

Polynomial bounds for chromatic number.  
IV. A near-polynomial bound for excluding the five-vertex path

Alex Scott<sup>1</sup>

Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour<sup>2</sup>

Princeton University, Princeton, NJ 08544

Sophie Spirkl<sup>3</sup>

University of Waterloo, Waterloo, Ontario N2L3G1, Canada

<sup>1</sup>Research supported by EPSRC grant EP/V007327/1.

<sup>2</sup>Supported by AFOSR grant A9550-19-1-0187, and by NSF grant DMS-1800053.

<sup>3</sup>We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC) [funding reference number RGPIN-2020-03912]. Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG) [numéro de référence RGPIN-2020-03912].

### Abstract

A graph  $G$  is  $H$ -free if it has no induced subgraph isomorphic to  $H$ . We prove that a  $P_5$ -free graph with clique number  $\omega \geq 3$  has chromatic number at most  $\omega^{\log_2(\omega)}$ . The best previous result, due to Esperet, Lemoine, Maffray, and Morel, was the exponential upper bound  $(5/27)3^\omega$ . There is a general conjecture, due to Esperet, that would imply a polynomial upper bound; and this would imply the celebrated Erdős-Hajnal conjecture for  $P_5$ , which is the smallest open case of the Erdős-Hajnal conjecture. Thus there is great interest in Esperet's conjecture for  $P_5$ -free graphs, and our result is an attempt to approach that.

# 1 Introduction

If  $G, H$  are graphs, we say  $G$  is  $H$ -free if no induced subgraph of  $G$  is isomorphic to  $H$ ; and for a graph  $G$ , we denote the number of vertices, the chromatic number, the size of the largest clique, and the size of the largest stable set by  $|G|, \chi(G), \omega(G), \alpha(G)$  respectively.

The Gyárfás-Sumner conjecture [6, 10] says:

**1.1 Conjecture:** *For every forest  $H$  there is a function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every  $H$ -free graph  $G$ .*

This is open in general, but has been proved [6] when  $H$  is a path, and for several other simple types of tree (see [8]).

A class of graphs is said to be  $\chi$ -bounded if the class is closed under taking induced subgraphs and there is a function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every graph  $G$  in the class (thus the Gyárfás-Sumner conjecture says that, for every forest  $H$ , the class of  $H$ -free graphs is  $\chi$ -bounded). Louis Esperet [4] made the following remarkable conjecture:

**1.2 Conjecture:** *Let  $\mathcal{G}$  be a  $\chi$ -bounded class. Then there is a polynomial function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every  $G \in \mathcal{G}$ .*

The two conjectures 1.1 and 1.2 would together imply the following:

**1.3 Conjecture:** *For every forest  $H$ , there exists  $c > 0$  such that  $\chi(G) \leq \omega(G)^c$  for every  $H$ -free graph  $G$ .*

This is a beautiful conjecture. In most cases where the Gyárfás-Sumner conjecture has been proved, the current bounds are very far from polynomial, and 1.3 has been only been proved for a much smaller collection of forests. In [9] we proved it for any  $P_5$ -free tree  $H$ , but it has not been settled for any tree  $H$  that contains  $P_5$ . In this paper we focus on the case  $H = P_5$ .

The best previously-known bound on the chromatic number of  $P_5$ -free graphs in terms of their clique number, due to Esperet, Lemoine, Maffray, and Morel [5], was exponential:

**1.4** *If  $G$  is  $P_5$ -free and  $\omega(G) \geq 3$  then  $\chi(G) \leq (5/27)3^{\omega(G)}$ .*

Here we make a significant improvement, showing a “near-polynomial” bound:

**1.5** *If  $G$  is  $P_5$ -free and  $\omega(G) \geq 3$  then  $\chi(G) \leq \omega(G)^{\log_2(\omega(G))}$ .*

(The cycle of length five shows that we need to assume  $\omega(G) \geq 3$ . When  $\omega(G) = 2$ , Sumner [10] showed that  $\chi(G) \leq 3$ .) Conjecture 1.3 when  $H = P_5$  is of great interest, because of a famous conjecture due to Erdős and Hajnal [2, 3], that:

