

Non-Homotopic Drawings of Multigraphs

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Abstract

A multigraph drawn in the plane is *non-homotopic* if no two edges connecting the same pair of vertices can be continuously deformed into each other without passing through a vertex, and is *k-crossing* if every pair of edges (self-)intersects at most k times. We prove that the number of edges in an n -vertex non-homotopic k -crossing multigraph is at most $6^{13n(k+1)}$, which is a big improvement over previous upper bounds.

We also study this problem in the setting of *monotone* drawings where every edge is an x -monotone curve. We show that the number of edges, m , in such a drawing is at most $2^{\binom{2n}{k+1}}$ and the number of crossings is $\Omega\left(\frac{m^{2+1/k}}{n^{1+1/k}}\right)$. For fixed k these bounds are both best possible up to a constant multiplicative factor.

1 Introduction

A central question of graph drawing research asks for the maximum number of edges in an n -vertex graph that admits a drawing satisfying certain properties. For example, it follows from Euler’s formula that planar graphs have at most $3n - 6$ edges (for $n \geq 3$), which is tight for planar triangulations. The situation for multigraphs (allowing parallel edges and self-loops) is less clear, since there are planar multigraphs with one vertex and arbitrarily many edges. Two constraints on the drawing are typically applied to avoid this. The first constraint is that the drawing is *non-homotopic*: no pair of parallel edges or loops can be continuously deformed into each other without passing through vertices. This eliminates the possibility of a planar multigraph with arbitrarily many edges. The second constraint is that the drawing is *k-crossing*: the number of crossings between each pair of edges is at most k and the number of self-intersections of each edge is at most k . This eliminates non-homotopic drawings on a fixed number of vertices with arbitrarily many edges where different edges wind round the same vertex increasingly many times, as illustrated in [Figure 1](#).

In 1996, Juvan, Malnič, and Mohar [[12](#), Thm. 3.5] proved that n -vertex multigraphs with a non-homotopic k -crossing drawing have a bounded number of edges. Following this, the maximum cardinality of a collection of curves that is non-homotopic and k -crossing on various surfaces has been well-studied. See, for example, the papers [[2–5](#), [7](#), [10](#), [15](#), [17](#), [21](#), [22](#)].

The first result of this paper is an improved upper bound on the number of edges in a non-homotopic k -crossing drawing of a multigraph.

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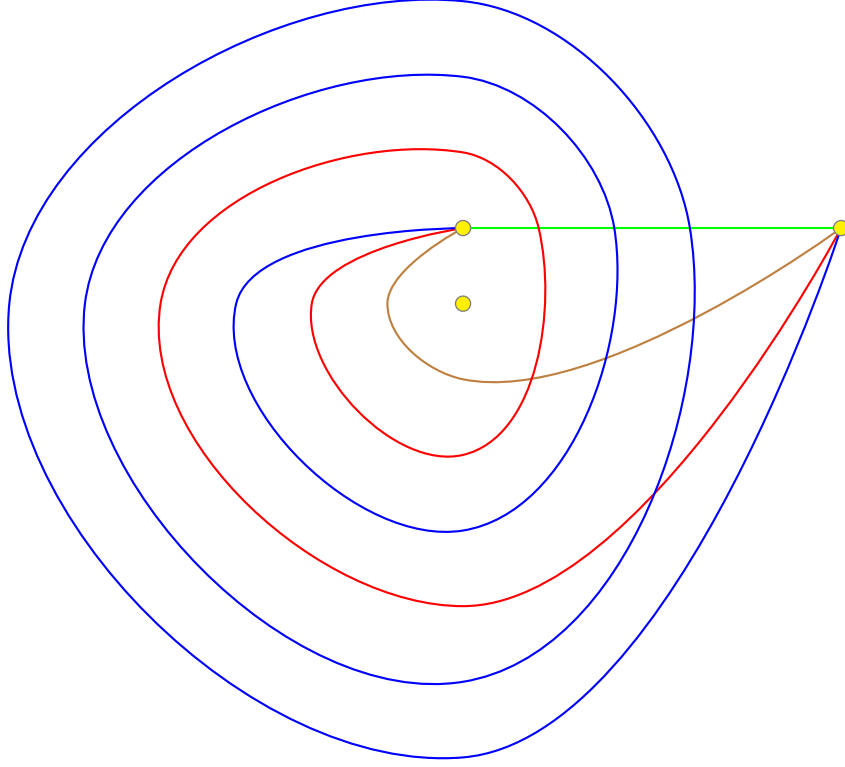


Figure 1: Non-homotopic edges (with no self-crossings) winding around the same vertex increasingly many times.

Theorem 1.1. *Let n, k be positive integers. Let G be an n -vertex multigraph that has a non-homotopic k -crossing drawing. Then*

$$|E(G)| \leq 6^{13n(k+1)}.$$

Theorem 1.1 improves on results from the literature, as we now explain. For integers $n, k \geq 1$, let S denote the set obtained from \mathbb{R}^2 by removing n distinct points, and fix $x \in S$. Let $f(n, k)$ be the maximum number of pairwise non-homotopic x -loops in S such that none of them passes through x , each of them has fewer than k self-intersections, and every pair of them cross fewer than k times. The argument of Juvan, Malnič, and Mohar [12] shows that

$$f(n, k) < (nk)^{\mathcal{O}(nk^2)}. \quad (1)$$

In the fixed n regime, Aougab and Souto [3] improved the dependence on k to

$$f(n, k) \leq 2^{\mathcal{O}(\sqrt{k})}. \quad (2)$$

Theorem 1.1 (in the special case that every edge is a loop incident to a single vertex) implies

$$f(n, k) \leq 6^{13nk}.$$

This improves the bound in (1) and improves the dependence on n in (2). Note that Pach, Tardos, and Tóth [17] proved the exponential lower bound,

$$f(n, k) \geq 2^{\sqrt{nk}/3}. \quad (3)$$

for $k \geq n \geq 2$.

Another common constraint applied to graph drawings is monotonicity: a drawing is *monotone* if every edge is an x-monotone curve. See [6, 8, 16, 18, 19] for previous work on crossings in monotone drawings. The second direction of this paper is to obtain a bound on the number of edges in a multigraph with a monotone non-homotopic k -crossing drawing, and to prove an associated crossing lemma. We also derive the corresponding results when every pair of parallel edges cross at most k times and every pair of incident edges cross at most k times (a pair of edges is *incident* if they share at least one common end-vertex). All of these results have optimal dependence on n .

Theorem 1.2. *Let $k < n$ be non-negative integers. For every n -vertex multigraph G and every monotone non-homotopic drawing D :*

- (a) *If each pair of edges cross at most k times in D , then $|E(G)| \leq 2 \cdot \binom{2n}{k+1}$.*
- (b) *If each pair of incident edges cross at most k times in D , then $|E(G)| \leq n \cdot \binom{2n}{k+1}$.*
- (c) *If each pair of parallel edges cross at most k times in D , then $|E(G)| \leq n(n-1) \cdot \binom{2n}{k+1}$.*

The condition $k < n$ is fully general: if an n -vertex multigraph G has a monotone drawing, then it has a homotopically equivalent monotone drawing in which every pair of edges cross fewer than n times.¹ In Section 3 we give examples that show the dependence upon n is best possible in all three parts.

One motivation for such extremal theorems is that they are essential ingredients in proofs of crossing number lower bounds. The famous ‘Crossing Lemma’ of Ajtai, Chvátal, Newborn, and Szemerédi [1] and Leighton [14] says that every drawing of any n -vertex simple graph (with no parallel edges and no loops) with $m \geq 4n$ edges has $\Omega\left(\frac{m^3}{n^2}\right)$ crossings. Székely [23] proved an analogous result for multigraphs where the lower bound depends on the edge multiplicity. Recent research has shown that under certain assumptions, no dependence on the multiplicity is needed [13, 17, 20]. Similarly, using Theorem 1.2, we prove the following three tight lower bounds on the number of crossings in monotone non-homotopic drawings where each pair of parallel edges/incident edges/edges cross at most k times.

Theorem 1.3. *For every non-negative integer k there are positive constants α_k , β_k , and γ_k such that the following holds. For every n -vertex m -edge multigraph G with $m \geq 4n$ and every monotone non-homotopic drawing D of G :*

- (a) *If each pair of edges cross at most k times in D , then D has at least $\alpha_k \frac{m^{2+1/k}}{n^{1+1/k}}$ crossings.*
- (b) *If each pair of incident edges cross at most k times in D , then D has at least $\beta_k \frac{m^{2+1/(k+1)}}{n^{1+1/(k+1)}}$ crossings.*
- (c) *If each pair of parallel edges cross at most k times in D , then D has at least $\gamma_k \frac{m^{2+1/(k+2)}}{n^{1+1/(k+2)}}$ crossings.*

In Section 3 we give an example to show that the results in Theorem 1.3 are best possible up to the values of α_k , β_k , and γ_k .

