A multidimensional Ramsey Theorem

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Abstract

Ramsey theory is a central and active branch of combinatorics. Although Ramsey numbers for graphs have been extensively investigated since Ramsey's work in the 1930s, there is still an exponential gap between the best known lower and upper bounds. For k-uniform hypergraphs, the bounds are of tower-type, where the height grows with k. Here we give a multidimensional generalisation of Ramsey's Theorem to cartesian products of graphs, proving that a doubly exponential upper bound is enough in every dimension. More precisely, we prove that for every $r, n, d \in \mathbb{N}$, in any r-colouring of the edges of the Cartesian product $\square^d K_N$ of d copies of K_N there is a copy of $\square^d K_n$ such that the edges in each direction are monochromatic, provided $N \geq 2^{2^{C(d)rn^d}}$. As an application of our new approach we also obtain improvements on the multidimensional Erdős-Szekeres Theorem proved by Fishburn and Graham 30 years ago. Their bound was recently improved by Bucić, Sudakov, Tran, who gave an upper bound that is triply exponential in four or more dimensions. We improve upon their results showing that a doubly exponential upper bound holds in any number of dimensions.

1 Introduction

The study of Ramsey theory is a longstanding and central part of combinatorics. As usual, for positive integers r, k, the Ramsey number $R_r(k)$ is the smallest n for which every r-colouring of K_n contains a monochromatic copy of K_k. Ramsey [15] showed in 1930 that these numbers exist. Since then, Ramsey numbers have been studied extensively, upper and lower bounds have been proved and many generalisations have been considered, see, e.g. [4, 5, 6, 12, 16, 20, 22]. Even for $r = 2$, the asymptotics are still not fully resolved. It is not hard to show that $R_2(k)$ grows at exponential rate, but the constant in the exponent is yet not known: the best current bounds are $(1 + o(1))\frac{\sqrt{2}(k+1)}{e}2^{k+1/2} \leq R_2(k + 1) \leq e^{-c\log^2 k}\binom{2k}{k}$ the lower bound due to Spencer [20] and upper bound by Sah [16]. For larger r, the bounds are less good, and even for $k = 3$, we do not understand the behaviour of $R_r(3)$ as $r \to \infty$. Very recently, Conlon and Ferber [7], Wigderson [23], and Sawin [17] found nice constructions which give the best known lower bounds for $R_r(k)$ when $r \geq 3$.

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In this paper, we prove a multidimensional generalisation of Ramsey's Theorem for cartesian products of graphs. Given two graphs H and G, we write $G \,\square\, H$ for the Cartesian product of H and G, namely the graph with vertex set $V(G) \times V(H)$ in which (x, y) is joined to (x', y') if and only if $x = x'$ and $yy' \in E(H)$ or $xx' \in E(G)$ and $y = y'$. The Cartesian product is associative, so it makes sense to write $G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d$ (without brackets) for the Cartesian product of d graphs; we write $\square^d G$ for the product $G \square \cdots \square G$ of d copies of G. Note that in a Cartesian product of d graphs $G_1, \ldots, G_d$, there is an edge between $v = (v_1, \ldots, v_d)$ and $w = (w_1, \ldots, w_d)$ if and only if there is some i such that $v_i w_i$ is an edge of G_i and $v_j = w_j$ for $j \neq i$, and in this case we will say that the edge vw is in *direction* i.

Given a colouring c of the edges of $\square^d K_n$, we say that c is *monochromatic in every direction* if for each $i \in \{1, \ldots, d\}$ there is some c_i such that all edges in direction i have colour c_i. For positive integers r, d, we define $R_r(d, n)$ to be the smallest N such that every r-colouring of the edges of $\square^d K_N$ contains a copy of $\square^d K_n$ that is monochromatic in every direction. Note that we cannot demand a copy of $\square^d K_n$ that has the *same* colour in every direction, as we are asking for a full-dimensional subgraph: for example, $\square^d K_N$ could be coloured with colour 1 for all edges in direction 1 and coloured with 2 for the edges in all other directions. It is easy to see that if we demand a monochromatic copy of $\square^\ell K_n$ then ℓ must be at most $\lceil d/r \rceil$. It will follow from Theorem 1.1 that this is also tight.

