Covering Complete Geometric Graphs by Monotone Paths

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Abstract

Given a set A of n points (vertices) in general position in the plane, the *complete geometric graph* $K_n[A]$ consists of all $\binom{n}{2}$ segments (edges) between the elements of A. It is known that the edge set of every complete geometric graph on n vertices can be partitioned into $O(n^{3/2})$ crossing-free paths (or matchings). We strengthen this result under various additional assumptions on the point set. In particular, we prove that for a set A of n *randomly* selected points, uniformly distributed in $[0,1]^2$, with probability tending to 1 as $n \to \infty$, the edge set of $K_n[A]$ can be covered by $O(n \log n)$ crossing-free paths and by $O(n\sqrt{\log n})$ crossing-free matchings. On the other hand, we construct n-element point sets such that covering the edge set of $K_n(A)$ requires a quadratic number of monotone paths.

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1 Introduction

A set of points in the plane is said to be (i) in *general position* if no 3 points are collinear; and (ii) in *convex position* if every point in the set is an extreme point of the convex hull. Given n points in general position in the plane, the graph obtained by connecting certain point-pairs by straight-line segments is called a *geometric graph* G. If no two segments (edges) of G cross each other, then G is said to be *crossing-free* or a *plane graph*.

In 2005, Araujo, Dumitrescu, Hurtado, Noy, and Urrutia [3] asked the following question: Can the edge set of every complete geometric graph on n vertices be partitioned into a small number of crossing-free matchings? In particular, they asked:

▶ **Problem 1.** *Does there exist a constant $c > 0$ such that every complete geometric graph on n vertices can be partitioned into at most cn plane matchings?*

It is easy to verify that when n is even (respectively odd), every complete geometric graph of n vertices in convex position can be decomposed into $n-1$ (respectively n) perfect plane matchings. On the other hand, the best known upper bound for point sets in general position is only $O(n^{3/2})$ [3], which seems to be far off.

Given a point-set A, let $K_n[A]$ denote the complete geometric graph induced by A. We prove the following.

▶ **Theorem 2.** *Let A be a set of n random points uniformly distributed in $[0,1]^2$, and let $n \to \infty$. Then, with probability tending to 1, the edge set of $K_n[A]$ can be covered by at most $O(n \log n)$ crossing-free paths, and by $O(n\sqrt{\log n})$ crossing-free matchings.*

This is better than the $O(n^{3/2})$ bound in [3], which holds for every point set in general position. The proof in fact gives the stronger result that there is a covering by $O(n \log n)$ monotone paths.

▶ **Definition 3.** *A polygonal path $\xi = (v_1, v_2, \ldots, v_t)$ in $\mathbb{R}^2$ is* monotone in direction $\mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ *(or is* $\mathbf{u}$-monotone*) if every directed edge of ξ has a positive inner product with $\mathbf{u}$, that is, $\langle \overrightarrow{v_i v_{i+1}}, \mathbf{u}\rangle > 0$ for $i = 1, \ldots, t-1$; here $\mathbf{0}$ is the origin. A path $\xi = (v_1, v_2, \ldots, v_t)$ is* monotone *if it is monotone in some direction $\mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Obviously, every monotone path is crossing-free.*

A matching is said to be monotone in direction $\mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ *if the edge projections onto* $\mathbf{u}$ *are disjoint intervals. A matching is said to be* monotone *if it is monotone in some direction $\mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Equivalently, a matching is monotone if its edge set is the subset of the edge set of a monotone path.*

For any constant $\alpha > 0$, a set of n points is α-dense *if the ratio between the longest and the shortest distance between two points in the set is at most $\alpha\sqrt{n}$.*

For dense sets, the $O(n^{3/2})$ upper bound [3] for decompositions into plane matchings can be strengthened by requiring that the paths are *monotone*.

▶ **Theorem 4.** *Let A be an α-dense point set in general position in the plane. Then the edge set of $K_n[A]$ can be covered by $O(n^{3/2})$ monotone paths, and such a covering can be computed in $O(n^2)$ time. Consequently, the edge set of $K_n[A]$ can be decomposed into $O(n^{3/2})$* monotone *matchings.*

From the opposite direction, we suspect that covering the edge set of the complete geometric graph of every dense point set requires a superlinear number of monotone paths.