We now compare Theorems 1.2 and 1.3 to related results in the literature. Here we conflate a vertex or edge with its image in a drawing. Pach and Tóth [20] and Kaufmann, Pach, Tóth, and Ueckerdt [13] defined a drawing of a multigraph to be:

- *separated* if any two parallel edges do not cross and the “lens” formed by their union has at least one vertex in its interior and at least one vertex in its exterior;

¹First project the monotone drawing to one that is homotopically equivalent where the vertices of G lie at the points $(1,0), (2,0), \dots, (n,0)$ on the x-axis. For an edge e in this drawing from $p_{x_1} = (x_1, 0)$ to $p_{x_2} = (x_2, 0)$, let p_j ($j = x_1 + 1, \dots, x_2 - 1$) be the point where e has x-coordinate j . Let \tilde{e} be the piecewise linear curve going through $p_{x_1}, p_{x_1+1}, \dots, p_{x_2}$. The new drawing of G where every edge e is replaced by \tilde{e} is homotopically equivalent to the original. Note that for any integer a , edges \tilde{e}_1 and \tilde{e}_2 are straight lines (or single points or not present) when restricted to the region $a \leq x \leq a+1$ and so cross at most once in this region. Hence, every pair of edges cross at most $n-1$ times.

- *single-crossing* if any pair of edges cross at most once;
- *locally star-like* if no two edges with a common end-vertex cross.

Note that every separated drawing is non-homotopic but there are non-homotopic drawings that are not separated, since parallel edges might cross in a non-homotopic drawing. Also note that locally star-like is the same as each pair of incident edges cross at most 0 times.

Pach and Tóth [20] defined a drawing to be *branching* if it is separated, single-crossing, and locally star-like. They proved that the number of edges, m , in a branching drawing on n vertices is at most $n(n - 2)$ and the number of crossings is $\Omega(\frac{m^3}{n^2})$. Kaufmann, Pach, Tóth, and Ueckerdt [13] proved that these results still hold without the single-crossing assumption. Theorem 1.2(b) and Theorem 1.3(b) with $k = 0$ match these bounds (up to multiplicative constants), replacing the separated assumption by the weaker non-homotopic assumption and adding the monotonicity assumption. Kaufmann, Pach, Tóth, and Ueckerdt [13] conjectured that every separated single-crossing (but not necessarily locally star-like) drawing on n vertices has $\mathcal{O}(n^2)$ edges. Fox, Pach, and Suk [7] proved a $\mathcal{O}(n^2 \log n)$ upper bound in this case. Theorem 1.2(a) in the $k = 1$ case gives a better bound on the number of edges without the assumption that parallel edges do not cross, and with the extra monotonicity assumption. The conjecture of Kaufmann, Pach, Tóth, and Ueckerdt [13] remains open.

The rest of the paper is structured as follows. We introduce formal notation in Section 1.1. In Section 2 we prove Theorem 1.1. The examples showing the tightness of Theorems 1.2 and 1.3 are given in Section 3. We prove Theorem 1.2 in Section 4 and Theorem 1.3 in Section 5. We finish with open problems in Section 6.

1.1 Notation

A *drawing* of a multigraph G is a function ϕ that maps each vertex $v \in V(G)$ to a point $\phi_v \in \mathbb{R}^2$ and maps each edge $e = vw \in E(G)$ to a continuous curve $\phi_e: [0, 1] \rightarrow \mathbb{R}^2$ with endpoints ϕ_v and ϕ_w , such that:

- every vertex is represented by a different point; that is, the points $(\phi_v: v \in V(G))$ are distinct;
- no edge passes through any vertex; that is, $\phi_e(t) \neq v$ for all $v \in V(G)$, $e \in E(G)$, and $t \in (0, 1)$;
- each pair of edges cross at a finite number of points, where a *crossing* is an unordered pair of distinct pairs $(e, s), (f, t) \in E(G) \times (0, 1)$ with $\phi_e(s) = \phi_f(t)$.

We note that if k edges meet at a point, then this contributes $\binom{k}{2}$ crossings. Parallel edges $e, f \in E(G)$ with end-vertices $v, w \in V(G)$ are *homotopic* with respect to a drawing ϕ if there is a homotopy of $\mathbb{R}^2 \setminus \{\phi_x: x \in V(G) \setminus \{v, w\}\}$ mapping ϕ_e to ϕ_f with ϕ_v and ϕ_w fixed. Here $\mathbb{R}^2 \setminus \{\phi(x): x \in V(G) \setminus \{v, w\}\}$ is the plane with $|V(G)| - 2$ puncture points. Intuitively, parallel edges e and f are homotopic if e can be continuously deformed into f without passing through any vertices and while keeping the end-points fixed. As illustrated in Figure 2, a drawing of a multigraph is *non-homotopic* if no two parallel edges are homotopic. It is natural to consider non-homotopic drawings, since homotopic edges can be considered to be equivalent. The non-homotopic assumption immediately eliminates the example of a planar multigraph with exactly one vertex and more than one edge.

A drawing ϕ of a multigraph G is *monotone* if ϕ_e is an x -monotone curve for each edge $e \in E(G)$; that is, ϕ_e intersects each vertical line in at most one point. Note that a monotone drawing has no self-loops and no edge self-intersects.

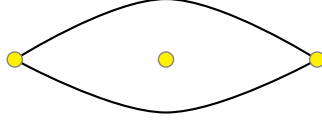


Figure 2: Non-homotopic parallel edges.

2 Non-Homotopic Drawings

In this section we prove [Theorem 1.1](#), which says that every non-homotopic k -crossing drawing on n vertices has at most $6^{13n(k+1)}$ edges. This result follows quickly from the next lemma.

Lemma 2.1. *Let n, k be positive integers. Let x, y be two (not necessarily distinct) vertices of an n -vertex multigraph G such that every edge of G is from x to y and G has a non-homotopic k -crossing drawing. Then*

$$|E(G)| \leq 6^{13n(k+1)} / n^2.$$

Proof of Theorem 1.1. Let G be an n -vertex multigraph as in the statement of [Theorem 1.1](#). For $x, y \in V(G)$, let $G_{x,y}$ be the subgraph of G consisting of the edges from x to y . Then $G_{x,y}$ satisfies the hypothesis of [Lemma 2.1](#), so $|E(G_{x,y})| \leq 6^{13n(k+1)} / n^2$. The result follows since $E(G) = \cup_{x,y} E(G_{x,y})$. \square

In fact, [Lemma 2.1](#) implies that [Theorem 1.1](#) holds even with the k -crossing assumption weakened to every pair of parallel edges cross at most k times, every pair of loops incident to the same vertex cross at most k times, and every edge has at most k self-intersections.

Before proving [Lemma 2.1](#) we need a simple lemma about plane graphs as well as some basic facts about the winding number.

Lemma 2.2. *Let G be a connected plane graph in which all but at most r vertices have degree at least three and every face is incident to at least seven edges. Then $|V(G)| \leq 10r - 28$.*

Proof. Let n, m, F denote the number of vertices, edges and faces of G , respectively. Consider all pairs (e, f) where f is a face of G and e is an edge of G that is incident to f . Every edge is in at most two such pairs and every face is in at least seven, so $7F \leq 2m$. Hence, by Euler's formula,

$$n + \frac{2}{7}m \geq n + F \geq m + 2.$$

Thus $m \leq \frac{7}{5}(n - 2)$. Finally, since G is connected and so has minimum degree at least one,

$$\frac{14}{5}(n - 2) \geq 2m = \sum_{v \in V(G)} \deg(v) \geq 3(n - r) + r = 3n - 2r.$$

Rearranging gives the required result. \square

The following facts about the winding number that can be found in most texts on complex analysis or algebraic topology (such as [9]). Let $\gamma: [0, 1] \rightarrow \mathbb{C}$ be a continuous closed curve (so $\gamma(0) = \gamma(1)$). For any $a \notin \text{Im } \gamma$ there are continuous functions $r: [0, 1] \rightarrow \mathbb{R}_{>0}$ and $\theta: [0, 1] \rightarrow \mathbb{R}$ such that $\gamma = a + re^{2\pi i \theta}$. The *winding number of γ about a* , denoted by $I(\gamma, a)$, is the integer $\theta(1) - \theta(0)$. This integer does not depend on the parameterisation (r, θ) . It satisfies the following properties.

- The winding number $I(\gamma, \cdot)$ is constant on each component of $\mathbb{C} \setminus \text{Im } \gamma$, is zero on the unbounded component and differs by one between adjacent components.

- If γ is the union of two continuous closed curves γ_1 and γ_2 , then $I(\gamma, a) = I(\gamma_1, a) + I(\gamma_2, a)$ for any $a \notin \text{Im } \gamma$.

The following two facts are also standard but do not always appear in textbooks. We include their proof for completeness.

Lemma 2.3. *Let a be a point in the plane and let γ , γ_0 , and γ_1 be closed curves in $\mathbb{R}^2 - a$.*

1. γ_0 and γ_1 are homotopic in $\mathbb{R}^2 - a$ if and only if they have the same winding number about a .
2. γ must self-intersect at least $|I(\gamma, a)| - 1$ times (counted with multiplicity²).

Proof. We first address part 1. Viewing γ_0 and γ_1 as continuous functions from $[0, 1]$ to \mathbb{C} , there are continuous functions $r_0, r_1: [0, 1] \rightarrow \mathbb{R}_{>0}$ and $\theta_0, \theta_1: [0, 1] \rightarrow \mathbb{R}$ such that $\gamma_j = a + r_j e^{2\pi i \theta_j}$. First suppose that γ_0 and γ_1 have the same winding number I , so $I = \theta_1(1) - \theta_1(0) = \theta_0(1) - \theta_0(0)$. Let $H: [0, 1]^2 \rightarrow \mathbb{C} \setminus \{a\}$ be given by $a + r e^{2\pi i \theta}$ where $r(\cdot, t) = t r_1 + (1 - t) r_0$ and $\theta(\cdot, t) = t \theta_1 + (1 - t) \theta_0$. Then H is continuous, $H(\cdot, 0) = \gamma_0$, and $H(\cdot, 1) = \gamma_1$. Also, $\theta(1, t) = \theta(0, t) + I$ and so $H(\cdot, t)$ is a closed curve for each t . Thus γ_0 and γ_1 are homotopic.

