss of generality, we may assume that P has endpoints $u_1 \in W_1$ and $u_2 \in W_2$ and all internal vertices of P are in W_0. Again, this includes the case of a single edge between $W_1 \cup W_3$ an $W_2 \cup W_4$. Suppose that v has at least two blue neighbours in each of W_1, W_2 and W_3, the other cases being similar. We may find $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$ with $\{u_1, u_2\} \cap \{w_1, w_2, w_3\} = \emptyset$ such that each of w_1, w_2 and w_3 are blue neighbours of v.

By Claim 6.2 we have the following paths:

- for all even $\ell \in [6, (1/2-19\delta)\,n]$, $B_G[W_2, W_4]$ contains a u_2-w_2 path P_ℓ of length ℓ;
- for all even $\ell' \in [6, (1/2-19\delta)\,n]$, $B_G[W_1, W_3]$ contains a u_1-w_1 path $P'_{\ell'}$ of length ℓ';
- for all odd $\ell'' \in [7, (1/2-19\delta)\,n]$, $B_G[W_1, W_3]$ contains a u_1-w_3 path $P''_{\ell''}$ of length ℓ''.

Then, for all even $\ell, \ell' \in [6, (1/2-19\delta)\,n]$, the path

$$u_2 P_\ell w_2 v w_1 P'_{\ell'} u_1$$

is a blue u_1-u_2 path of length $2+\ell+\ell'$ which is internally disjoint from P. Similarly, for all even $\ell \in [6, (1/2-19\delta)\,n]$ and odd $\ell'' \in [6, (1/2-19\delta)\,n]$, the path

$$u_2 P_\ell w_2 v w_3 P''_{\ell''} u_1$$

is a blue u_1-u_2 path of length $2+\ell+\ell''$ which is internally disjoint from P.

Hence, for all $L \in [14, (1-38\delta)\,n]$, there is a blue u_1-u_2 path of length L which is internally disjoint from P. Since $|P| \le |W_0|+2 \le 2\delta n$, this gives blue cycles of length L for all $L \in [2\delta n+14, (1-38\delta)\,n]$. Since $2\delta n + 14 < (1/2-29\delta)\,n$, we are done.

Thus, we may assume that each vertex in W_0 has blue degree at most $(1/2+8\delta)\,n+3$, and so red degree at least $(1/4-8\delta)\,n-3$. Let C_1 be the red component of $G-\{v_R\}$ containing $W_1 \cup W_2$ and C_2 be the red component of $G-\{v_R\}$ containing $W_3 \cup W_4$. We know that $R_G[W_1 \cup W_2]$ and $R_G[W_3 \cup W_4]$ are connected, and the minimal red degree condition on W_0 ensures that there are at most two components in $R_G[V-\{v_R\}]$. As v_R has red degree at least $(1/4-8\delta)\,n-3$, it has at least $(1/8-5\delta)\,n$ red neighbours in at least one of C_1 or C_2. Let C'_i be the set C_i, with v_R added if it has at least $(1/8-5\delta)\,n$ red neighbours in C_i.

Then $|C'_1|+|C'_2| \ge n$ and so we may assume without loss of generality that $|C'_1| \ge \lceil n/2 \rceil$. All vertices in C'_1 have degree in $R_G[C'_1]$ at least $(1/8-5\delta)\,n$. Further, all vertices in $C'_1 \setminus |W_0|$ have degree in $R_G[C'_1]$ at least $|C'_1|-6\delta n$. As $|C'_1| \le (1/2+8\delta)\,n$ and $|W_0| \le \delta n$, the condition of Theorem 2.6 holds on $R_G[C'_1]$ and so $R_G[C'_1]$ is hamiltonian. But we also have

$$e(R_G[C_1']) \geq \frac{1}{2}\left(|C_1'| - 6\delta n\right)\left(|C_1'| - |V_0|\right)$$
$$> \frac{1}{4}|C_1'|^2.$$

Hence, by Theorem 1.2, $R_G[C_1']$ is pancyclic and we are done with Case 1.