▶ **Conjecture 5.** *For some (possibly, for every) dense n-element point set A in general position in the plane, covering the edge set of $K_n[A]$ requires a superlinear number of monotone paths.*

In contrast to random or dense point sets, arbitrary point sets require a quadratic number of monotone paths in the worst-case.

▶ **Theorem 6.** *Let A be a set of n points in general position in the plane. Then the edge set of the complete geometric graph $K_n[A]$ can be covered by $n^2/6 + O(n)$ monotone paths. On the other hand, there exist n-element point sets that require at least $n^2/15$ monotone paths.*

Monotone paths are ubiquitous and have been studied extensively over the years, particularly in the field of optimization [6]. On the other hand, crossing-free paths and matchings have been an attraction for geometers [8, 9, 10]. Note that every monotone path is crossing-free. Clearly, by taking the odd, resp. even numbered edges, every monotone path decomposes into two monotone matchings.

The rest of this paper is organized as follows. In Sections 2, 3, and 4, we prove Theorems 2, 3, and 5, respectively. The last section contains some concluding remarks and open problems. All point sets appearing in this paper are assumed to be in general position, and the logarithms are in base 2.

2 Proof of Theorem 2

Let $A = a_1, \dots, a_n$ be a random sequence of n points, independently and uniformly distributed in $U = [0,1]^2$. For any unit vector $\mathbf{v} \in \mathbb{R}^2$, we can sort A according to the projections $\langle a_i, \mathbf{v}\rangle$. Observe that sorting with respect to $\mathbf{v}$ and $-\mathbf{v}$ are equivalent. Let $\theta = \alpha(\log n)/n$ be a small angle, where $\alpha > 0$ is a suitable small constant, so that $N = \pi/\theta$ is an integer. Let $\mathcal{V}$ be a set of N unit vectors evenly spaced in $[0, \pi)$ so that $(1,0) \in \mathcal{V}$.

Consider the following procedure for covering the edge set of $K_n[A]$ by (i) monotone paths and (ii) crossing-free matchings. For each vector $\mathbf{v} \in \mathcal{V}$, sort the elements of A according to their projection on $\mathbf{v}$, and label them in increasing order as $a_1^{\mathbf{v}}, \dots, a_n^{\mathbf{v}}$. Now proceed as follows:

- To produce a collection of monotone paths: for each $\mathbf{v} \in \mathcal{V}$, and $1 \le i \le j \le \beta \log n$, we consider the sequence $(a_k^{\mathbf{v}} : k \equiv i \bmod j)$. This defines a monotone path for each choice of i, j and $\mathbf{v}$.
- To produce a collection of non-crossing matchings: we partition A into $t := \lceil n/(\beta \log n)\rceil$ intervals $B_1, B_2 \dots B_t$ of size at most $\lceil \beta \log n \rceil$. For each set of the form $B_i \cup B_{i+1}$, by the $O(n^{3/2})$ aforementioned bound in [3], we can cover the complete graph on $B_i \cup B_{i+1}$ with at most $(2\lceil\beta \log n\rceil)^{3/2}$ matchings. We use these to produce a total of $2(2\lceil\beta \log n\rceil)^{3/2}$ matchings: first pair up the matchings from $B_1 \cup B_2, B_3 \cup B_4, \dots$, and then pair up the matchings from $B_2 \cup B_3, B_4 \cup B_5, \dots$.

For each choice of $\mathbf{v}$, the first bullet produces a set of at most $(\beta^2/2)\log^2 n$ paths; and the second bullet produces a set of at most $2(2\lceil\beta \log n\rceil)^{3/2}$ matchings. Since there are $N = n\pi/\alpha \log n$ choices of $\mathbf{v}$, we use a total of at most $O(n \log n)$ paths and at most $O(n\sqrt{\log n})$ matchings.

All that remains is to check that the paths and matchings we have produced do in fact cover all pairs of points from A. Note that, for each of the orderings we define, we have covered all pairs that are at most $\beta \log n$ apart in the ordering. Thus it is enough to show that, for every pair of points $a, b \in A$, there is some $\mathbf{v}$ such that a and b are at most $\beta \log n$ apart in the corresponding ordering of A.