Next suppose that γ_0 and γ_1 are homotopic. Then there is a continuous function $H: [0, 1]^2 \rightarrow \mathbb{C} \setminus \{a\}$ where $H(\cdot, t)$ is a closed curve for each t , $H(\cdot, 0) = \gamma_0$, and $H(\cdot, 1) = \gamma_1$. Now $\tilde{\theta} := \frac{1}{2\pi} \arg(H - a): [0, 1]^2 \rightarrow \mathbb{R}/\mathbb{Z}$ is continuous and \mathbb{R} is a covering space for \mathbb{R}/\mathbb{Z} so, by the homotopy lifting property (see, for example, [11, Prop 1.30]), there is a continuous $\theta: [0, 1]^2 \rightarrow \mathbb{R}$ such that $\tilde{\theta}(x, t) \equiv \theta(x, t) \pmod{1}$ for all x and t . The function $r := |H - a|: [0, 1]^2 \rightarrow \mathbb{R}_{>0}$ is continuous. Thus we may write $H = a + r e^{2\pi i \theta}$ for continuous $r: [0, 1]^2 \rightarrow \mathbb{R}_{>0}$ and $\theta: [0, 1]^2 \rightarrow \mathbb{R}$. Since $H(\cdot, t)$ is a closed curve, $f(t) := \theta(1, t) - \theta(0, t) \in \mathbb{Z}$ for all t . Note that f is continuous and so constant. Thus $I(\gamma_1, a) = \theta(1, 1) - \theta(0, 1) = f(1) = f(0) = \theta(0, 1) - \theta(0, 0) = I(\gamma_0, a)$ and so γ_0 and γ_1 have the same winding number about a .

We now address part 2. We will prove the following equivalent statement by induction on ℓ : if γ self-intersects ℓ times, then $|I(\gamma, a)| \leq \ell + 1$. First suppose that $\ell = 0$. Then γ is a simple closed curve. By the Jordan Curve Theorem, γ splits the plane into two regions: an interior region bounded by γ and an exterior unbounded region. $I(\gamma, \cdot)$ is zero on the unbounded region and is ± 1 on the interior region (since the interior and exterior regions are adjacent). Now suppose that $\ell \geq 1$. Let b be a point of self-intersection of γ . We may partition γ into two closed curves $\gamma^{(0)}$ and $\gamma^{(1)}$ that meet at b . Let $\gamma^{(i)}$ have ℓ_i self-intersection. Then $\ell \geq \ell_0 + \ell_1 + 1$ (the $+1$ is for the self-intersection at b). Finally, by induction on ℓ ,

$$|I(\gamma, a)| = |I(\gamma^{(0)}, a) + I(\gamma^{(1)}, a)| \leq |I(\gamma^{(0)}, a)| + |I(\gamma^{(1)}, a)| \leq (\ell_0 + 1) + (\ell_1 + 1) \leq \ell + 1. \quad \square$$

We are finally ready to prove [Lemma 2.1](#).

Proof of Lemma 2.1. We introduce some notation. Suppose that H is a graph drawn in the plane and $E' \subseteq E(H)$ is a set of edges. We write $\mathcal{C}(E')$ for the union of the curves given by the edges of E' . This includes all end-points of edges of E' . For example, if H is a K_3 with vertices $\{(0,0), (1,0), (0,1)\}$ and straight edges, then $\mathcal{C}(E(H))$ is the perimeter of the right-angled triangle with vertices $\{(0,0), (1,0), (0,1)\}$.

We will first prove the result under the assumption that $x \neq y$ and at the end of the proof we will explain how the strategy is adapted to handle the $x = y$ case. Let the vertices of G be $x, y, v_1, \dots, v_{n-2}$.

Fix a non-homotopic drawing of G in which every pair of edges cross at most k times. Recall that all edges of G are from x to y . By small applying a small perturbation we may assume that

²If γ passes through the same point k times, then this contributes $\binom{k}{2}$ self-intersection which is consistent with the definition of crossing in [Section 1.1](#)

no three edges meet at a point. For $i = 1, \dots, n - 2$, let e_i be an edge of G passing closest to v_i (the e_i may not be distinct). Let p_i be a point on e_i closest to v_i (the p_i may not all be distinct, some of them may be crossings and some of them may be x or y). By the minimality of the length of line-segment $p_i v_i$, no edge of G crosses the interior of $p_i v_i$.

We are going to build a ‘frame’ F which we will use to encode the homotopy of edges of G . This will be a connected plane graph with $V(G) \subseteq V(F)$. We first build a frame F_n that will be a plane tree containing all vertices of G . We use f to refer to edges of a frame and e for edges of G .

As illustrated in Figure 3, start with F_1 which is the empty graph on vertex set $\{x\}$. In step 1, let f_1 be a shortest sub-curve of $e_1 \cup p_1 v_1$ from x to v_1 . Note that f_1 is a non-self-intersecting curve from x to v_1 . Furthermore, all edges of $G - e_1$ intersect f_1 at most k times. Let F_2 be the plane graph obtained from F_1 by adding the edge f_1 and the vertex v_1 . In step i , for $i = 2, \dots, n - 2$, do the following:

1. If $v_i \in V(F_i)$, then move straight onto step $i + 1$.
2. Else if $v_i \in \mathcal{C}(E(F_i))$, then let F_{i+1} be the plane tree obtained from F_i by adding vertex v_i (this subdivides an edge of F_i into two). Move straight onto step $i + 1$.
3. Else let f_i be the shortest sub-curve of $e_i \cup p_i v_i$ that goes from $\mathcal{C}(E(F_i))$ to v_i . Let v'_i be the point where f_i meet $\mathcal{C}(E(F_i))$.
4. If $v'_i \in V(F_i)$, then let F_{i+1} be the plane tree obtained from F_i by adding edge f_i and vertex v_i .
5. If $v'_i \notin V(F_i)$, then let F_{i+1} be the plane tree obtained from F_i by adding vertex v'_i (this subdivides an edge of F_i into two), adding edge f_i , and adding vertex v_i .

This defines F_1, \dots, F_{n-1} . If $y \in V(F_{n-1})$, then take $F_n = F_{n-1}$. Else if $y \in \mathcal{C}(E(F_{n-1}))$, then let F_n be the plane graph obtained from F_{n-1} by adding vertex y (this subdivides an edge of F_{n-1} into two). Else let e_{n-1} be any edge of G , and let f_{n-1} be the shortest sub-curve of e_{n-1} that goes from $\mathcal{C}(E(F_{n-1}))$ to y (which is an end-point of e_{n-1}). Let v'_{n-1} be the point where f_{n-1} meets $\mathcal{C}(E(F_{n-1}))$. If $v'_{n-1} \in V(F_{n-1})$, then let F_n be the plane graph obtained from F_{n-1} by adding edge f_{n-1} and vertex y . If $v'_{n-1} \notin V(F_{n-1})$, then let F_n be the plane graph obtained from F_{n-1} by adding vertex v'_{n-1} (this subdivides an edge of F_{n-1} into two), adding edge f_{n-1} , and adding vertex y .

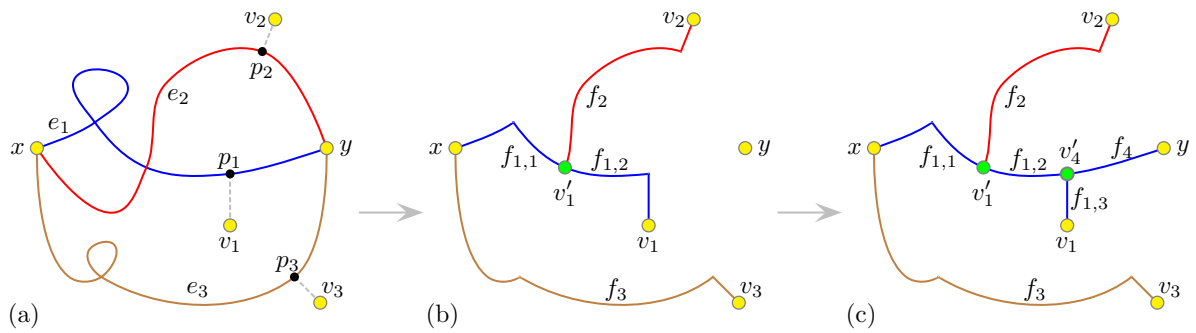


Figure 3: Example of the $n = 5$ case: (a) drawing of G , (b) F_{n-1} and (c) F_n , where $f_{1,1}$ $f_{1,2}$ and $f_{1,3}$ are obtained by subdividing f_1 .

We note the following properties of F_n :

- Since each step introduces at most two vertices, F_n is a plane tree with at most $2n$ vertices.
- Since step i introduces v_i (if it is not already present), $V(G) \subseteq V(F)$.
- Since v'_i has degree at least three when introduced, all vertices in $V(F) \setminus V(G)$ have degree at least three.