**1.6 Conjecture:** *For every graph  $H$  there exists  $c > 0$  such that  $\alpha(G)\omega(G) \geq |G|^c$  for every  $H$ -free graph  $G$ .*

This is open in general, despite a great deal of effort; and in view of [1], the smallest graph  $H$  for which 1.6 is undecided is the graph  $P_5$ . Every forest  $H$  satisfying 1.3 also satisfies the Erdős-Hajnal conjecture, and so showing that  $H = P_5$  satisfies 1.3 would be a significant result.

We use standard notation throughout. When  $X \subseteq V(G)$ ,  $G[X]$  denotes the subgraph induced on  $X$ . We write  $\chi(X)$  for  $\chi(G[X])$  when there is no ambiguity.

## 2 The main proof

Let  $f$  be a function from the nonnegative integers to the nonnegative real numbers. We say

- $f$  is *non-decreasing* if  $f(y) \geq f(x)$  for all integers  $x, y \geq 0$  with  $y > x \geq 0$ ;
- $f$  is a *binding* function for a graph  $G$  if it is non-decreasing and  $\chi(H) \leq f(\omega(H))$  for every induced subgraph  $H$  of  $G$ ; and
- $f$  is a *near-binding* function for  $G$  if  $f$  is non-decreasing and  $\chi(H) \leq f(\omega(H))$  for every induced subgraph  $H$  of  $G$  different from  $G$ .

In this section we show that if a function  $f$  satisfies a certain inequality, then it is a binding function for all  $P_5$ -free graphs. Then at the end we will give a function that satisfies the inequality, and deduce 1.5.

A *cutset* in a graph  $G$  is a set  $X$  such that  $G \setminus X$  is disconnected. A vertex  $v \in V(G)$  is *mixed* on a set  $A \subseteq V(G)$  or a subgraph  $A$  of a graph  $G$  if  $v$  is not in  $A$  and has a neighbour and a non-neighbour in  $A$ . It is *complete* to  $A$  if it is adjacent to every vertex of  $A$ . We begin with the following:

**2.1** *Let  $G$  be  $P_5$ -free, and let  $f$  be a near-binding function for  $G$ . Let  $G$  be connected, and let  $X$  be a cutset of  $G$ . Then*

$$\chi(G \setminus X) \leq f(\omega(G) - 1) + \omega(G)f(\lfloor \omega(G)/2 \rfloor).$$

**Proof.** We may assume (by replacing  $X$  by a subset if necessary) that  $X$  is a minimal cutset of  $G$ ; and so  $G \setminus X$  has at least two components, and every vertex in  $X$  has a neighbour in  $V(B)$ , for every component  $B$  of  $G \setminus X$ . Let  $B$  be one such component; we will prove that  $\chi(B) \leq f(\omega(G) - 1) + \omega(G)f(\lfloor \omega(G)/2 \rfloor)$ , from which the result follows.

Choose  $v \in X$  (this is possible since  $G$  is connected), and let  $N$  be the set of vertices in  $B$  adjacent to  $v$ . Let the components of  $B \setminus N$  be  $R_1, \dots, R_k, S_1, \dots, S_\ell$ , where  $R_1, \dots, R_k$  each have chromatic number more than  $f(\lfloor \omega(G)/2 \rfloor)$ , and  $S_1, \dots, S_\ell$  each have chromatic number at most  $f(\lfloor \omega(G)/2 \rfloor)$ . Let  $S$  be the union of the graphs  $S_1, \dots, S_\ell$ ; thus  $\chi(S) \leq f(\lfloor \omega(G)/2 \rfloor)$ . For  $1 \leq i \leq k$ , let  $Y_i$  be the set of vertices in  $N$  with a neighbour in  $V(R_i)$ , and let  $Y = Y_1 \cup \dots \cup Y_k$ .