We first note a geometric fact. Let $a, b \in A$ be any two (distinct) points in $U = [0,1]^2$. We rotate the picture, and assume for convenience that ab is a vertical segment, and that the unit square has been rotated by some angle. Let U_γ denote the square rotated by γ; we have $\text{Area}(U_\gamma) = 1$ for every γ. Consider the geometric locus $R(a,b)$ of the points $p \in U_\gamma$ on lines making an angle of at most θ with ab and which intersect ab, see Fig. 1.

▶ **Lemma 7.** *The area of $R(a,b)$ is at most 4θ.*

Proof. Consider four lines incident to a and b respectively, and making (clockwise or counterclockwise) angles of θ with ab. The boundary of the locus is made by these four lines and two polygonal arcs on the boundary of the square. The locus is easily seen to be contained in an axis-parallel rectangle of width $2\sqrt{2}\tan\theta$ and height $\sqrt{2}$. As such, its area is bounded from above by $4\tan\theta$. Excluding the two isosceles triangles based on the left and right vertical sides yields an improved area bound of $\frac{3}{4} \cdot 4\tan\theta = 3\tan\theta \le 4\theta$ (here we assume that n is sufficiently large). ◀

We are now ready to complete the proof. We reveal two points $a, b \in A$ (and note that $a \ne b$ with probability 1). Choose a vector $\mathbf{v} \in \mathcal{V}$ that makes an angle at most θ with the direction orthogonal to ab. Then any point of A that comes between a and b in the ordering generated by $\mathbf{v}$ must lie in the region $R(a,b)$, and the lemma shows that, for any given point,

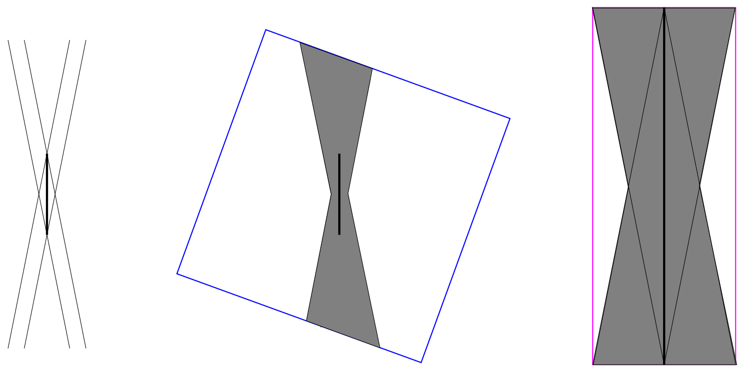

Figure 1 Left: four lines making an angle of θ with ab through a and b. Center: the geometric locus (shaded) associated with a vertical segment. Right: a bounding rectangle for the locus associated with a long vertical segment.

the probability that the point lies in this region is at most 4θ. The probability that at least $\beta \log n$ points of A lie in $R(a,b)$ is therefore at most

$$\binom{n}{\lceil \beta \log n \rceil}(4\theta)^{\lceil \beta \log n \rceil} \le \left(\frac{en}{\lceil \beta \log n \rceil}\right)^{\lceil \beta \log n \rceil}(4\theta)^{\lceil \beta \log n \rceil} \le \left(\frac{4\alpha e}{\beta}\right)^{\beta \log n},$$

where we used the estimate $\binom{n}{k} \le (en/k)^k$. This probability is $o(1/n^2)$, provided we have chosen α and β sensibly. There are $O(n^2)$ pairs: so, by the union bound, with high probability, for every pair $a, b \in A$ there is a direction in which a and b are separated by at most $\beta \log n$ other points. This completes the proof. ◀

3 Proof of Theorem 4

Consider an α-dense point set A. By scaling, we can assume that A is contained in the unit square $U = [0,1]^2$, and the distance between any two points of A is at least $\sqrt{2} \cdot \alpha^{-1} n^{-1/2}$.