It is not too hard to prove that $R_r(d, n)$ exists by an iterated application of Ramsey's Theorem. However, this gives an upper bound of tower-type. Our main goal is to show that a doubly exponential bound on n^d suffices.

Theorem 1.1. *Let d be a positive integer. There exists $C_d > 0$ such that for every n, r the following holds. For $N \geq r^{r^{C_d r n^d}}$, every r-colouring of $\square^d K_N$ contains a copy of $\square^d K_n$ which is monochromatic in every direction. That is, $R_r(d, n) \leq r^{r^{C_d r n^d}}$.*

As an immediate corollary, we see that any r-edge colouring of $\square^d K_N$ contains a *monochromatic* copy of $\square^\ell K_n$ for $\ell = \lceil d/r \rceil$.

Corollary 1.2. *Let n, d, r be positive integers and $\ell = \lceil d/r \rceil$. For $N \geq r^{r^{C_d r n^d}}$, every r-edge-colouring of $\square^d K_N$ contains a monochromatic copy of $\square^\ell K_n$. The value of ℓ is tight.*

Another foundational result in Ramsey theory appears in a paper Erdős and Szekeres [6] from 1935: any sequence of $n^2 + 1$ distinct real numbers contains either an increasing or decreasing subsequence of length $n + 1$. This simple result was one of the starting seeds for the development of Ramsey theory (see, for instance, [19] for proofs and applications). There are a number of different ways to generalise the Erdős-Szekeres Theorem to higher dimensions (see, for example, [2, 3, 10, 11, 13, 14, 18, 21]). Perhaps the most natural approach was developed thirty years ago by Fishburn and Graham [9]. A *d-dimensional array* is an injective function f from $A_1 \times \ldots \times A_d$ to $\mathbb{R}$ where $A_1, \ldots A_d$ are non-empty subsets of $\mathbb{Z}$; we say f has *size* $|A_1| \times \cdots \times |A_d|$; if $|A_i| = n$ for each i, it will be convenient to say that f has *size* $[n]^d$. A multidimensional array is said to be *monotone* if for each direction all the 1-dimensional subarrays in that direction are increasing or decreasing. In other words, for every i, one of the following holds:

- For every choice of a_j, $j \neq i$, the function $f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_d)$ is increasing in x.
- For every choice of a_j, $j \neq i$, the function $f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_d)$ is decreasing in x.

Let $M_d(n)$ be the smallest N such that a d-dimensional array on $[N]^d$ contains a monotone d-dimensional subarray of size $[n]^d$. Fishburn and Graham [9] showed that $M_d(n)$ exists but their upper bounds were a tower of height d. Recently, Bucić, Sudakov, and Tran [1] proved considerably better upper bounds on $M_d(n)$, showing doubly exponential bounds in n^{d-1} for 2 and 3 dimensions, and triply exponential bounds in 4 or higher dimensions.

Theorem 1.3. *(Bucić, Sudakov, and Tran [1])*

i) $M_2(n) \leq 2^{2^{(2+o(1))n}}$,

ii) $M_3(n) \leq 2^{2^{(2+o(1))n^2}}$,

iii) $M_d(n) \leq 2^{2^{2^{O_d(n^{d-1})}}}$, *for* $d \geq 4$.

Bucić, Sudakov, and Tran [1] asked whether a better bound could be proved in four and higher dimensions, speculating that the triply exponential bound could be reduced to a doubly exponential bound. Using the methods from our proof of Theorem 1.1, we resolve their question, showing that a doubly exponential upper bound holds in all dimensions.