Case 2: for every $i = 1, 2, 3, 4$, there is at most one red edge in $G[W_i]$.

By moving at most four vertices attached to a red edge in $G[W_i]$ to W_0, we may assume that all edges of $G[W_i]$ are blue for all $i = 1, 2, 3, 4$.

By (iv) and Corollary 1.3, G contains blue cycles of length ℓ for all $\ell \in [3, |W_1|]$. But there are also two vertex disjoint blue edges between W_1 and W_3 because there was a blue edge between U_1 and U_3 in H. Using (iv), we can join greedily their endpoints in W_i, $i \in \{1, 3\}$, by a blue path of any length between 2 and $|W_i| - 10\delta n - 2$. By concatenating these two paths and two edges, we get a cycle of any length between $|W_1|$ and $|W_1| + |W_3| - 20\delta - 4 > (1/2 - 29\delta)n$. Hence, G contains blue cycles of all lengths between 3 and $|W_1| + |W_3| - 20\delta n$. Moreover, the same argument gives that for any two vertices u, w in $W_1 \cup W_3$, there are blue u-w paths of every length between 8 and $|W_1| + |W_3| - 20\delta n$. The same is true in $B_G[W_2 \cup W_4]$.

Consequently, there are no two internally disjoint paths between $W_1 \cup W_3$ and $W_2 \cup W_4$ in B_G. Hence, there exists a vertex v_B such that, in $G - v_B$, there are no blue paths from $W_1 \cup W_3$ to $W_2 \cup W_4$.

Now we essentially follow the proof in Case 1, with colours red and blue interchanged: There must exist two internally disjoint red paths from $W_1 \cup W_2$ to $W_3 \cup W_4$, otherwise we would get a contradiction with Claim 6.1. Consequently, we join their endpoints by red paths in $R_G[W_1 \cup W_2]$ and $R_G[W_3 \cup W_4]$ to get a red cycle of length at least $(1 - 38\delta)\,n$.

If there is a vertex in W_0 with at least $(1/2 + 8\delta)\,n + 3$ red neighbours, then we are done as in Case 1. Hence, we may assume that every vertex of W_0 has at least $(1/4 - 8\delta)\,n - 3$ blue neighbours. Hence, we may partition W_0 into sets $W_1', \ldots, W_4'$, so that each vertex in W_i' has at least $(1/16 - 3\delta)n$ blue neighbours in W_i. It follows that either $|W_1 \cup W_1' \cup W_3 \cup W_3'| \geq \lceil n/2 \rceil$ or $|W_2 \cup W_2' \cup W_4 \cup W_4'| \geq \lceil n/2 \rceil$.

Without loss of generality, suppose that $|W_1 \cup W_1' \cup W_3 \cup W_3'| \geq \lceil n/2 \rceil$. By removing vertices, if necessary, we may assume that $|W_1 \cup W_1' \cup W_3 \cup W_3'| = \lceil n/2 \rceil$. We construct a blue cycle on $\lceil n/2 \rceil$ vertices as follows. Take two vertex disjoint blue edges u_1u_3, v_1v_3 such that $u_1, v_1 \in W_1$ and $u_3, v_3 \in W_3$. Take any two vertices $w_1 \in W_1$ and $w_3 \in W_3$ distinct from u_1, v_1, u_3, v_3. By (iv) and by the definition of W_i', one can greedily construct blue u_i-w_i path P_i containing all the vertices of W_i', avoiding v_i, and not having more than $3\delta n$ vertices. Then, by (iv), the induced sub-graph $B_G[(W_i \setminus V(P_i)) \cup \{w_i\}]$ satisfies the assumptions of Corollary 2.7, and so it must contain a blue v_i-w_i path P_i'. By concatenating paths P_1, P_1', P_3, P_3' and edges u_1u_3, v_1v_3, we obtain a blue cycle on $\lceil n/2 \rceil$ vertices. Clearly, by omitting some vertices from W_1', W_3' and W_1, we can obtain a blue cycle of any length between $(1/2 - 29\delta)\,n$ and $\lceil n/2 \rceil$. $\square$

7. Monochromatic circumference

In this section we shall look at the monochromatic circumference of a graph. We begin by proving Theorem 1.8.