The covering procedure is similar to Phase I in the proof of Theorem 2, but with a twist. Let $\varphi = cn^{-1/2}$ be a small angle, for a suitable small constant $c > 0$, so that $N = 2\pi/\varphi$ is an integer. Let $\mathcal{V}$ be a set of N unit vectors evenly spaced around the origin so that $(1,0) \in \mathcal{V}$. For each vector $\mathbf{v} \in \mathcal{V}$, compute the $\mathbf{v}$-monotone spanning path ξ_v induced by A. Let the path be $a_1 a_2 \dots a_n$ in the order of monotonicity. Set

$$\beta = \frac{8\alpha^2 c + 4\sqrt{2} \cdot \alpha}{\pi},$$

and for each $1 \le j \le i \le \beta\sqrt{n}$, add the path $a_j a_{i+j} a_{2i+j} \dots$ to the set $\mathcal{P}$ of covering paths. Note that for each vector $\mathbf{v} \in \mathcal{V}$, there are at most $1 + 2 + \dots + \beta\sqrt{n} < \beta^2 n$ paths to be added to $\mathcal{P}$, all of which are monotone and, therefore, $\mathcal{P}$ contains at most $\frac{2\pi}{c}\sqrt{n} \cdot \beta^2 n = \Theta(n^{3/2})$ monotone paths.

We claim that the paths in $\mathcal{P}$ cover the complete geometric graph on A. That is, for any pair of points $a, b \in A$, we show that the segment ab appears in at least one path in $\mathcal{P}$. We construct the region $R(a,b)$ analogous to the one in the proof of Theorem 2 but with respect to φ instead. By Lemma 7, its area is at most 4φ. Assume that $R(a,b)$ contains k points of A. The disks of radius $r = \frac{\sqrt{2}}{2} \cdot \alpha^{-1} n^{-1/2}$ centered at these k points are disjoint and they cannot exceed the region $R(a,b)$ extended by a margin of r, whose area is then at most $4(\varphi + r)$. The total area of the k disks is at most $4(\varphi + r)$, and therefore

$$k\pi r^2 \le 4(\varphi + r), \text{ thus}$$
$$k\pi \le 8\alpha^2 n \left(\frac{c}{\sqrt{n}} + \frac{\sqrt{2}}{2\alpha\sqrt{n}} \right) \text{ or}$$
$$k \le \frac{8\alpha^2 c + 4\sqrt{2} \cdot \alpha}{\pi} \cdot \sqrt{n} = \beta\sqrt{n}.$$

This means that in any $\mathbf{v}$-monotone spanning path induced by A, where $\mathbf{v} \in \mathcal{V}$ makes an angle of at most φ with ab, there are at most $\beta\sqrt{n}$ points between a, b and thus, the segment ab appears in one of the monotone paths constructed from ξ_v.

Since every monotone path can be decomposed into two monotone matchings (the odd- and even-numbered edges in the path), the edge set of $K_n[A]$ can be covered by $\Theta(n^{3/2})$ monotone matchings.

For the analysis of the algorithm, we have $\Theta(\sqrt{n})$ vectors and for each vector v we first sort the points according to their v component in $O(n \log n)$ time. Finally, by tracing the points in the sorted array, each point may update at most $O(\sqrt{n})$ monotone paths corresponding to v. Overall, the running time of the algorithm is

$$\Theta(\sqrt{n}) \left[O(n \log n) + n \cdot O(\sqrt{n}) \right] = O(n^2).$$

◀

4 Proof of Theorem 6

Lower bound. Assume that n is divisible by 3, i.e., $n = 3k$. Partition the n points in our set P into three groups A, B, C, of the same size, close to the vertices of a unit equilateral triangle, where each group consists of roughly equidistant points, but the distances are different; see Fig. 2. Let the diameters of the 3 groups of k vertices be

$$\mathrm{diam}(C) \ll \mathrm{diam}(B) \ll \mathrm{diam}(A) \ll 1.$$

The number of (undirected) inter-group edges, namely edges in

$$E_0 := E(A \times B) \cup E(A \times C) \cup E(B \times C),$$

is $3k^2 = n^2/3$. Let $\mathcal{P}$ be a covering of $K_n[P]$ by *monotone* paths. We claim that every path $\xi \in \mathcal{P}$ contains at most five edges in E_0. Consequently, covering all the edges of the tripartite graph on $P = A \cup B \cup C$ requires at least $3k^2/5 = n^2/15$ monotone paths.