- $\mathcal{C}(E(F_n))$ is a subset of the union of $\mathcal{C}(\{e_1, \dots, e_{n-1}\})$ and the line-segments $p_i v_i$ ($i = 1, \dots, n-2$). In particular, each edge of $G - \{e_1, \dots, e_{n-1}\}$ intersects $\mathcal{C}(E(F_n))$ at most $k(n-1)$ times.

We now construct a new frame F building on F_n . F_n is a plane tree and so has one (infinite) face. Let C be a face of the current frame F , and let $f, f' \in E(F)$ be (not necessarily distinct) edges incident to C . We say f sees f' across C if there is an edge e of G such that, at some point on e 's journey from x to y , e crosses f and then crosses f' and in between e is in the interior of C and crosses no other edge of F .³

We carry out the following process. Let F be the current frame which is initially F_n . Let E' be a subset of $E(G)$ initialised as $\{e_1, \dots, e_{n-1}\}$. Iteratively do the following.

- As illustrated in Figure 4, suppose there is an edge f and a face C of F such that f sees at least nine other edges of F across C . Label nine of these edges f_1, f_2, \dots, f_9 in any order.
- Since f sees f_j across C , there is an edge e'_j of G that crosses f and then f_j and in between is in side C and crosses no other edges of F . Let c'_j be the shortest sub-curve of e'_j from f to f_j whose interior is inside C and crosses no other edges of F .
- Since f sees nine distinct edges, there is some choice of j such that c'_j splits C into two faces both of which are incident to at least seven edges (a part of f , a part of f_j , c'_j , and four of $\{f_1, \dots, f_9\} \setminus \{f_j\}$). Let e be this c'_j and e' be e'_j .
- Add the edge e to F together with a vertex at the point where e meets f and a vertex at the point where e meets f_j (this subdivides each of f and f_j exactly once).
- Add e' to E' .

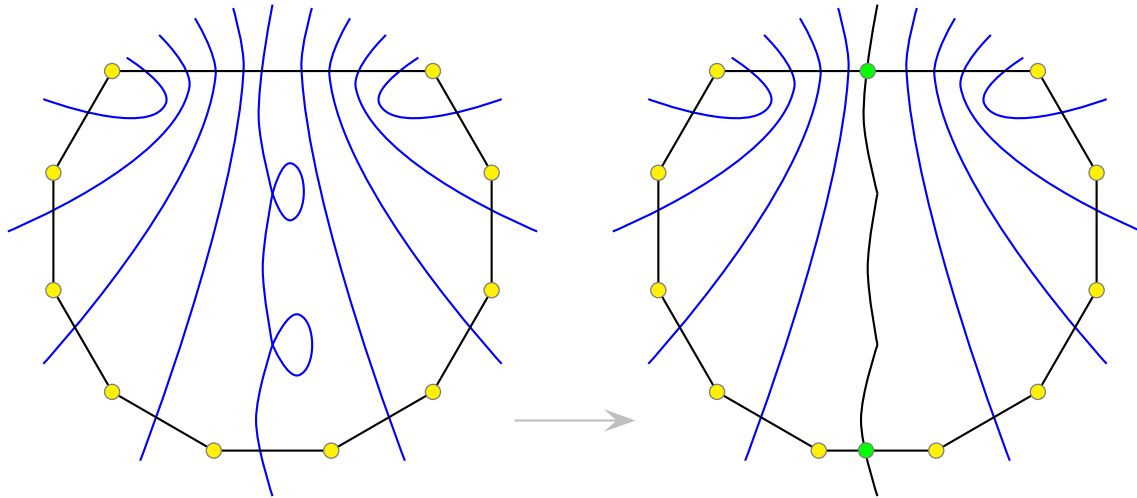


Figure 4: Building F .

The process stops when, for every edge $f \in E(F)$ and face C of F , f sees at most eight other edges across C . We now show the number of steps in this process is at most $5(n-1)$ and the final frame has at most $10n$ edges. First note that if F_n has at most nine edges, then no steps will be carried out and we have the required bounds. Otherwise, F_n is a plane tree with at least ten edges in which at most n vertices have degree less than three. Every step introduces two vertices of degree three and splits a face into two faces both of which are incident to at least seven edges. Hence, every frame in the process is a connected plane graph, has at most n vertices of degree less than 3 and every face of F is incident to at least seven edges. Thus, by Lemma 2.2, every

³To be precise: using the ϕ notation defined in Section 1.1, there are $s < t$ in $[0, 1]$ such that $\phi_e(s)$ is in f , $\phi_e(t)$ is in f_j , and $\phi_e((s, t))$ is inside C and disjoint from $\mathcal{C}(E(F))$.

frame within the process has at most $10n - 28 \leq 10n$ vertices. Since each step adds two vertices, the process must stop within $5n - 14 \leq 5(n - 1)$ steps.

We note some properties of the final frame F and final set E' .

- F is a simple plane graph on at most $10n$ vertices. Each step introduces at most three new edges (the edge across the face plus one for each of the two subdivisions) and F_n had at most $2n$ edges, so F has at most $2n + 3 \cdot 5(n - 1) \leq 17n$ edges.
- For every edge f and face C of F , f sees at most nine edges across F (it might see itself).
- Since the process stops within $5(n - 1)$ steps, $|E'| \leq (n - 1) + 5(n - 1) = 6(n - 1)$.
- $\mathcal{C}(E(F))$ is a subset of the union of $\mathcal{C}(E')$ and the line-segments $p_i v_i$. In particular, each edge in $E(G) - E'$ intersects $\mathcal{C}(E(F))$ at most $6(n - 1)k$ times.

We are now going to encode each edge of $G - E'$ using the edges of F so that if two edges of $G - E'$ had the same encoding, then they would be homotopic. Now, F is a simple connected plane graph with one *outer face* C_∞ . The rest of its faces are *inner faces*. An edge f of F is an *outer edge* if it is incident to the outer face of F . Otherwise f is an *inner edge*.

We will encode each edge $e \in E(G) - E'$ by writing down the sequence of edges of F that e crosses together with some further information. Consider two consecutive edges $f_1, f_2 \in E(F)$ in the sequence for e . If e goes from f_1 to f_2 across an inner face C , then f_1, f_2 determine this portion of e up to homotopy. However, this is not the case if e goes from f_1 to f_2 across the outer face C_∞ – we need to further specify how many times this portion of e loops around F and whether this looping is anticlockwise or clockwise.

To this end, for each pair of outer edges f, f' of F , fix:

- a simple curve $P_{f,f'} \subseteq \mathcal{C}(E(F))$ that joins f to f' along the outer face C_∞ ,
- a point $p_{f,f'}$ in the plane which is sufficiently close to (but not on) the curve $P_{f,f'}$ so that the perpendicular segment from $p_{f,f'}$ to $P_{f,f'}$ does not intersect any edge of G or F .

If x is incident to the outer face C_∞ , then, for each outer edge f of F , we fix a curve $P_{x,f} \subseteq \mathcal{C}(E(F))$ that joins x to f along C_∞ and fix a point $p_{x,f}$ in the plane that is sufficiently close to but not on $P_{x,f}$. If y is incident to C_∞ , then we similarly define $P_{y,f}$ and $p_{y,f}$ for each outer edge f .

Now if an edge e goes from f_1 to f_2 across C_∞ , then we will encode this portion of the journey (call it e') as follows. We form a closed curve consisting of e' , P_{f_1,f_2} as well as parts of f_1, f_2 that join these up, we orient this curve so that e goes from f_1 to f_2 and write down the resulting winding number about p_{f_1,f_2} . By [Lemma 2.3](#), for a point p in the plane, two closed curves in $\mathbb{R}^2 - p$ are homotopic if and only if they have the same winding number around p . Since the closed curve created here never enters an inner face of F , the winding number we write down determines e' up to homotopy. Similarly, if an edge e goes from x to an outer edge f across C_∞ (call this portion e') we will form a closed curve consisting of e' , $P_{x,f}$ as well as the part of f that joins these up and write down the winding number of this closed curve about $p_{x,f}$. Again we do similarly if an edge e goes from an outer edge f to y .

We are now ready to define, for an edge $e \in E(G) - E'$, the *signature of e* , denoted by $\sigma(e)$. It is a sequence $a_1, f_1, a_2, f_2, \dots, a_\ell, f_\ell, a_{\ell+1}$ where:

- The edge e starts at x and crosses edges $f_1, \dots, f_\ell \in E(F)$ in that order before arriving at y . The f_i need not be distinct.
- If e goes from f_{i-1} to f_i by crossing an internal face, then a_i is defined to be $*$.
- If e goes from f_{i-1} to f_i by crossing C_∞ , then a_i is the winding number defined above.
- If e goes from x to f_1 by crossing an internal face, then a_1 is defined to be $*$.

- If e goes from x to f_1 by crossing C_∞ , then a_1 is the winding number defined above for the portion of e going from x to f_1 .
- If e goes from f_ℓ to y by crossing an internal face, then $a_{\ell+1}$ is defined to be $*$.
- If e goes from f_ℓ to y by crossing C_∞ , then $a_{\ell+1}$ is the winding number defined above for the portion of e going from f_ℓ to y .