(1) *For  $1 \leq i \leq k$ , every vertex in  $Y_i$  is complete to  $R_i$ .*

Let  $y \in Y_i$ . Thus  $y$  has a neighbour in  $V(R_i)$ ; suppose that  $y$  is mixed on  $R_i$ . Since  $R_i$  is connected, there is an edge  $ab$  of  $R_i$  such that  $y$  is adjacent to  $a$  and not to  $b$ . Now  $v$  has a neighbour in each component of  $G \setminus X$ , and since there are at least two such components, there is a vertex  $u \in V(G) \setminus (X \cup V(B))$  adjacent to  $v$ . But then  $u-v-y-a-b$  is an induced copy of  $P_5$ , a contradiction. This proves (1).

(2)  $\chi(Y) \leq (\omega(G) - 1)f(\lfloor \omega(G)/2 \rfloor)$ .

Let  $1 \leq i \leq k$ . Since  $f(\lfloor \omega(G)/2 \rfloor) < \chi(R_i) \leq f(\omega(R_i))$ , and  $f$  is non-decreasing, it follows that  $\omega(R_i) > \omega(G)/2$ . By (1),  $\omega(G[Y_i]) + \omega(R_i) \leq \omega(G)$ , and so  $\omega(G[Y_i]) < \omega(G)/2$ . Consequently  $\chi(Y_i) \leq f(\lfloor \omega(G)/2 \rfloor)$ , for  $1 \leq i \leq k$ . Choose  $I \subseteq \{1, \dots, k\}$  minimal such that  $\bigcup_{i \in I} Y_i = Y$ . From the minimality of  $I$ , for each  $i \in I$  there exists  $y_i \in Y_i$  such that  $y_i \notin Y_j$  for each  $j \in I \setminus \{i\}$ ;

and so the vertices  $y_i$  ( $i \in I$ ) are all distinct. For each  $i \in I$  choose  $r_i \in V(R_i)$ . For all distinct  $i, j \in I$ , if  $y_i, y_j$  are nonadjacent, then  $r_i-y_i-v-y_j-r_j$  is isomorphic to  $P_5$ , a contradiction. Hence the vertices  $y_i$  ( $i \in I$ ) are all pairwise adjacent, and adjacent to  $v$ ; and so  $|I| \leq \omega(G) - 1$ . Thus  $\chi(Y) = \chi(\bigcup_{i \in I} Y_i) \leq (\omega(G) - 1)f(\lfloor \omega(G)/2 \rfloor)$ . This proves (2).

All the vertices in  $N \setminus Y$  are adjacent to  $v$ , and so  $\omega(G[N \setminus Y]) \leq \omega(G) - 1$ . Moreover, for  $1 \leq i \leq k$ , each vertex of  $R_i$  is adjacent to each vertex in  $Y_i$ , and  $Y_i \neq \emptyset$  since  $B$  is connected, and so  $\omega(R_i) \leq \omega(G) - 1$ . Since there are no edges between any two of the graphs  $G[N \setminus Y], R_1, \dots, R_k$ , their union ( $Z$  say) has clique number at most  $\omega(G) - 1$  and so has chromatic number at most  $f(\omega(G) - 1)$ . But  $V(B)$  is the union of  $Y, V(S)$  and  $V(Z)$ ; and so

$$\chi(B) \leq f(\omega(G) - 1) + (\omega(G) - 1)f(\lfloor \omega(G)/2 \rfloor) + f(\lfloor \omega(G)/2 \rfloor).$$

This proves 2.1. ■

**2.2** Let  $\Omega \geq 1$ , and let  $f$  be a non-decreasing function from the set of nonnegative integers to the set of nonnegative real numbers, satisfying the following:

- $f$  is a binding function for every  $P_5$ -free graph  $H$  with  $\omega(H) \leq \Omega$ ; and
- $f(w - 1) + (w + 2)f(\lfloor w/2 \rfloor) \leq f(w)$  for each integer  $w > \Omega$ .