To verify the claim, observe that the directed edges in ξ can be of six types:

1. $A \to B$, and the corresponding opposite orientation $B \to A$
2. $A \to C$, and the corresponding opposite orientation $C \to A$
3. $B \to C$, and the corresponding opposite orientation $C \to B$

Let us trace the edges of ξ from one end of the path to the other, and record the sequence of groups in the order they are visited. (When two consecutive vertices belong to the same group, we do not repeat the corresponding symbol in this sequence.)

Due to the monotonicity of ξ, the resulting sequence satisfies the following conditions:

(i) There is no cycle of the form $A \to B \to C \to A$.

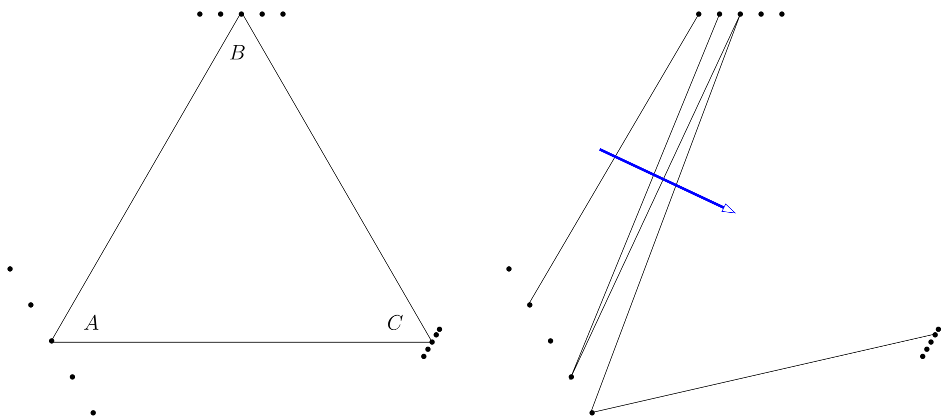

Figure 2 Covering the edge set of a tripartite graph by monotone paths. The figure shows the edges in E_0 corresponding to a monotone path ξ; note that this edge set may be disconnected. A direction of monotonicity is drawn in blue color.

(ii) For any directed edge type U, let U^r denote the type of an edge with opposite orientation. There are no two types, U and V, such that ξ has edges belonging to all four types, U, U^r, V, and V^r. This follows from the fact that there is no straight-line on which the orthogonal projection of four such edges would be disjoint.

(iii) For any two groups, $X, Y \in \{A, B, C\}$, there are at most *four* edges between X and Y in ξ. Moreover, if $\mathrm{diam}(X) \gg \mathrm{diam}(Y)$ and there are four edges between X and Y in ξ, then ξ starts or ends at a point in X.

This implies that there are no monotone paths consisting of more than *five* edges in E_0, and the only monotone paths covering five edges are of the following form:

1. $A \to B \to A \to B \to A \to C$ (shown in Fig. 2).
2. $A \to C \to A \to C \to A \to B$
3. $B \to C \to B \to C \to B \to A$

Upper bound. Since every 2-edge path is monotone, one can decompose the edge set of $K_n[A]$ into $\lceil \frac{1}{2}\binom{n}{2} \rceil$ 2-edge paths. We can improve upon this bound by using 3-edge paths. Since not every 3-edge path is monotone, we need special 3-edge paths that are monotone. A 3-edge path is a *zig-zag* path if its two endpoints lie in opposite open halfplanes determined by the middle edge of the path. It is easy to see that any 3-edge zig-zag path is monotone.

The key is a result by Wilson [20] about decomposition of the edge set of complete graphs into complete subgraphs of the same size. The case we are interested in is as follows.