Theorem 1.4. *For every $d \geq 2$, there is $C_d > 0$, such that for every positive n, $M_d(n) \leq 2^{n^{C_d n^{d-1}}}$.*

We note that the arguments of [1] make significant use of the fact that the Erdős-Szekeres Theorem has a polynomial bound: it is iterated n^{d-1} times to obtain the bounds for $d \geq 4$ in Theorem 1.3. Ramsey's Theorem, however, has an exponential bound and so a different approach is required to avoid getting a tower-type upper bound in Theorem 1.1. In our proof, we first find a new intermediate notion of a structured object which we call *d-consistent* and then we show that a density-type argument on these structured objects allows us to get a monochromatic $\square^d K_n$. We then use this new approach to improve the upper bound in Theorem 1.3, leading to Theorem 1.4.

We prove Theorem 1.1 in Section 2 and Theorem 1.4 in Section 3. We conclude with some further discussion in Section 4.

2 Upper bound on $R_r(d, n)$

To simplify notation we will identify the vertex set of K_N with $[N] = \{1, \ldots, N\}$ and the vertex set of $K_N \,\square \cdots \square\, K_N$ with $[N]^d$. Suppose that, for each $i \in [d]$ we have a graph G_i and a subgraph $H_i \subseteq G_i$. For $a \in V(G_d)$, we write $H_1 \,\square \cdots \square\, H_{d-1} \,\square\, a$ for the copy of $H_1 \,\square \cdots \square\, H_{d-1} \subseteq G_1 \,\square \cdots \square\, G_d$ with vertex set $V(H_1) \times \cdots \times V(H_{d-1}) \times \{a\}$ (thus all vertices have a as their dth coordinate; we will usually omit the braces around $\{a\}$ for simplicity). We define subgraphs such as $a_1 \,\square \cdots \square\, a_{d-1} \,\square\, H_d$ analogously. We will sometimes refer to induced subgraphs by their vertex sets: thus, for an edge-coloured graph G we say that $S \subset V(G)$ *contains a monochromatic* K_k if there is some set $T \subset S$ such that the induced subgraph $G[T]$ is a monochromatic copy of K_k.

We begin with a short proof for the case $d = 2$, and then give a (more involved) argument for the general case $d \geq 3$.

Proof of Theorem 1.1 for $d = 2$. Note that $R_r(1, t)$ is just the usual Ramsey number for K_t, which is smaller than r^{rt}. Let $N := r^{r^{10rn^2}}$, and consider an r-colouring of the edges of $K_N \,\square\, K_N$. Fix a set $S \subset N$ of size $r^{r^{3rn^2+1}}$. By Ramsey's Theorem, for each $i \in [N]$ we can find a monochromatic copy of $K_{r^{3rn^2}}$ in $S \times i$. Note that there are at most $r \cdot \binom{|S|}{r^{3rn^2}} \leq r^{r^{7rn^2}}$ choices for the vertex set of each monochromatic copy of $K_{r^{3rn^2}}$ and its colour. Since $N/r^{r^{7rn^2}} \geq r^{2rn}$, a pigeonhole argument shows that there is a set A_1 of size r^{3rn^2} and a set $A_2 \subset [N]$ of size at least r^{2rn} such that, for each $a_2 \in A_2$ the set $A_1 \times a_2$ induces a monochromatic copy of $K_{r^{3rn^2}}$ (all with the same choice of colour). Applying Ramsey's Theorem once again, we can find in each set $b \times A_2$ a monochromatic copy of K_n (as $|A_2| = r^{2rn}$). As there are at most $r \cdot \binom{|A_2|}{n} \leq r^{2rn^2}$ choices for the vertex set and colour of this K_n, we can apply the pigeonhole principle again: there is a set $A_1' \subset A_1$ of size at least $|A_1|/r^{2rn^2} \geq k$ and a set $A_2' \subset A_2$ of size n such that, for every $a_1 \in A_1'$, the set $a_1 \times A_2'$ forms a monochromatic copy of K_n (and all with the same choice of colour). It is clear that $A_1' \times A_2'$ forms a copy of $K_n \,\square\, K_n$ that is monochromatic in both directions. Hence $R_r(2,n) \leq N$, as we wanted to show. $\square$