Then  $f$  is a binding function for every  $P_5$ -free graph  $G$ .

**Proof.** We prove by induction on  $|G|$  that if  $G$  is  $P_5$ -free then  $f$  is a binding function for  $G$ . Thus, we may assume that  $G$  is  $P_5$ -free and  $f$  is near-binding for  $G$ . If  $G$  is not connected, or  $\omega(G) \leq \Omega$ , it follows that  $f$  is binding for  $G$ , so we assume that  $G$  is connected and  $\omega(G) > \Omega$ . Let us write  $w = \omega(G)$  and  $m = \lfloor w/2 \rfloor$ . If  $\chi(G) \leq f(w)$  then  $f$  is a binding function for  $G$ , so we assume, for a contradiction, that:

(1)  $\chi(G) > f(w - 1) + (w + 2)f(m)$ .

(2) Every cutset  $X$  of  $G$  satisfies  $\chi(X) > 2f(m)$ .

If some cutset  $X$  satisfies  $\chi(X) \leq 2f(m)$ , then since  $\chi(G \setminus X) \leq f(w - 1) + wf(m)$  by 2.1, it follows that  $\chi(G) \leq f(w - 1) + (w + 2)f(m)$ , contrary to (1). This proves (2).

(3) If  $P, Q$  are cliques of  $G$ , both of cardinality at least  $w/2$ , then  $G[P \cup Q]$  is connected.

Suppose not; then there is a minimal subset  $X \subseteq V(G) \setminus (P \cup Q)$  such that  $P, Q$  are subsets of different components  $(A, B)$  say of  $G \setminus X$ . From the minimality of  $X$ , every vertex  $x \in X$  has a neighbour in  $V(A)$  and a neighbour in  $V(B)$ . If  $x$  is mixed on  $A$  and mixed on  $B$ , then since  $A$  is connected, there is an edge  $a_1a_2$  of  $A$  such that  $x$  is adjacent to  $a_1$  and not to  $a_2$ ; and similarly there is an edge  $b_1b_2$  of  $B$  with  $x$  adjacent to  $b_1$  and not to  $b_2$ . But then  $a_2-a_1-x-b_1-b_2$  is an induced copy of  $P_5$ , a contradiction; so every  $x \in X$  is complete to at least one of  $A, B$ . The set of vertices in  $X$  complete to  $A$  is also complete to  $P$ , and hence has clique number at most  $m$ , and hence has

chromatic number at most  $f(m)$ ; and the same for  $B$ . Thus  $\chi(X) \leq 2f(m)$ , contrary to (2). This proves (3).

Let  $a \in V(G)$ , and let  $N(a)$  be the set of neighbours of  $a$  in  $G$ . Let  $B$  be a component of  $G \setminus (N(a) \cup \{a\})$ ; we will show that  $\chi(B) \leq (w - m + 2)f(m)$ .

A subset  $Y$  of  $V(B)$  is a *joint* of  $B$  if there is a component  $C$  of  $B \setminus Y$  such that  $\chi(C) > f(m)$  and  $Y$  is complete to  $C$ . If  $\emptyset$  is not a joint of  $B$  then  $\chi(B) < f(m)$  and the claim holds, so we may assume that  $\emptyset$  is a joint of  $B$ ; let  $Y$  be a joint of  $B$  chosen with  $Y$  maximal.