▶ **Lemma 8** (Wilson 1975). *Let n be a large enough positive integer. Then the edge set of K_n can be decomposed into subgraphs of the form K_6, if and only if n satisfies two divisibility conditions, namely $5 | n-1$ and $\binom{6}{2} | \binom{n}{2}$.*

Moreover, if n does not satisfy the above divisibility conditions, then one can still pack $n^2/30 - O(n)$ K_6's into K_n, leaving out only linearly many edges.

Our approach is packing as many K_6 as possible into $K_n[A]$, then decomposing each K_6 into five 3-edge zig-zag paths, and considering each of the remaining edges as a single path. This way we can guarantee a decomposition of the edge set of $K_n[A]$ into $\binom{n}{2}/\binom{6}{2} \times 5 + O(n) = n^2/6 + O(n)$ monotone paths.

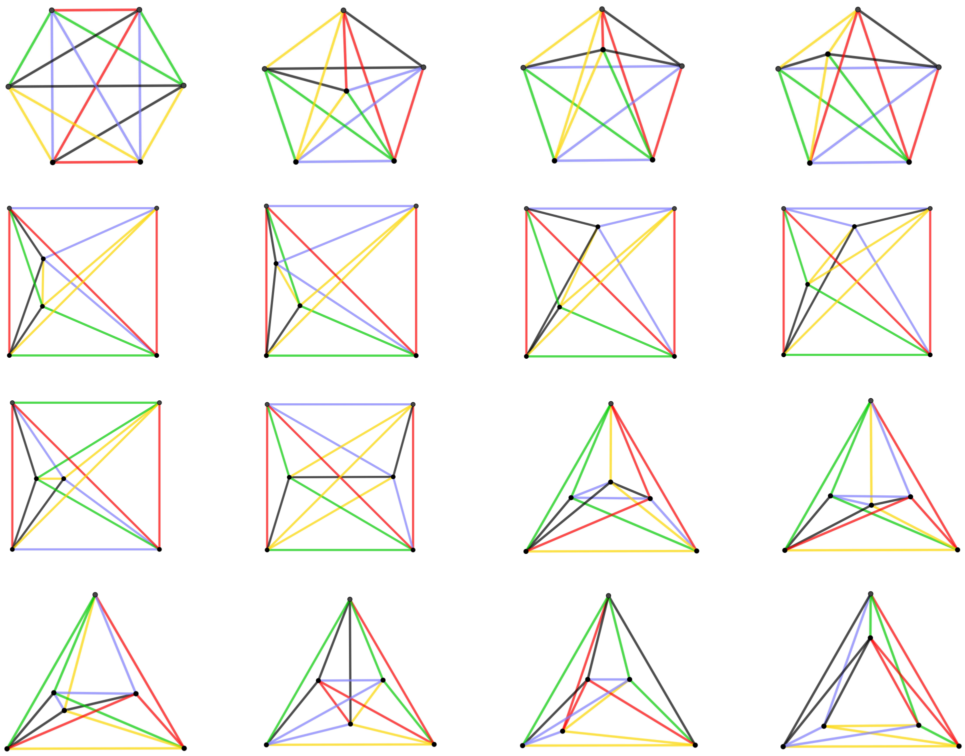

Figure 3 Covering the edge set of the complete graph on six points by 3-edge zig-zag paths.

It suffices to show that for a 6-point set A, the edge set of $K_6[A]$ can be decomposed into five 3-edge zig-zag paths. It is known that there are 16 order types for such sets, see, e.g., [16]. Since the existence of a zig-zag path decomposition depends only on the order type and not on the specific geometric realization, it suffices to provide zig-zag path decompositions for each type, as we do in Fig. 3. ◀

A similar bipartite version of the lower bound construction with $n = 2k$ points and $|E_0| = k^2$ (undirected) inter-group edges only yields a lower bound of $n^2/16$: If $\mathcal{P}$ is a covering of $K_n[A]$ by monotone paths, it has the property that every path $\xi \in \mathcal{P}$ contains at most four edges in E_0. Consequently, covering all the edges of the bipartite graph requires at least $k^2/4 = n^2/16$ monotone paths.