We now turn to the general argument for $d \geq 3$. Consider an r-edge-coloured product of complete graphs. We say a 1-dimensional K_k is 1-*consistent* if it is monochromatic. For $d \geq 2$, we say $\square^d K_k$ is d-*consistent* if the following two conditions hold:

- for every $a \in [k]$, the subgraph $(\square^{d-1} K_k) \,\square\, a$ has the same colour pattern: for every edge $\mathbf{xy}$ in $\square^{d-1} K_k$, the colour of the edge between $\mathbf{x} \times a$ and $\mathbf{y} \times a$ is the same for every $a \in [k]$; and
- for some (and thus for every) a, the subgraph $\square^{d-1} K_k \,\square\, a$ is $(d-1)$-*consistent.*

In other words, for each $i \in [d]$, the 'i-dimensional subspaces' $(\square^i K_N) \,\square\, a_{i+1} \,\square \cdots \square\, a_d$ have the same colouring for every choice of $a_{i+1}, \ldots, a_d$.

The proof splits into two lemmas. Given suitable $N \gg k \gg n$ and an r-edge colouring of $\square^d K_N$, we first find a d-*consistent* subgraph $H = \square^d K_k$. We then show that H contains a monochromatic copy of $\square^d K_n$.

Lemma 2.1. *For every $d \geq 1$ there is a constant $g(d)$ such that the following holds. Let r, k be positive integers and $0 < \epsilon < 1/2$. Suppose that $N \geq \epsilon^{-g(d)k^{d-1}} \cdot r^{g(d)rk^d}$ and $\square^d K_N$ has a d-consistent r-edge-colouring. Then every set S of at least ϵN^d vertices contains a copy of $\square^d K_k$ which is monochromatic in every direction.*

Proof. We argue by induction on d that $g(d) = (12+2r)^{d-1}$ will do. It is clear that when $d = 1$ it is sufficient to have $N \geq r^{rk}/\epsilon$, which we do as $g(1) = 1$. So we assume that $d \geq 2$ and we have handled smaller cases. Let T be the set of elements $\mathbf{v} \in [N]^{d-1}$ such that $\mathbf{v} \times [N]$ contains at least $\epsilon N/2$ elements of S. A counting argument shows that $|T| \geq \epsilon N^{d-1}/2$. Let $A \subset [N]$ be a random subset of size $(10/\epsilon) r^{rk}$. For each $\mathbf{v} \in T$, with probability at least $2/3$ the set $\mathbf{v} \times A$ contains at least r^{rk} elements of S. So we can choose A so that the set

$$T' := \{\mathbf{v} \in T : |(\mathbf{v} \times A) \cap S| \geq r^{rk}\}$$

has size at least $(2/3)|T| \geq (\epsilon/3) N^{d-1}$. By Ramsey's Theorem, for each $\mathbf{v} \in T'$, there is a monochromatic copy of K_k contained in $(\mathbf{v} \times A) \cap S$, say on vertices $B_\mathbf{v} \subset A$. There are at most $r\binom{(10/\epsilon)r^{rk}}{k} \leq \epsilon^{-9k} r^{rk^2+1}$ choices for $B_\mathbf{v}$ and the colour of the corresponding copy of K_k, so there are $B \subset A$ and $U \subset T'$ such that

$$|U| \geq \epsilon^{9k} r^{-rk^2-1} |T'| \geq (\epsilon/3) \epsilon^{9k} r^{-rk^2-1} N^{d-1}$$