As discussed above, the signature of e determines e up to homotopy, so the number of edges in $E(G) - E'$ is at most the number of possible signatures. We look in some more detail at the value of a_i when e goes from f_{i-1} to f_i by crossing C_∞ ; denote this portion of e by e' . By [Lemma 2.3](#), a closed curve with winding number r about a point must self-intersect at least $|r| - 1$ times. Hence, the closed curve consisting of e', P_{f_{i-1}, f_i} as well as the parts of f_{i-1}, f_i joining these up must self-intersect at least $|a_i| - 1$ times. But none of these self-intersections can occur on $\mathcal{C}(E(F))$ as the interior of e' is within C_∞ . Thus e' must self-intersect at least $|a_i| - 1$ times. Note the following properties of $\sigma(e)$:

- Since each edge in $E(G) - E'$ intersects $\mathcal{C}(E(F))$ at most $6(n-1)k$ times, $\ell \leq 6(n-1)k$.
- Since each edge self-intersects at most k times, $\sum_i \max\{|a_i| - 1, 0\} \leq k$ where we ignore summands corresponding to $*$.
- There are $|E(F)| \leq 17n$ choices for f_1 .
- Since for every edge f and face C of F , f sees at most nine edges across F , there are, given f_i , at most nine choices for f_{i+1} .

We now upper bound the number of possible sequences satisfying these properties. Fix ℓ . The number of options for f_1, f_2, \dots, f_ℓ is at most $17n \cdot 9^{\ell-1}$. For each i , let $b_i = \max\{|a_i| - 1, 0\}$ if $a_i \neq *$ and let $b_i = 0$ if $a_i = *$. The b_i are $\ell + 1$ non-negative integers with sum at most k and so there are at most $\binom{k+\ell+1}{k}$ choices for $b_1, \dots, b_{\ell+1}$. Given b_i there are at most four options for a_i (if $b_i = 0$, then a_i could be $-1, 0, 1$, or $*$). Hence, the number of options for $a_1, \dots, a_{\ell+1}$ is at most $4^\ell \binom{k+\ell+1}{k}$. Thus, for fixed ℓ , the number of signatures is at most

$$17n \cdot 9^{\ell-1} \cdot 4^\ell \binom{k+\ell+1}{k} \leq 2n \cdot 6^{2\ell} \binom{6nk}{k} \leq 2n \cdot 6^{12(n-1)k} \cdot (6en)^k.$$

There are fewer than $6nk$ choices for ℓ , and $|E(G)|$ is at most the number of signatures plus $|E'|$. So

$$\begin{aligned} |E(G)| &\leq 6nk \cdot 2n \cdot 6^{12(n-1)k} \cdot (6en)^k + |E'| \leq 12n^2k \cdot (6en)^k \cdot 6^{12(n-1)k} + 6(n-1) \\ &\leq 12n^2(k+1) \cdot (6en)^k \cdot 6^{12(n-1)k}. \end{aligned}$$

Hence, it suffices to prove that

$$12n^4(k+1) \cdot (6en)^k \leq 6^{nk+13n+12k}.$$

But $n^4 \leq 6^n$, $12(k+1) \leq 6^{3k}$, and $(6en)^k \leq (6^{n+2})^k = 6^{nk+2k}$, as required.

Now we outline how the above argument is altered to prove the result in the case $x = y$ (when all edges are x -loops). Let the vertices of G be x, v_1, \dots, v_{n-1} . We build the frame F_n exactly as before but with v_{n-1} replacing y when we build F_n from F_{n-1} . Again F_n satisfies the following properties:

- F_n is a plane tree with at most $2n$ vertices.
- $V(G) \subseteq V(F_n)$ and all vertices in $V(F_n) \setminus V(G)$ have degree at least three.
- Each edge of $G - \{e_1, \dots, e_{n-1}\}$ intersects $\mathcal{C}(E(F_n))$ at most $k(n-1)$ times.

We construct the frame F from F_n exactly as before. The process stops when, for every edge $f \in E(F)$ and face C of F , f sees at most eight other edges across C . Again, by [Lemma 2.2](#), the

number of steps in this process is at most $5(n - 1)$ and the final frame has at most $10n$ vertices. The final frame F and final set of edges E' satisfy the same properties as before.

We now encode each edge $e \in E(G) - E'$. Direct each such edge and encode them as before by writing down the signature $a_1, f_1, \dots, a_\ell, f_\ell, a_{\ell+1}$ where the f_i are the sequence of edges of F that e crosses and the a_i are either winding numbers (if e is crossing the unbounded face C_∞) or are $*$ (if e is crossing an internal face). The only difference now is that there is no need for the curves $P_{y,f}$ and points $p_{y,f}$.

We then finish the argument as before. \square

Theorem 1.1 applies for $k \geq 1$. The $k = 0$ case (with no crossings) is handled by the following elementary result.

Proposition 2.4. *Every non-homotopic plane multigraph on $n \geq 2$ vertices has at most $4n - 4$ edges.*

Proof. Let G be a non-homotopic plane multigraph on $n \geq 2$ vertices with m edges. By adding edges, we may assume that G is connected.

A loop of G is *trivial* if it has no vertices in its interior. Any two trivial loops incident to the same vertex are homotopic, so G has at most n trivial loops. Let G' be obtained from G by deleting all the trivial loops, and let $m' := |E(G')|$. Then $m \leq m' + n$. We claim that $m' \leq 3n - 4$ which will suffice. Since G is connected, so too is G' . Define the *length* of a face F to be the number of edges (counting multiplicity) in a walk around F . As an example, the length of the face F shown in [Figure 5](#) is 3 (the edge e is traversed twice in a walk around F).

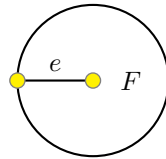


Figure 5: A face F of length 3.

Consider a face F of length at most 2. If F has a vertex in its interior, then, since G' is connected, there is an edge from an interior vertex to a vertex on the boundary of F and so F has length at least 3. Thus F has no vertex in its interior. The boundary of F must either be a loop or a pair of parallel edges between two vertices u and v . Since G' contains no trivial loops and parallel edges are not homotopic, F must be the outer face of G' . In particular, every face of G' except the outer face has length at least 3. Consider the sum of the lengths of the faces of G' . Each edge contributes 2 to this sum and so the sum is equal to $2m'$. On the other hand, each face, except the outer face, contributes 3. Since $n \geq 2$ and G' is connected, the outer face contributes at least 1. In particular, the sum is at least $3f - 2$ where f is the number of faces of G' . Thus $3f - 2 \leq 2m'$. By Euler's formula, $2 + m' - n \leq f \leq \frac{2}{3}(m' + 1)$, and so $m' \leq 3n - 4$, as desired. \square

The bound in [Proposition 2.4](#) is tight. For $n = 2$, take the multigraph on $\{u, v\}$ consisting of edge uv , trivial loops at u and v as well as a loop at u that has all the other edges in its interior. For $n \geq 3$, start with any simple plane triangulation on n vertices, with the outer face bounded by the triangle (u, v, w) . Add a parallel edge uv so that w is no longer on the outer face boundary. Add a loop incident to u so that v is no longer on the outer face boundary. Finally, for each vertex x add a loop incident to x drawn in an internal face incident to x . The resulting drawing is non-homotopic, and with $(3n - 6) + 1 + 1 + n = 4n - 4$ edges.

3 Construction of Monotone Drawings

Here we give a construction showing the tightness of [Theorems 1.2](#) and [1.3](#). For integers $1 \leq k < n$ define the multigraph $G_{n,k}$ as follows, and illustrated in [Figure 6](#). It has vertex set $V(G_{n,k}) = \{v_1, \dots, v_n\}$ where each vertex v_i is drawn at $(i, 0)$ in the plane. For integers $1 \leq a_0 < a_1 < \dots < a_\ell \leq n$ where $1 \leq \ell \leq k$, add to $G_{n,k}$ an edge $e = e_{a_0, a_1, \dots, a_\ell}$ with endpoints v_{a_0} and v_{a_ℓ} , drawn as a degree- $(k+1)$ monic polynomial

$$y = P_e(x) := x^{k-\ell} (x - a_0) (x - a_1 - \frac{1}{2}) (x - a_2 - \frac{1}{2}) \cdots (x - a_{\ell-1} - \frac{1}{2}) (x - a_\ell)$$

with domain $a_0 \leq x \leq a_\ell$. This is a monotone curve crossing the x-axis at $(a_i + \frac{1}{2}, 0)$ for each $i \in \{1, \dots, \ell - 1\}$. The resulting multigraph $G_{n,k}$ has $\sum_{\ell=1}^k \binom{n}{\ell+1} = \Theta_k(n^{k+1})$ edges.

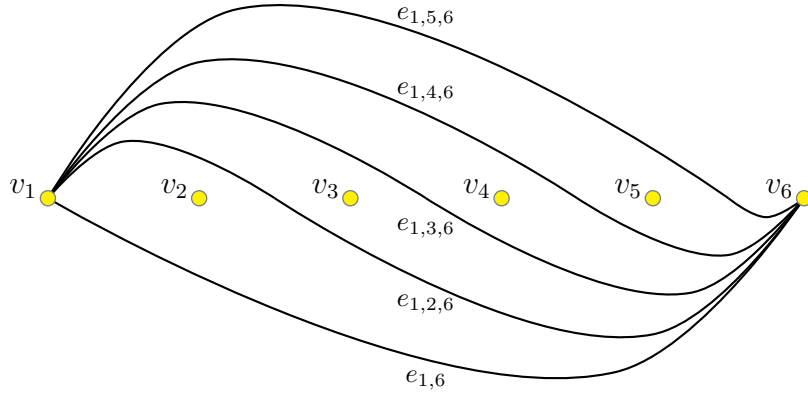


Figure 6: Edges between v_1 and v_6 in $G_{n,2}$.