Proof of Theorem 1.8. As in the proof of Theorem 1.6, we consider the reduced graph H, which has order k and minimal degree at least $(3/4-\delta)\,k$. Applying Lemma 4.1 to H, we have one of the following.

(i) There is a component of R_H or B_H which contains a matching on at least $(2/3+\delta)\,k$ vertices.
(ii) There is a set S of order at least $(2/3-\delta/2)\,k$ such that either $\Delta(R_H[S]) \leq 10\delta k$ or $\Delta(B_H[S]) \leq 10\delta k$.
(iii) There is a partition $V(H) = U_1 \cup \cdots \cup U_4$ with $\min_i |U_i| \geq (1/4-3\delta)\,k$ such that there are no blue edges from $U_1 \cup U_2$ to $U_3 \cup U_4$ and no red edges from $U_1 \cup U_3$ to $U_2 \cup U_4$.

In the first case, we use Lemma 2.5(a) to find a monochromatic cycle of length at least $(2/3+\delta/2)\,n$. In the second case, assume without loss of generality that $\Delta(R_H[S]) \leq 10\delta k$. Then, by Lemma 4.4, G contains a blue cycle of length ℓ for all $\ell \in [3,(2/3-\delta)\,n]$. In the third case, Lemma 4.5 implies that G contains a monochromatic cycle of length at least $(1-38\delta)\,n \geq (2/3+\delta)\,n$. □

We will make the following definition.

Definition. For $0 < c < 1$, let $\Phi = \Phi_c$ be the supremum of values ϕ such that any graph G of sufficiently large order n with $\delta(G) > cn$ and a 2-colouring $E(G) = E(R) \cup E(B)$ has monochromatic circumference at least ϕn.

For $c \geq 3/4$, Theorem 1.8 implies that $\Phi_c \geq 2/3$. However, the example given after Theorem 1.8 shows that $\Phi_c \leq 2/3$ for all c. We can also find upper and lower bounds for Φ_c when $c < 3/4$, and we collect them into the following theorem.

Theorem 7.1. *For all $c \geq 3/4$, we have $\Phi_c = 2/3$. For all $c \in (0,1)$, we have $\Phi_c \geq c/2$. Also, there are the following upper bounds on Φ_c.*

$$\Phi_c \leq \begin{cases} \frac{1}{2} & c \in [\frac{3}{5}, \frac{3}{4}) \\ \frac{2}{5} & c \in [\frac{5}{9}, \frac{3}{5}) \\ \frac{1}{r} & c < \frac{2r-1}{r^2} \text{ for all } r \geq 3. \end{cases}$$

Note that, as $c \to 0$, we may use the last upper bound to show that $\Phi_c/(c/2) \to 1$. Hence, asymptotically, as $c \to 0$, the upper and lower bounds on Φ_c agree.

Proof of Theorem 7.1. For $c \in (0,1)$, every 2-edge coloured graph with $\delta(G) > cn$ has at least $cn^2/2$ edges. Hence, there are at least $cn^2/4$ edges of one colour. We may deduce from Theorem 2.11 that, in that colour, there is a cycle of length at least $cn/2$. Hence $\Phi_c \geq c/2$ for all $c \in (0,1)$. Next, we prove the upper bounds on Φ_c.

For $c \in [5/9, 3/5)$, let t be an integer such that $t > 1/(3-5c)$ and let $n = 5t$. We define a graph G'_t as follows. Let S_1 and S_2 be sets of order $2t$ and T be a set of order t. Let R be the union of the complete graph on S_1 and the complete graph on S_2. Then R has circumference $2t$. Let B be the union of the complete graph on T and the complete bipartite graph between T and $S_1 \cup S_2$. Then, any two consecutive vertices of a cycle in B must contain a vertex of T and hence B has circumference at most $2t$. Let G'_t be the union of R and B. Therefore $\delta(G) = 3t - 1 > 5tc = c|G'_t|$ and so $\Phi_c \leq 2/5$.