and $B_\mathbf{v} = B$ for every $\mathbf{v} \in U$. Now let $\tilde{\epsilon} = (\epsilon/3)\epsilon^{9k} r^{-rk^2-1} \geq \epsilon^{12k} r^{-rk^2-1}$, so $|U| \geq \tilde{\epsilon} N^{d-1}$. We now apply the inductive hypothesis to $[N]^{d-1}$, with the colouring inherited from $[N]^{d-1} \times a$ (which by consistency is the same for any $b \in B$) with the subset U playing the role of S. Since $N \geq \epsilon^{-g(d)k^{d-1}} r^{g(d)rk^d} \geq \tilde{\epsilon}^{-g(d-1)k^{d-2}} \cdot r^{g(d-1)rk^{d-1}}$, we obtain a $(d-1)$-dimensional product P of copies of K_k which is monochromatic in every direction. Then $P \times B$ gives a d-dimensional product of copies of K_k which is monochromatic in every direction (as $P \times b$ is coloured in the same way for every $b \in B$, and all sets $\mathbf{v} \times B$ ($\mathbf{v} \in B$) give monochromatic copies of K_k with the same colour). $\square$

Lemma 2.2. *For every positive integer d there is $f(d)$ such that for every r, k the following holds. Let $N := r^{f(d)rk^d}$. Then in every r-colouring of $\square^d K_N$ there is a d-consistent $\square^d K_k$.*

Proof. We argue by induction on d that $f(d) = 2^{d-1}d!$ will do. For $d = 1$, this is true by Ramsey's Theorem. Now, let $M := r^{f(d-1)rk^{d-1}}$ and consider the subgraph $(\square^{d-1} K_M) \,\square\, K_N$. By induction any r-colouring of $\square^{d-1} K_M$ contains a $(d-1)$-consistent $\square^{d-1} K_k$. Therefore, for every $a \in [N]$ the subgraph $\square^{d-1} K_M \times a$ contains a $(d-1)$-consistent $\square^{d-1} K_k \times a$. As there are most $r^{(d-1)k^{2+(d-2)}}$ possible r-colourings of $\square^{d-1} K_k$ and at most $\binom{M}{k}^{d-1}$ possible vertex sets, there are at most $r^{(d-1)k^{2+(d-2)}}\binom{M}{k}^{d-1} \leq r^{(d-1)k^d} M^{k(d-1)}/k < r^{f(d)rk^d}/k = N/k$ possible combinations. Thus, by the pigeonhole principle, there is a $(d-1)$-consistent colour pattern c of $\square^{d-1} K_k$ and a set $A \subset [N]$ of size k, such that for every $a \in A$, the subgraph $\square^{d-1} K_k \times a$ has colour pattern c, and all these boxes lie on the same vertex set in the first $d-1$ dimensions, as we wanted to show. $\square$

Proof of Theorem 1.1. We will show that the statement holds with $C_d = 3dg(d-1) + f(d) + 1$, where f and g are any functions satisfying the previous two lemmas. Let $N := r^{r^{C_d rn^d}}$. Let $t := r^{3g(d-1)rn^d}$ and $u := r^{rn}$. Applying Lemma 2.2 to $\square^d K_N$, we obtain a d-consistent copy B of $\square^d K_t$. Relabelling, and restricting the final coordinate to u choices, we may assume that we have a d-consistent colouring of $\square^{d-1} K_t \,\square\, K_u$.

For every $a_1 \times \cdots \times a_{d-1}$, where each $a_i \in [t]$, we can apply Ramsey's Theorem to $a_1 \,\square \cdots \square\, a_{d-1} \,\square\, K_u$ to get a monochromatic copy of K_n. There are $\binom{u}{n} \cdot r$ choices of colour and coordinates C for this copy; setting $\epsilon = 1/(ru^n)$ we see that there is some set S of at least ϵt^{d-1} vertices in $\square^{d-1} K_t$ for which both colour and coordinates agree.