(4) *If  $v \in N(a)$  has a neighbour in  $V(C)$ , then the set of vertices in  $V(C)$  nonadjacent to  $v$  has chromatic number at most  $f(m)$ .*

Let  $N_C(v)$  be the set of neighbours of  $v$  in  $V(C)$ , and  $M = V(C) \setminus N_C(v)$ ; and suppose that  $\chi(M) > f(m)$ . Let  $C'$  be a component of  $G[M]$  with  $\chi(C') > f(m)$ , and let  $Z$  be the set of vertices in  $N_C(v)$  that have a neighbour in  $V(C')$ . Thus  $Z \neq \emptyset$ , since  $N_C(v), V(C') \neq \emptyset$  and  $C$  is connected. If some  $z \in Z$  is mixed on  $C'$ , let  $p_1 p_2$  be an edge of  $C'$  such that  $z$  is adjacent to  $p_1$  and not to  $p_2$ ; then  $a-v-z-p_1-p_2$  is an induced copy of  $P_5$ , a contradiction. So every vertex in  $Z$  is complete to  $V(C')$ ; but also every vertex in  $Y$  is complete to  $V(C)$  and hence to  $V(C')$ , and so  $Y \cup Z$  is a joint of  $B$ , contrary to the maximality of  $Y$ . This proves (4).

(5)  $\chi(C) \leq (w - m + 1)f(m)$ .

Let  $X$  be the set of vertices in  $N(a)$  that have a neighbour in  $V(C)$ . Since  $C$  is a component of  $B \setminus Y$  and hence a component of  $G \setminus (X \cup Y)$ , and  $a$  belongs to a different component of  $G \setminus (X \cup Y)$ , it follows that  $X \cup Y$  is a cutset of  $G$ . By (2),  $\chi(X \cup Y) > 2f(m)$ . Since  $C$  has clique number at least  $m + 1$  and every vertex in  $Y$  is complete to  $V(C)$ , it follows that  $G[Y]$  has clique number at most  $w - m - 1 \leq m$ , and so has chromatic number at most  $f(m)$ ; and so  $\chi(X) > f(m)$ . Consequently there is a clique  $P \subseteq X$  with cardinality  $w - m$ . The set of vertices of  $C$  complete to  $P$  has clique number at most  $m$ , and so has chromatic number at most  $f(m)$ ; and for each  $v \in P$ , the set of vertices of  $C$  nonadjacent to  $v$  has chromatic number at most  $f(m)$  by (4). Thus  $\chi(C) \leq (|P| + 1)f(m) = (w - m + 1)f(m)$ . This proves (5).

(6)  $\chi(B) \leq (w - m + 2)f(m)$ .

By (3), every clique contained in  $V(B) \setminus (V(C) \cup Y)$  has cardinality less than  $w/2$ , and so

$$\chi(B \setminus (V(C) \cup Y)) \leq f(m);$$

and hence  $\chi(B \setminus Y) \leq (w - m + 1)f(m)$  by (5), since there are no edges between  $C$  and  $V(B) \setminus (V(C) \cup Y)$ . But  $\chi(Y) \leq f(m)$  as we saw before, and so  $\chi(B) \leq (w - m + 2)f(m)$ . This proves (6).

By (6),  $G \setminus N(a)$  has chromatic number at most  $(w - m + 2)f(m)$ . But  $G[N(a)]$  has clique number at most  $w - 1$  and so chromatic number at most  $f(w - 1)$ ; and so  $\chi(G) \leq f(w - 1) + (w - m + 2)f(m)$ , contrary to (1). This proves 2.2. ■

Now we deduce 1.5, which we restate:

**2.3**  $\chi(G) \leq \omega(G)^{\log_2(\omega(G))}$  for every  $P_5$ -free graph  $G$  with  $\omega(G) \geq 3$ .

**Proof.** Define  $f(0) = 0$ ,  $f(1) = 1$ ,  $f(2) = 3$ , and  $f(x) = x^{\log_2(x)}$  for every real number  $x \geq 4$ . Let  $G$  be  $P_5$ -free. If  $\omega(G) \leq 2$  then  $\chi(G) \leq 3 = f(2)$ , by a result of Sumner [10]; if  $\omega(G) = 3$  then  $\chi(G) \leq 5 \leq f(3)$ , by an application of the result 1.4 of Esperet, Lemoine, Maffray, and Morel [5]; and if  $\omega(G) = 4$  then  $\chi(G) \leq 15 \leq f(4)$ , by another application of 1.4. Consequently every  $P_5$ -free graph  $G$  with clique number at most four has chromatic number at most  $f(\omega(G))$ .