Suppose that two edges $e = e_{s, a_1, \dots, a_{\ell-1}, t}$ and $f = e_{s, b_1, \dots, b_{\ell'-1}, t}$ between vertices v_s and v_t are homotopic. Then the signs of P_e and P_f must agree at each integer i strictly between s and t . Every crossing of one of the polynomials with the x-axis occurs at $x = m + \frac{1}{2}$ for $m \in \{s+1, \dots, t-1\}$ and all such crossings except $t - \frac{1}{2}$ correspond to a change in sign of the polynomial. Hence P_e and P_f must cross the x-axis at the same places (except possibly for $t - \frac{1}{2}$). In particular, the sets $\{a_1, \dots, a_{\ell-1}\}$ and $\{b_1, \dots, b_{\ell'-1}\}$ are the same or differ by the element $t - 1$. If they are the same, then $e = f$. If they differ by the element $t - 1$, then WLOG $\ell' = \ell - 1$, $a_1 = b_1, \dots, a_{\ell-2} = b_{\ell-2}$, and, $a_{\ell-1} = t - 1$. But then $xP_e(x) = (x - t + \frac{1}{2})P_f(x)$ and so P_e and P_f differ in sign at $x = t - 1$ and so e and f are not homotopic. Therefore, the drawing of $G_{n,k}$ given is indeed non-homotopic.

Consider two distinct edges e and f of G . Both P_e and P_f are monic polynomials of degree $k+1$, so their difference is a polynomial of degree at most k , which has at most k roots. Thus the edges e and f cross at most k times. Now suppose that edges e and f are incident at vertex v_a . Then both e and f are monic polynomials of degree $k+1$ with a root at a , so their difference is a polynomial of degree at most k with a root at a . The root at a is not a crossing point of e and f , so e and f cross at most $k - 1$ times. Similarly if e and f are parallel, then they cross at most $k - 2$ times. In particular,

- $G_{n,k}$ has a monotone non-homotopic drawing in which each pair of edges cross at most k times,
- $G_{n,k+1}$ has a monotone non-homotopic drawing in which each pair of incident edges cross at most k times,
- $G_{n,k+2}$ has a monotone non-homotopic drawing in which each pair of parallel edges cross at most k times.

This shows that the dependence on n is best possible in [Theorem 1.2](#) except for the case $k = 0$ in part (a). For this particular case we note that any straight-line plane triangulation gives a monotone non-homotopic drawing with $3n - 6$ edges and no crossings.

Next we address the tightness of [Theorem 1.3](#). Let $t \in \{2k, \dots, n\}$. Let $G_{n,k,t}$ be the sub-multigraph of $G_{n,k}$ consisting of those edges e_{a_0, \dots, a_ℓ} with $a_\ell - a_0 \leq t$. Each vertex has degree between $\sum_{\ell=1}^k \binom{t}{\ell}$ and $2 \sum_{\ell=1}^k \binom{t}{\ell}$, that is, has degree $\Theta_k(t^k)$. Thus $G_{n,k,t}$ has $m = \Theta_k(nt^k)$ edges.

Fix an edge $e = e_{a_0, \dots, a_\ell}$. Suppose an edge $f = e_{b_0, \dots, b_\ell}$ crosses e . Then $b_0 < a_\ell$ and $b_\ell > a_0$. Since $a_\ell - a_0 \leq t$ and $b_\ell - b_0 \leq t$, we have $b_0 \geq b_\ell - t > a_0 - t$ and $b_0 < a_\ell \leq a_0 + t$. In particular, there are at most $2t$ choices for b_0 . Each vertex has degree $\Theta_k(t^k)$, so e crosses at most $\mathcal{O}_k(t^{k+1})$ other edges. Hence the number of crossings in $G_{n,k,t}$ is at most $m \cdot \mathcal{O}_k(t^{k+1}) = \mathcal{O}_k(nt^{2k+1})$. But $nt^{2k+1} = (nt^k)^{2+1/k} / n^{1+1/k} = \Theta_k(m^{2+1/k} / n^{1+1/k})$, so $G_{n,k,t}$ has $\mathcal{O}_k(m^{2+1/k} / n^{1+1/k})$ crossings.

Thus $G_{n,k,t}$ shows [Theorem 1.3\(a\)](#) is tight where m can be chosen independently of n by varying t . $G_{n,k+1,t}$ and $G_{n,k+2,t}$ show that parts (b) and (c) are tight, respectively.

4 Number of Edges in Monotone Drawings

In this section we prove [Theorem 1.2](#). The main work is part (a); the other two parts then follow quickly.

Proof of [Theorem 1.2\(b\)](#) and (c) assuming [Theorem 1.2\(a\)](#). Let G be an n -vertex multigraph with a monotone non-homotopic drawing in which each pair of incident edges cross at most k times. For each vertex x , let G_x denote the sub-multigraph of G consisting of those edges incident to x . Then G_x is an n -vertex multigraph with a monotone non-homotopic drawing in which each pair of edges cross at most k times. By [Theorem 1.2\(a\)](#), $|E(G_x)| \leq 2 \cdot \binom{2n}{k+1}$, implying

$$|E(G)| = \frac{1}{2} \sum_{x \in V(G)} |E(G_x)| \leq n \cdot \binom{2n}{k+1},$$

which proves [Theorem 1.2\(b\)](#).

Now let G be an n -vertex multigraph with a monotone non-homotopic drawing in which each pair of parallel edges cross at most k times. For each pair of vertices $\{x, y\} \in \binom{V(G)}{2}$, let $G_{x,y}$ denote the sub-multigraph of G consisting of those edges between x and y . By [Theorem 1.2\(a\)](#), $|E(G_{x,y})| \leq 2 \cdot \binom{2n}{k+1}$. Thus,

$$|E(G)| = \sum_{\{x,y\} \in \binom{V(G)}{2}} |E(G_{x,y})| \leq n(n-1) \cdot \binom{2n}{k+1},$$

which proves [Theorem 1.2\(c\)](#). □

It remains to prove [Theorem 1.2\(a\)](#). To do so we first encode the edges in the monotone non-homotopic k -crossing drawing of G by a sequence of length n and then bound the number of such sequences. We consider sequences in which each entry is in $\{+, 0, -, *\}$. Such a sequence is a *monotone drawing sequence* if it is of one of the two following forms:

$$\begin{array}{c} \underbrace{*, \dots, *}_{\geq 0}, 0, \underbrace{\pm, \dots, \pm}_{\geq 0}, 0, \underbrace{*, \dots, *}_{\geq 0} \\ \underbrace{\pm, \dots, \pm}_{\geq 0}, 0, \underbrace{*, \dots, *}_{\geq 0} \end{array}$$

where \pm denotes that the entry is either a $+$ or a $-$ (independently between different symbols). In other words, there are two types of sequences: those with two 0's which have a sequence of $+$ and $-$ symbols between them and $*$'s at both ends, and those with one 0 which has a sequence of $+$ and $-$ symbols before and a sequence of $*$'s after.

Partially order the four symbols, $- < 0 < +$, where $*$ is incomparable with the others. Given two monotone drawing sequences (a_i) and (b_i) we say that (a_i) is below (b_i) at j if $a_j < b_j$ and (a_i) is above (b_i) at j if $a_j > b_j$. Note that if a_j or b_j is a $*$, then the sequences are incomparable or equal at j .

Two sequences (a_i) and (b_i) cross k times if there are indices $i_0 < i_1 < \dots < i_k$ such that (a_i) is below (b_i) at i_0, i_2, i_4, \dots and (a_i) is above (b_i) at i_1, i_3, i_5, \dots or vice versa. Note that if (a_i) and (b_i) have length n , then they cross at most $n - 1$ times.

Lemma 4.1. *If an n -vertex multigraph G has a monotone drawing in the plane such that no edges are homotopic and every pair of edges cross at most k times, then there exists a set of $|E(G)|$ monotone drawing sequences of length n any two of which cross at most k times.*

Proof. We may perturb the vertices so that no two vertices have the same x-coordinate. Let v_1, \dots, v_n be the vertices of G ordered by increasing x-coordinate. Let L_i be the vertical line through v_i . Consider each edge e with endpoints v_i and v_j with $i < j$. For $\ell \in \{1, \dots, i-1\} \cup \{j+1, \dots, n\}$, let $s_{e,\ell} = *$. Let $s_{e,i} = 0$ and $s_{e,j} = 0$. For $\ell \in \{i+1, \dots, j-1\}$, let $s_{e,\ell} = +$ if e crosses L_ℓ above v_ℓ , and let $s_{e,\ell} = -$ if e crosses L_ℓ below v_ℓ . Since e is x-monotone, $s_{e,\ell}$ is well-defined. Let $s_e := (s_{e,1}, \dots, s_{e,n})$. By construction, s_e is a monotone drawing sequence. Fix distinct edges $e, f \in E(G)$. Since e and f are not homotopic, $s_e \neq s_f$. Also, if there are indices $i < j$ with s_e above s_f at i and s_e below s_f at j , then e and f must have a crossing whose x-coordinate is strictly between i and j . Hence, s_e and s_f cross at most k times. \square

Lemma 4.2. *Let $k < n$ be positive integers. Let \mathcal{A} be a set of monotone drawing sequences of length n , each pair of which cross at most k times. Then*

$$|\mathcal{A}| \leq 2 \cdot \binom{2n}{k+1}.$$

Proof. Let $g(n, k)$ denote the size of the largest set of length n monotone drawing sequences where every pair crosses at most k times. In this definition we allow $k \geq n$, in which case $g(n, k) = g(n, n-1)$, since no two monotone drawing sequences of length n cross n times. We also allow $k = -1$, in which case $g(n, -1) = 1$ (as no two sequences cross at most -1 times).