Finally, we apply Lemma 2.1 to $S \subset \square^{d-1} K_t$ and obtain a copy K of $\square^{d-1} K_n$ contained in S that is monochromatic in every direction (we need $t \geq \epsilon^{-g(d-1)n^{d-2}} r^{g(d-1)rn^{d-1}}$, which holds by our choice of constants, as $\epsilon^{-g(d-1)n^{d-2}} = (ru^n)^{g(d-1)n^{d-2}} = r^{(rn^2+1)g(d-1)n^{d-2}}$). Then $K \,\square\, C$ is a copy of $\square^d K_n$ that is monochromatic in every direction. $\square$

3 Upper bound on $M_d(n)$

We note first that an immediate application of Theorem 1.1 gives an upper bound of $M_d(n) \leq 2^{2^{C_d n^d}}$. However, we wish to get a dependence of $d-1$ in the exponent in Theorem 1.4, and so prove it separately.

The proof follows along the same route as the proof of Theorem 1.1. However, we need an modified version of Lemma 2.1 where the exponent is slightly better. We will assume all the arrays hereafter are injective (which we can ensure by a small perturbation to the values). We say a 1-dimensional array is *consistent* if it is monotone. For $d \geq 2$, we say an array $f : [N]^d \to \mathbb{R}$ is d-consistent if the following two conditions hold:

- for every $a \in [N]$, f restricted to $[N]^{d-1} \times \{a\}$ has the same order pattern i.e. for every $\mathbf{x}, \mathbf{y} \in [N]^{d-1}$, if $f(\mathbf{x} \times a) < f(\mathbf{y} \times a)$, for some a then the same holds for all $a \in [N]$; and
- for some (and thus for every) a, the $(d-1)$-dimensional subarray $f : [N]^{d-1} \times a \to \mathbb{R}$ is $(d-1)$-*consistent*.

As in Section 2, we will show that being d-consistent with the appropriate parameters is enough for our purpose.

Lemma 3.1. *For every $d \geq 1$, there is a constant $g(d)$ such that the following holds. Let r, k be positive integers and $0 < \epsilon < 1/2$. Suppose that $N \geq \epsilon^{-g(d)k^{d-1}} \cdot k^{g(d)k^{d-1}}$ and the array $f : [N]^d \to \mathbb{R}$ is d-consistent. Then every $S \subseteq [N]^d$ of size at least ϵN^d contains a monotone subarray of size $[k]^d$.*

Proof. We follow the proof of Lemma 2.1, with k^2 playing the role of r^{rk}, $g(d) = 2 \cdot 15^{d-1}$, and with the remark that in the 1-dimensional case, the Erdős-Szekeres Theorem implies that it is enough to have $N \geq \epsilon^{-1}(2k)^2$. We include the full details here for completeness.

We argue by induction on d that $g(d) = 2 \cdot 15^{d-1}$ will do. It is clear that when $d = 1$ it is sufficient to have $N \geq k^2/\epsilon$, which we do as $g(1) = 2$. So we assume that $d \geq 2$ and we have handled smaller cases. Let T be the set of elements $\mathbf{v} \in [N]^{d-1}$ such that $\mathbf{v} \times [N]$ contains at least $\epsilon N/2$ elements of S. A counting argument shows that $|T| \geq \epsilon N^{d-1}/2$. Let $A \subset [N]$ be a random subset of size $(10/\epsilon)k^2$. For each $\mathbf{v} \in T$, with probability at least $2/3$ the set $\mathbf{v} \times A$ contains at least k^2 elements of S. So we can choose A so that the set

$$T' := \{\mathbf{v} \in T : |(\mathbf{v} \times A) \cap S| \geq k^2\}$$

has size at least $(2/3)|T| \geq (\epsilon/3)N^{d-1}$. By the Erdős-Szekeres Theorem, for each $\mathbf{v} \in T'$, there is a monotone subarray of size k contained in $(\mathbf{v} \times A) \cap S$, say on vertices $B_{\mathbf{v}} \subset A$. There are at most $2\binom{(10/\epsilon)k^2}{k} \leq \epsilon^{-9k}k^{2k}$ choices for $B_{\mathbf{v}}$ and the direction of monotonicity (i.e. increasing or decreasing), so there are $B \subset A$ and $U \subset T'$ such that