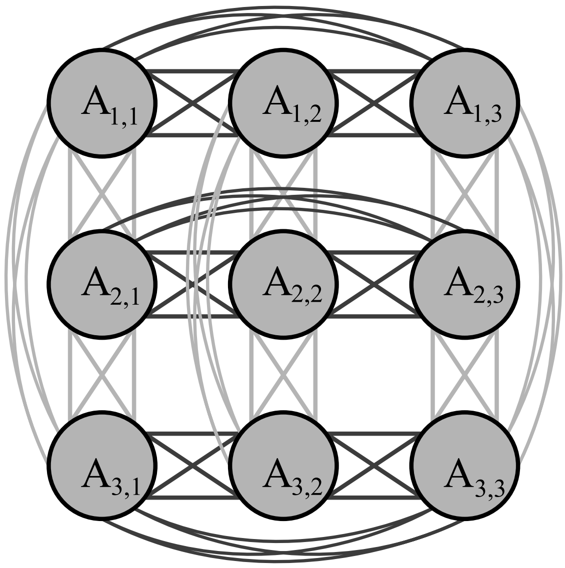

Figure 1. The graph $G_t^{(3)}$

Assume now that $c \in (0, \frac{2r-1}{r^2})$ for a given $r \geq 2$. We aim to show that $\Phi_c \leq 1/r$. Note that when $r = 2$ this gives the bound $\Phi_c \leq 1/2$ for $c < 3/4$. Let t be an integer such that $t > 1/(2r - 1 - r^2c)$ and let $n = tr^2$. Define a family $\{A_{i,j} : 1 \leq i \leq r, 1 \leq j \leq r\}$ of sets of order t. We define the following graphs on vertex set $\bigcup_{i,j} A_{i,j}$:

$$E(B) = \{uv : u \in A_{i,j}, v \in A_{i,j'} \text{ for some } 1 \leq i \leq r \text{ and } j \neq j'\};$$
$$E(R) = \{uv : u \in A_{i,j}, v \in A_{i',j} \text{ for some } 1 \leq j \leq r.\}$$

Let $G_t^{(r)}$ be the union of the graphs R and B, as illustrated in Figure 1, for the case $r = 3$. Then $\delta(G''_{r,t}) = (2r-1)t - 1 > ctr^2 = c|G_t^{(r)}|$. However, as all monochromatic components have order rt, there are no monochromatic cycles of length greater than n/r. Hence $\Phi_c \leq 1/r$. $\square$

8. Conclusion

Theorem 1.6 is a 2-colour version of the uncoloured (or 1-coloured) result of Bondy that all graphs with order $n \geq 3$ and minimum degree at least $n/2$ are either pancyclic or isomorphic to $K_{n/2,n/2}$. We may hope to generalise to k colours. In this case, we let $E(G) = \bigcup_{i=1}^{k} E(G_i)$ be an edge colouring, where each G_i is a spanning subgraph of G, representing the edges coloured i. Our extremal graph was found by letting both R and B be subgraphs of the extremal graph in the uncoloured case, and we again use this method to find k-coloured graphs with high minimum degree but no odd cycles.

Definition. Let $n = 2^k p$ and let G be isomorphic to the 2^k-partite graph with classes all of order p. A *k-bipartite k-edge colouring* of G is a k-edge colouring $E(G) = \bigcup_{i=1}^{k} E(G_i)$ such that each G_i is bipartite.