We claim that

$$f(x-1) + (x+2)f(\lfloor x/2 \rfloor) \leq f(x)$$

for each integer  $x > 4$ . If that is true, then by 2.2 with  $\Omega = 4$ , we deduce that  $\chi(G) \leq f(\omega(G))$  for every  $P_5$ -free graph  $G$ , and so 1.5 holds. Thus, it remains to show that

$$f(x-1) + (x+2)f(\lfloor x/2 \rfloor) \leq f(x)$$

for each integer  $x > 4$ . This can be verified by direct calculation when  $x = 5$ , so we may assume that  $x \geq 6$ .

The derivative of  $f(x)/x^4$  is

$$(2\log_2(x) - 4)x^{\log_2(x)-5},$$

and so is nonnegative for  $x \geq 4$ . Consequently

$$\frac{f(x-1)}{(x-1)^4} \leq \frac{f(x)}{x^4}$$

for  $x \geq 5$ . Since  $x^2(x^2 - 2x - 4) \geq (x-1)^4$  when  $x \geq 5$ , it follows that

$$\frac{f(x-1)}{x^2 - 2x - 4} \leq \frac{f(x)}{x^2},$$

that is,

$$f(x-1) + (2x+4)f(x)/x^2 \leq f(x),$$

when  $x \geq 5$ . But when  $x \geq 6$  (so that  $f(x/2)$  is defined and the first equality below holds), we have

$$f(\lfloor x/2 \rfloor) \leq f(x/2) = (x/2)^{\log_2(x/2)} = (x/2)^{\log_2(x)-1} = (2/x)(x/2)^{\log_2(x)} = (2/x^2)f(x),$$

and so

$$f(x-1) + (x+2)f(\lfloor x/2 \rfloor) \leq f(x)$$

when  $x \geq 6$ . This proves 2.3. ■

## References

- [1] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, “Erdős-Hajnal for graphs with no five-hole”, submitted for publication, [arXiv:2102.04994](https://arxiv.org/abs/2102.04994).
- [2] P. Erdős and A. Hajnal, “On spanned subgraphs of graphs”, *Graphentheorie und Ihre Anwendungen* (Oberhof, 1977).

- [3] P. Erdős and A. Hajnal, “Ramsey-type theorems”, *Discrete Applied Math.* **25** (1989), 37–52.
- [4] L. Esperet, *Graph Colorings, Flows and Perfect Matchings*, Habilitation thesis, Université Grenoble Alpes (2017), 24, <https://tel.archives-ouvertes.fr/tel-01850463/document>.
- [5] L. Esperet, L. Lemoine, F. Maffray, and G. Morel, “The chromatic number of  $\{P_5, K_4\}$ -free graphs”, *Discrete Math.* **313** (2013), 743–754.
- [6] A. Gyárfás, “On Ramsey covering-numbers”, in *Infinite and Finite Sets, Vol. II* (Colloq., Keszthely, 1973), *Coll. Math. Soc. János Bolyai* **10**, 801–816.
- [7] A. Gyárfás, “Problems from the world surrounding perfect graphs”, *Proceedings of the International Conference on Combinatorial Analysis and its Applications*, (Pokrzywna, 1985), *Zastos. Mat.* **19** (1987), 413–441.
- [8] A. Scott and P. Seymour, “A survey of  $\chi$ -boundedness”, *J. Graph Theory* **95** (2020), 473–504, [arXiv:1812.07500](https://arxiv.org/abs/1812.07500).
- [9] A. Scott, P. Seymour and S. Spirkl, “Polynomial bounds on chromatic number. III. Excluding a double star”, in preparation.
- [10] D. P. Sumner, “Subtrees of a graph and chromatic number”, in *The Theory and Applications of Graphs*, (G. Chartrand, ed.), John Wiley & Sons, New York (1981), 557–576.