$$|U| \geq \epsilon^{9k}k^{-2k}|T'| \geq (\epsilon/3)\epsilon^{9k}k^{-2k}N^{d-1}$$

and $B_{\mathbf{v}} = B$ for every $\mathbf{v} \in U$. Now let $\tilde{\epsilon} = (\epsilon/3)\epsilon^{9k}k^{-2k} \geq \epsilon^{12k}k^{-2k}$, then $|U| \geq \tilde{\epsilon}N^{d-1}$. We now apply the inductive hypothesis to the subset $U \subseteq [N]^{d-1}$, with the values inherited from $[N]^{d-1} \times b$ (which by consistency are the same for any $b \in B$). Since $N \geq \epsilon^{-g(d)k^{d-1}}k^{g(d)k^{d-1}} \geq \tilde{\epsilon}^{-g(d-1)k^{d-2}} \cdot k^{g(d-1)k^{d-2}}$, we obtain a $(d-1)$-dimensional monotone subarray of size $[k]^{d-1}$. Then $P \times B$ gives a monotone subarray of size $[k]^d$. □

It is thus enough to find a d-consistent array. The following lemma is analogue of Lemma 2.2 for arrays.

Lemma 3.2. *For every positive integer d there is $f(d)$ such that for every k the following holds. Let $N := k^{f(d)k^{d-1}}$. Then in every d-dimensional array of size $[N]^d$ there is a d-consistent subarray of size $[k]^d$.*

Proof. We argue by induction on d that $f(d) = 2^{d-1}d! + 1$ will do. For $d = 1$, this is true by the Erdős-Szekeres' Theorem. Now let $M := k^{f(d-1)k^{d-2}}$. By induction any array on $[M]^{d-1}$ contains a $(d-1)$-consistent array of size $[k]^{d-1}$. Therefore, for every $a \in [N]$ the $(d-1)$-dimensional array on $[M]^{d-1} \times \{a\}$ contains a $(d-1)$-consistent array on $A_1 \times \cdots \times A_{d-1} \times \{a\}$, where $|A_i| = n$ for all $i \in [d-1]$. As there are most $k^{(d-1)k^{1+(d-2)}}$ possible orderings of an array of size k^{d-1}, and at most $\binom{M}{k}^{d-1}$ choices for the sets A_i, there are at most $k^{(d-1)k^{1+(d-2)}}\binom{M}{k}^{d-1} < k^{(d-1)k^{d-1}}M^{k(d-1)}/k \leq k^{f(d)k^{d-1}}/k = N/k$ possible combinations. Thus, by the pigeonhole principle there are an ordering $\mathcal{O}$ of $[k]^{d-1}$ and a set $A \subset [N]$ of size k, where $[k]^{d-1} \times \{a\}$ have the same ordering $\mathcal{O}$ for every $a \in A$ and all these $(d-1)$-dimensional arrays lie on the same vertex set in the first $d-1$ dimensions, as we wanted to show. □

Proof of Theorem 1.4. We will show that the statement holds with $C_d = 4dg(d-1) + f(d)$, where f and g are any functions satisfying Lemma 3.1 and Lemma 3.2. Let $N := n^{n^{C_d n^{d-1}}}$. Let $t := n^{3g(d-1)n^{d-1}}$ and

$u := n^2$. Applying Lemma 3.2 to $[N]^d$, with t playing the role of k, we obtain a d-consistent array on $B_1 \times \cdots \times B_d := B$, where $|B_i| = t$ for all $i \in [d]$. Relabelling, and restricting the final coordinate to u choices, we may assume that we have a d-consistent array on $[t]^{d-1} \times [u]$. For every $a_1 \times \cdots \times a_{d-1} \in [t]^{d-1}$, we can apply the Erdős-Szekeres Theorem to the 1-dimensional array on $a_1 \times \cdots \times a_{d-1} \times [u]$ to get a monotone subarray of size n. By a simple counting argument there are at most $\binom{u}{n} \cdot 2$ choices for the coordinates of the subarray and whether it is increasing or decreasing, and hence there is some choice of these which occurs on a fraction of at least $(2 \cdot \binom{u}{n})^{-1} := \hat{\epsilon}$ of the vertices of $[t]^{d-1}$, say on a set S. let $A \subseteq [u]$ be the common choice of coordinates for the monotonic subarrays corresponding to S.