As in the 2-coloured case, we can deduce that a k-bipartite k-edge colouring of the 2^k-partite graph with classes all of order p induces a labelling U_α, $\alpha \in \{1,2\}^k$, of the classes such that, for all i, the graph G_i is bipartite with classes

$$\bigcup_{\alpha:\alpha_i=1} U_\alpha$$

and

$$\bigcup_{\alpha:\alpha_i=2} U_\alpha.$$

Note that this implies that, if α and β in $\{1,2\}^k$ differ only in the ith place, then all edges between U_α and U_β are coloured with i. As this graph has minimum degree $\left(1 - 2^{-k}\right) n$, we make the following conjecture.

Conjecture 8.1. *Let $n \geq 3$, and k be an integer. Let G be a graph of order n with $\delta(G) \geq \left(1 - 2^{-k}\right) n$. If $E(G) = \bigcup_{i=1}^{k} E(G_i)$ is a k-edge colouring, then either:*

- *for all $\ell \in \left[\min\{2^k, 3\}, \lceil n/2^{k-1} \rceil\right]$ there is some $1 \leq i \leq k$ such that $C_\ell \subseteq G_i$, or;*
- *$n = 2^k p$, G is the complete 2^k-partite graph with classes of order p, and the colouring is a k-biparitite k-edge colouring.*

Note that the case when $k = 1$ is Bondy's Theorem, and Theorem 1.6 is the case $k = 2$ for large n. We pose the following problem about the monochromatic circumference.

Problem 8.2. *What is the value of Φ_c for $c < 3/4$?*

Note that Theorem 7.1 shows that $\Phi_c = 2/3$ for all $c \geq 3/4$. In this case, we make the following conjecture with an exact bound on the monochromatic circumference.

Conjecture 8.3. *Let G be a graph of order n with $\delta(G) \geq 3n/4$. Let $n = 3t + r$, where $r \in \{0, 1, 2\}$. If $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring, then G has monochromatic circumference at least $2t + r$.*

Note that Theorem 1.8 is an asymptotic version of this conjecture. By considering the graph $F_{t,2t+r}$ as defined in Section 1, we see that this conjecture is best possible. For the latest progress on Conjecture 8.3, the reader should consult [4].

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References

[1] F. Benevides and J. Skokan. The 3-colored Ramsey number of even cycles. *J. Comb. Theory, Ser. B*, 99:690–708, 2009.
[2] C. Berge. Sur le couplage maximum d'un graphe. *C.R. Acad. Sci. Paris*, 247:258–259, 1958. (in French).
[3] B. Bollobás. *Extremal Graph Theory.* Academic Press Inc., 1978.
[4] B. Bollobás, F. Benevides, T. Łuczak, and J. Skokan. Graphs with large minimum degrees arrow even cycles. manuscript, 2011.
[5] J. Bondy. Pancyclic graphs I. *J. Comb. Theory, Ser. B*, 11:80–84, 1971.
[6] J. Bondy and M. Simonovits. Cycles of even lengths in graphs. *J. Comb. Theory, Ser. B*, 16:97–105, 1974.
[7] V. Chvátal. On Hamilton's ideals. *J. Comb. Theory, Ser. B*, 12:163–168, 1972.
[8] G. Dirac. Some theorems on abstract graphs. *Proc. London Math. Soc.*, 2:69–81, 1952.
[9] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar.*, 10:337–356, 1959.
[10] E. Györi, V. Nikiforov, and R. Schelp. Nearly bipartite graphs. *Disc. Math.*, 272:187–196, 2003.
[11] J. Komlós and M. Simonovits. Szemerdi's regularity lemma and its applications in graph theory. *Bolyai Society Mathematical Studies 2, Combinatorics, Paul Erds is Eighty*, 2:295–352, 1996.
[12] H. Li, V. Nikiforov, and R. Schelp. A new type of Ramsey-Turán problems. *Disc. Math.*, 310:3579–3583, 2010.
[13] E. Szemerédi. Regular Partitions of Graphs. *Colloques Internationaux C.N.R.S - Problèmes Combinatoires et Théorie des Graphes*, 260:399–401, 1976.
[14] W. Tutte. A factorization of linear graphs. *J. London Math. Soc.*, 22:107–111, 1947.