Let \mathcal{A} be a set of monotone drawing sequences of length $n+1$, such that every pair crosses at most k times. The proof will proceed by deleting the first entry of each sequence in \mathcal{A} and analysing how many sequences elide.

Let \mathcal{B} be those sequences of \mathcal{A} which, upon deleting their first entry, are no longer monotone drawing sequences. The only sequence in \mathcal{B} is $0, *, *, \dots, *$ and so $|\mathcal{B}| \leq 1$. Let $\mathcal{A}_+, \mathcal{A}_0, \mathcal{A}_-, \mathcal{A}_*$ be those sequences in $\mathcal{A} - \mathcal{B}$ which start with a $+$, 0 , $-$, $*$, respectively. Let \mathcal{A}'_+ be those sequences obtained by deleting the first entry (which, of course, is a $+$) from sequences in \mathcal{A}_+ . Similarly define $\mathcal{A}'_0, \mathcal{A}'_-, \mathcal{A}'_*$. Note that each of these is a set of monotone drawing sequences of length n . Let $\mathcal{A}' = \mathcal{A}'_+ \cup \mathcal{A}'_0 \cup \mathcal{A}'_- \cup \mathcal{A}'_*$ be those sequences obtained by deleting the first entry from a sequence in $\mathcal{A} - \mathcal{B}$. Note \mathcal{A}' is a set of monotone drawing sequences of length n where every pair of sequences crosses at most k times and so $|\mathcal{A}'| \leq g(n, k)$.

In any monotone drawing sequence, the symbol $*$ can be only be succeeded or preceded by a 0 or $*$. In particular, the first entry of a sequence in \mathcal{A}'_* is a $*$ or a 0 . If a sequence in $\mathcal{A}'_+ \cup \mathcal{A}'_0 \cup \mathcal{A}'_-$ starts with a $*$, then the original sequence (before deleting the first entry) must

have been $0, *, *, \dots, *$ and this was discounted when we removed \mathcal{B} . Further, the only sequence in $\mathcal{A}'_+ \cup \mathcal{A}'_0 \cup \mathcal{A}'_-$ starting with a 0 is $0, *, *, \dots, *$. Hence, $|\mathcal{A}'_* \cap (\mathcal{A}'_+ \cup \mathcal{A}'_0 \cup \mathcal{A}'_-)| \leq 1$.

We now consider those sequences that are in $\mathcal{A}'_+ \cap (\mathcal{A}'_0 \cup \mathcal{A}'_-)$ and show that every pair of them crosses at most $k - 1$ times. Indeed, suppose that $(a_i), (b_i) \in \mathcal{A}'_+ \cap (\mathcal{A}'_0 \cup \mathcal{A}'_-)$ cross k times. Let j be the first index where $a_j \neq b_j$. We first show that neither a_j nor b_j is a $*$. Suppose that $a_j = *$. Since $(a_i) \in \mathcal{A}'_+ \cup \mathcal{A}'_0 \cup \mathcal{A}'_-$, its first entry is a $+$, 0 , or $-$. Hence, from the definition of monotone drawing sequences, there is some $\ell < j$ with $a_\ell = 0$. By the minimality of j , b_ℓ is also 0 . But $(b_i) \in \mathcal{A}'_+ \cup \mathcal{A}'_0 \cup \mathcal{A}'_-$, so every entry after b_ℓ is a $*$ and so $b_j = *$, contradicting $a_j \neq b_j$.

Since a_j, b_j are distinct and neither is $*$, (a_i) is either above or below (b_i) at j . Without loss of generality (a_i) is below (b_i) at j . But (a_i) and (b_i) agree before j and cross k times, so there are indices $j = i_1 < i_2 < \dots < i_{k+1}$ with (a_i) below (b_i) at i_1, i_3, \dots and (a_i) above (b_i) at i_2, i_4, \dots . As $(a_i) \in \mathcal{A}'_+$, the sequence obtained by appending a $+$ to the start of (a_i) is in \mathcal{A} , while, since $(b_i) \in \mathcal{A}'_0 \cup \mathcal{A}'_-$, the sequence obtained by appending a 0 or $-$ (depending on whether $(b_i) \in \mathcal{A}'_0$ or $(b_i) \in \mathcal{A}'_-$) to the start of (b_i) is in \mathcal{A} . These two sequences cross $k + 1$ times, a contradiction.

Hence, $\mathcal{A}'_+ \cap (\mathcal{A}'_0 \cup \mathcal{A}'_-)$ is a set of monotone drawing sequences of length n , every pair of which cross at most $k - 1$ times and so $|\mathcal{A}'_+ \cap (\mathcal{A}'_0 \cup \mathcal{A}'_-)| \leq g(n, k - 1)$. Similarly, $|\mathcal{A}'_- \cap (\mathcal{A}'_0 \cup \mathcal{A}'_+)| \leq g(n, k - 1)$. Hence, the number of sequences in at least two of $\mathcal{A}'_+, \mathcal{A}'_0, \mathcal{A}'_-, \mathcal{A}'_*$ is at most $2g(n, k - 1) + 1$. By the inclusion-exclusion principle we have

$$\begin{aligned} |\mathcal{A}| &= |\mathcal{A}'_+| + |\mathcal{A}'_0| + |\mathcal{A}'_-| + |\mathcal{A}'_*| + |\mathcal{B}| \leq |\mathcal{A}'| + 2g(n, k - 1) + 2 \\ &\leq g(n, k) + 2g(n, k - 1) + 2. \end{aligned}$$

Hence, $g(n + 1, k) \leq g(n, k) + 2g(n, k - 1) + 2$. It suffices to show that $g(n, k) \leq 2 \cdot \binom{2n}{k+1}$ for all $-1 \leq k \leq n - 1$. There are some easy base cases:

- $g(n, -1) = 1$ for all n (any two sequences cross at least 0 times);
- $g(1, k) = 1$ for all $k \geq 0$ (the only monotone drawing sequence of length 1 is 0);
- $g(2, k) = 4$ for all $k \geq 0$ (the only monotone drawing sequences of length 2 are $+, 0; 0, 0; -, 0$; and $0, *$).

These and the recurrence give $g(n, 0) \leq 4n - 4$ for $n \geq 2$. Hence, the required inequality holds whenever $n \leq 2$ or $k \leq 0$. Suppose the result holds for $n \geq 2$. First suppose that $1 \leq k \leq n - 1$. Then

$$\begin{aligned} g(n + 1, k) &\leq g(n, k) + 2g(n, k - 1) + 2 \leq 2\binom{2n}{k+1} + 4\binom{2n}{k} + 2 \\ &\leq 2\binom{2n}{k+1} + 4\binom{2n}{k} + 2\binom{2n}{k-1} = 2\binom{2n+2}{k+1}. \end{aligned}$$

Finally suppose that $k = n$. Now

$$\begin{aligned} g(n + 1, n) &\leq g(n, n) + 2g(n, n - 1) + 2 = 3g(n, n - 1) + 2 \\ &\leq 6\binom{2n}{n} + 2 \leq 2\binom{2n+2}{n+1}, \end{aligned}$$

where the last inequality holds as $\binom{2n+2}{n+1}\binom{2n}{n}^{-1} = (4n + 2)/(n + 1) > 3$ for $n \geq 2$. \square

Proof of Theorem 1.2(a). Let G be an n -vertex multigraph that has a monotone non-homotopic drawing in which every pair of edges cross at most k times, where $k < n$ are integers. By Lemma 4.1, there exists a set \mathcal{A} of $|E(G)|$ monotone drawing sequences of length n , each pair of which cross at most k times. Applying Lemma 4.2 to \mathcal{A} gives

$$|E(G)| = |\mathcal{A}| \leq 2 \cdot \binom{2n}{k+1}. \quad \square$$

5 Crossing Number Lemma for Monotone Drawings

In this section we prove a tight lower bound on the number of crossings in monotone non-homotopic drawings, [Theorem 1.3](#), which we restate for the reader's convenience.

Theorem 1.3. *For every non-negative integer k there are positive constants α_k , β_k , and γ_k such that the following holds. For every n -vertex m -edge multigraph G with $m \geq 4n$ and every monotone non-homotopic drawing D of G :*

- (a) *If each pair of edges cross at most k times in D , then D has at least $\alpha_k \frac{m^{2+1/k}}{n^{1+1/k}}$ crossings.*
- (b) *If each pair of incident edges cross at most k times in D , then D has at least $\beta_k \frac{m^{2+1/(k+1)}}{n^{1+1/(k+1)}}$ crossings.*
- (c) *If each pair of parallel edges cross at most k times in D , then D has at least $\gamma_k \frac{m^{2+1/(k+2)}}{n^{1+1/(k+2)}}$ crossings.*

The following definitions are due to Kaufmann, Pach, Tóth, and Ueckerdt [13]. A *drawing style* is a collection of drawings of multigraphs. A drawing style D is *edge-deletion-closed* if for every drawing of a multigraph G in style D , for every edge e of G , the drawing of $G - e$ obtained from ϕ by deleting e is in style D .