Finally, we apply Lemma 3.1 to $S \subseteq [t]^{d-1}$. By the choices of $t, u, \hat{\epsilon}$ we have that $t \geq \hat{\epsilon}^{-g(d-1)n^{d-2}} \cdot n^{g(d-1)n^{d-2}}$, and so we obtain a monotone $(d-1)$-dimensional array T of size $[u]^{d-1}$. Since B is d-consistent, $T \times \{a\}$ is monotone for every $a \in A$ (with the same choice of direction of monotonicity). By construction, this gives a monotone d-dimensional array on $T \times A$. $\square$

4 Concluding remarks

In Theorem 1.1, we have given a doubly exponential upper bound on the d-dimensional Ramsey numbers. From below, we have only a singly exponential bound (which follows easily by considering random colourings). It would be very interesting to close the gap. In particular, it would be good to know whether there is a simple exponential upper bound, or whether the numbers grow more quickly.

Problem 4.1. *Fix $r, d \geq 2$. Is $R_2(2, n)$ superexponential in n?*

The same gap between lower and upper bounds is seen in the multimensional Erdős-Szekeres Theorem. Bucić, Sudakov and Tran [1] gave a doubly exponential upper bound for $d = 2, 3$; and Theorem 1.4 gives a doubly exponential upper bound for $d \geq 4$ (improving on the previous triply exponential upper bound [1]). But a frustrating gap between singly and doubly exponential bounds remains.

It is easy to see that $M_2(n) \leq R_2(2, n)$ for every n. It would interesting to see if there is some relation the other way, and the two functions have similar behaviour (such as both singly exponential or both double exponential). For example, is the following true?

Problem 4.2. *Is $\log R_2(2, n)$ bounded from above by some polynomial in $\log(M_2(n))$?*

We finish the paper by pointing out that our results also give improvements for multidimensional lexicographic-monotone array. In their paper, Fishburn and Graham [9] introduced another natural generalisation for monotone sequences and the Erdős-Szekeres theorem, which they called a *lex-monotone array*. A d-dimensional array f is said to be *lex-monotone* if the following holds. There exists a permutation $\tau \in S_{[d]}$ and a sign vector $s \in \{-1, 1\}^d$, such that $f(\mathbf{x}) < f(\mathbf{y})$ if and only if the vector $(s_{\tau(i)} x_{\tau(i)})_{i \in [d]}$ is smaller in lexicographic ordering than vector $(s_{\tau(i)} y_{\tau(i)})_{i \in [d]}$, that is, if there is $i \in [d]$ such that $s_{\tau(j)} x_{\tau(j)} = s_{\tau(j)} y_{\tau(j)}$ for every $j < i$, and $s_{\tau(i)} x_{\tau(i)} < s_{\tau(i)} y_{\tau(i)}$. Let $L_d(n)$ be the smallest N such that a d-dimensional array on $[N]^d$ contains a lex-monotone d-dimensional subarray of size $[n]^d$. Fishburn and Graham gave a tower bound of order $d-1$ on $L_d(n)$, which was improved by Bucić, Sudakov and Tran to a tower of order 5 for $d \geq 4$ (and triple exponential for $d = 3$). We note that using Theorem 1.4 together with Theorem 1.2 from [1] we obtain a triple exponential upper bound for all $d \geq 3$, that is, $L_d(n) \leq 2^{2^{2^{C_d n^{d-2}}}}$.

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