5 Concluding remarks

1. For $k \geq 2$, let $f(k, n)$ denote the maximum number of edges of a geometric graph on n vertices that contains no k pairwise disjoint edges. Early results of Erdős [7], Kupitz [11], and Alon and Erdős [2] show that $f(2, n) = n$ and $f(3, n) = O(n)$. The first polynomial upper bound for $f(k, n)$, for $k > 3$, was established by Pach and Törőcsik [15], and was improved by Tóth and Valtr [19]; see also [13, Chap. 14]. The current best result, $f(k, n) = O(k^2 n)$ is due to Tóth [18].

It is conjectured that $f(k, n) = O(kn)$ [5, Chap. 9.5]. If this conjecture is true, our methods would yield an $O(n \log n)$ (unfortunately, still superlinear) upper bound on the number of monotone matchings sufficient for covering the edge set of $K_n[A]$, analogous to covering this set by crossing-free matchings in [3].

2. Obenaus and Orthaber [12] gave a negative answer to the question of whether every complete geometric graph on n vertices (n even) can be partitioned into $n/2$ spanning trees (see [4]). Furthermore, their negative answer extends to the weaker question of whether every complete geometric graph on n vertices (n even) can be partitioned into $n/2$ plane subgraphs. See also [1].

On the other hand, it is possible that every complete geometric graph on n vertices can be partitioned into $n/2 + o(n)$ plane subgraphs. Pach, Saghafian, and Schnider [14] proved that a complete convex geometric graph on n vertices cannot be decomposed into fewer than $n - 1$ plane star-forests.

Perhaps the first step towards solving Problem 1 would be to answer the following question.

▶ **Problem 9.** *Does there exist a constant $c > 1/2$ with the property that, for infinitely many values of n, there are n-element point sets A in the plane such that every covering of the edge set of $K_n[A]$ by crossing-free paths requires at least cn paths?*

3. Pinchasi and Yerushalmi [17] showed that, given any n-element point set A in the plane where n is odd, the edge set of $K_n[A]$ can be partitioned into $(n^2 - 1)/8$ convex polygons whose vertices belong to A. This bound is tight. For the case of even n, $n^2/8 + n/4$ convex polygons suffice and $\lceil n^2/8 \rceil$ are needed. Since every convex polygon can be decomposed into two monotone paths, we essentially re-obtain the upper bound on monotone paths that comes from covering by two-edge paths: if n is odd, $(n^2 - 1)/4$ monotone paths suffice, whereas if n is even, $n^2/4 + n/2$ monotone paths suffice. These bounds are superseded by our Theorem 6.

4. Another interesting class of paths is zig-zag paths. A path is called *zig-zag* if any three consecutive edges form a 3-edge zig-zag path, as defined earlier. Note that this definition does not ensure that the path is monotone or even plane.

We show that for any finite point set A in general position in the plane, one can cover the edge set of $K_n[A]$ by plane Hamiltonian zig-zag paths. Equivalently, for any pair of points $a, b \in A$, the segment ab appears in at least one such path. To see this, assume that the line through a, b separates $A \setminus \{a, b\}$ into two disjoint parts A_1, A_2. The edge ab is an edge of the convex hull of $A_i \cup \{a, b\}$, for $i = 1, 2$. Starting with ab, we construct a plane zig-zag path that traverses all points in $A_1 \cup \{a, b\}$, as follows. At the first step, let $b_1 \in A_1$ be the point such that the angle $\angle abb_1$ is as small as possible. We continue this process by selecting b_i at the i-th step such that the angle $\angle b_{i-2}b_{i-1}b_i$ is minimized among all remaining points in A_1, where $b_0 = b$. It is not hard to verify that once all points in A_1 are included, the resulting path is a plane zig-zag path. Similarly, starting with ba we can construct a plane zig-zag path that traverses all points in $A_2 \cup \{a, b\}$. The union of these two paths is a plane zig-zag Hamiltonian path, as claimed.

While the edge set of any complete geometric graph can be covered by plane zig-zag (Hamiltonian) paths, the minimum number of (not necessarily Hamiltonian) plane zig-zag paths required for covering the edge set of $K_n[A]$ is not known.

▶ **Problem 10.** *What is the smallest number $z = z(n)$ such that the edge set of every complete geometric graph on n vertices can be covered by z plane zig-zag paths?*

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