Let ϕ be a drawing of a multigraph G , and let v be a vertex of G . Let B be a disc of positive radius centred at $\phi(v)$ containing no crossing points between edges in G . Consider the following operation: delete v and the parts of the edges incident to v within B , insert two vertices v_1 and v_2 in the interior of B , for each edge e incident to v , join the end of e at the boundary of B to either v_1 or v_2 , so that no edges cross within B . The resulting multigraph G' and drawing ϕ' of G' is a *vertex split* of G at v .



Figure 7: A vertex-split.

A drawing style D is *split-compatible* if for every drawing of a multigraph G in style D and for every vertex split G' of G , within the disc B the edges can be embedded to obtain a drawing of G' in style D . The *planarisation* of a drawing ϕ of multigraph G is the drawing ϕ' obtained from ϕ as follows: first make local adjustments to ϕ so that no three edges cross at a single point (while not changing the number of crossings), then insert a new vertex at each crossing point. We obtain a drawing ϕ' of a planar graph G' , in which each added vertex has degree 4.

The following lemma is implicitly proved by Kaufmann, Pach, Tóth, and Ueckerdt [13] (building on the work of Pach and Tóth [20], which employs the bisection-width method).

Lemma 5.1 ([13]). *Suppose D is an edge-deletion-closed and split-compatible drawing style, and there are constants $c_1, c_2 > 0$ and $b > 1$ such that for every multigraph G :*

- *if G has a drawing in style D with no crossings, then $|E(G)| \leq c_1|V(G)|$;*
- *if G has a drawing in style D , then $|E(G)| \leq c_2|V(G)|^b$.*
- *if G has a drawing in style D with C crossings, then G has a drawing in style D with at most C crossings whose planarisation is separated.*

Then there exists a constant $\alpha = \alpha(b, c_1, c_2) > 0$ such that for every multigraph G with $|E(G)| > (c_1 + 1)|V(G)|$, every drawing of G in style D has at least $\alpha|E(G)|^{2+1/(b-1)} / |V(G)|^{1+1/(b-1)}$ crossings.

Let $M_k/N_k/O_k$ denote the collection of monotone non-homotopic multigraph drawings in which each pair of edges/incident edges/parallel edges cross at most k times, respectively.

Lemma 5.2. $M_k, N_k,$ and O_k are edge-deletion-closed split-compatible drawing styles. Further if a multigraph G has a drawing in style $M_k/N_k/O_k$ with C crossings, then G has a drawing in style $M_k/N_k/O_k$ with at most C crossings whose planarisation is separated.

Proof. (The proof of this result is very similar to the proof of an analogous result of Kaufmann et al. [13].) It is immediate that $M_k, N_k,$ and O_k are edge-deletion-closed. We now show that they are all split-compatible. Let ϕ be a drawing of a multigraph G in style $M_k, N_k,$ or O_k . Orient each edge of G left-to-right. Let v be a vertex of G . Let $E^-(v)$ and $E^+(v)$ be the sets of incoming and outgoing edges incident to v . Let B be a disc of positive radius centred at $\phi(v)$ containing no crossing points between edges in G . As illustrated in Figure 8, let G' be a multigraph obtained by splitting G at v within disc B , where v is replaced by v_1 and v_2 . Consider each edge of G' to inherit the orientation of the corresponding edge in G . Let E_i be the set of edges of G' incident to v_i . By assumption, no two edges in $E^-(v_1) \cup E^+(v_1) \cup E^+(v_2) \cup E^-(v_2)$ cross within B . Thus, the edges incident with v_1 or v_2 appear as $E^-(v_1), E^+(v_1), E^+(v_2), E^-(v_2)$ in the cyclic ordering determined by where the edges cross the boundary of B . Thus the edges incident to v_1 or v_2 can be drawn within B so that no two edges cross, each edge is x -monotone, and the resulting drawing of G' is non-homotopic. Since each pair of parallel/incident edges in the new drawing were parallel/incident in the original, the styles $M_k, N_k,$ and O_k are all split-compatible.

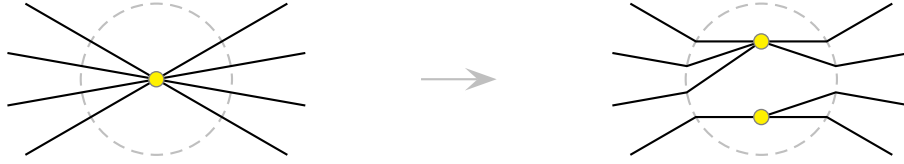


Figure 8: Splitting a vertex in a monotone drawing.

Consider a drawing of a multigraph G in style $M_k/N_k/O_k$ with C crossings. By local adjustments, we may assume that no three edges cross at a single point (while maintaining monotonicity and not creating any crossings). We now show that G has a drawing in the same style with at most C crossings whose planarisation is separated.

Consider the drawing of G in style $M_k/N_k/O_k$ with the smallest number of crossings. This minimal number is some $c \leq C$. Suppose there is a simple closed curve γ formed by portions of two edges e_1 and e_2 , where there is no vertex of G in the interior of γ . Assume that the interior of γ is inclusion-minimal among all such curves. This implies that every time an edge enters γ by crossing e_1 , it must leave γ by crossing e_2 and vice versa.

As illustrated in Figure 9, redraw the diagram replacing e_1 with e'_1 which is the same as e_1 outside γ and is $e_2 \cap \gamma$ on γ and replacing e_2 with e'_2 which is the same as e_2 outside γ and is very close to $e_1 \cap \gamma$ inside γ . By the observation about edges entering and leaving γ , any crossing

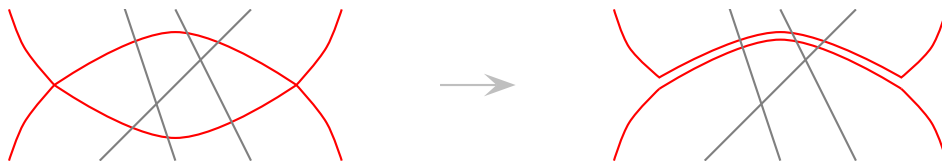


Figure 9: Rerouting a monotone drawing.

between an edge e and e'_1 corresponds to a crossing between e and e_1 and similarly for e_2 . Thus the number of crossings between pairs of parallel edges/incident edges/edge in the resulting drawing is at most k . Furthermore, the resulting drawing has less than c crossings (a crossing between e_1 and e_2 has been removed), is x -monotone, and is non-homotopic (since no vertex is in the interior of γ). This contradiction shows that every simple closed curve γ formed by portions of two edges of G has a vertex of G in its interior. A similar proof shows that γ has a vertex of G in its exterior. Therefore, the planarisation of this drawing is separated. \square

Lemma 5.2 shows that **Lemma 5.1** is applicable to drawing style $M_k/N_k/O_k$. **Theorem 1.2(a)** with $k = 0$ shows that we may take $c_1 = 4$ for all three styles. **Theorem 1.2(a)** shows we may take $b = k + 1$ and $c_2 = \frac{2^{k+2}}{(k+1)!}$ for M_k . **Theorem 1.2(b)** shows we may take $b = k + 2$ and $c_2 = \frac{2^{k+1}}{(k+1)!}$ for N_k . **Theorem 1.2(c)** shows we may take $b = k + 2$ and $c_2 = \frac{2^k}{(k+1)!}$ for O_k . All three parts of **Theorem 1.3** immediately follow.

6 Open Problems

Let $h(n, k)$ denote the maximum number of edges in an n -vertex multigraph that has a non-homotopic k -crossing drawing. **Theorem 1.1** establishes the upper bound $h(n, k) \leq 2^{\mathcal{O}(nk)}$. For lower bounds, (3) gives $h(n, k) \geq 2^{\Omega(\sqrt{nk})}$ while the graphs $G_{n,k}$ from **Section 3** give $h(n, k) \geq \sum_{\ell=1}^k \binom{n}{\ell+1}$. What is the true growth rate of $h(n, k)$? For fixed n and k growing, (2) and (3) show that $h(n, k) = 2^{\Theta(\sqrt{k})}$. Improved bounds in other regimes would be very interesting.

In the monotone setting we have established the correct growth rate in terms of n . Let $h_{\text{mon}}(n, k)$ denote the maximum number of edges in a monotone non-homotopic k -crossing drawing of an n -vertex multigraph. **Theorem 1.2(a)** and the graphs $G_{n,k}$ give $\sum_{\ell=1}^k \binom{n}{\ell+1} \leq h_{\text{mon}}(n, k) \leq 2 \cdot \binom{2n}{k+1}$. Hence, $h_{\text{mon}} = \Theta_k(n^{k+1})$ but the growth rate in terms of k is unclear. It seems likely that the lower bound is closer to the truth. Is there some absolute constant c such that $h_{\text{mon}}(n, k) \leq c \sum_{\ell=0}^k \binom{n}{\ell+1}$?

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