Pure pairs. VI. Excluding an ordered tree

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Abstract

A pure pair in a graph G is a pair (Z_1, Z_2) of disjoint sets of vertices such that either every vertex in Z_1 is adjacent to every vertex in Z_2, or there are no edges between Z_1 and Z_2. It is known that, for every forest F, every graph G with at least two vertices that does not contain F or its complement as an induced subgraph has a pure pair (Z_1, Z_2) with $|Z_1|, |Z_2|$ linear in $|G|$.

Here we investigate what we can say about pure pairs in an ordered graph G, when we exclude an ordered forest F and its complement as induced subgraphs. Fox showed that there need not be a linear pure pair; but Pach and Tomon showed that if F is a monotone path then there is a pure pair of size $c|G|/\log |G|$. We generalise this to all ordered forests, at the cost of a slightly worse bound: we prove that, for every ordered forest F, every ordered graph G with at least two vertices that does not contain F or its complement as an induced subgraph has a pure pair (Z_1, Z_2) with $|Z_1|, |Z_2| \geq |G|^{1-o(1)}$.

1 Introduction

In this paper, all graphs are finite and with no loops or parallel edges, and $|G|$ denotes the number of vertices of G. Two disjoint sets are *complete* to each other if every vertex of the first is adjacent to every vertex of the second, and *anticomplete* if there are no edges between them. A pair (Z_1, Z_2) of subsets of $V(G)$ is *pure* if Z_1 is either complete or anticomplete to Z_2. A graph G is H*-free* if no induced subgraph of G is isomorphic to H; and if $\mathcal{F}$ is a family of graphs then a graph is $\mathcal{F}$*-free* if it is F-free for all $F \in \mathcal{F}$. We denote the complement graph of H by $\overline{H}$. A *hereditary class* or *ideal* of graphs is a class of graphs closed under taking induced subgraphs and under isomorphism.

A class $\mathcal{G}$ of graphs has the *strong Erdős-Hajnal property* if there is some $\epsilon > 0$ such that every graph $G \in \mathcal{G}$ with at least two vertices contains a pure pair (A, B) such that $|A|, |B| \geq \epsilon|G|$. Let us consider the class $\mathcal{G}$ of graphs defined by excluding a finite set $\mathcal{F}$ of graphs: by considering sparse random graphs, it is easy to show that if the class of $\mathcal{F}$-free graphs has the strong Erdős-Hajnal property then $\mathcal{F}$ must contain a forest; and by considering complements, it follows also that $\mathcal{F}$ must contain the complement of a forest. In an earlier paper [3], with Maria Chudnovsky, we proved that this is enough to obtain the strong Erdős-Hajnal property:

1.1 *For every forest F, there exists $\epsilon > 0$ such that every graph G with at least two vertices that is both F-free and $\overline{F}$-free contains a pure pair (Z_1, Z_2) with $|Z_1|, |Z_2| \geq \epsilon|G|$.*

We also proved the stronger result that, for sparse graphs, it is enough to exclude just a forest:

1.2 *For every forest F, there exists $\epsilon > 0$ such every F-free graph G with $|G| \geq 2$ has either*

- *a vertex with degree at least $\epsilon|G|$; or*
- *an anticomplete pair (Z_1, Z_2) with $|Z_1|, |Z_2| \geq \epsilon|G|$.*

Again, considering a sparse random graph shows that this does not hold unless F is a forest.

In this paper we will be concerned with ordered graphs. Let us say an *ordered graph* is a graph with a linear order on its vertex set, and if H is an ordered graph, $\overline{H}$ denotes the complement graph with the same vertex order. Every induced subgraph inherits an order on its vertex set in the natural way: let us say an ordered graph G *contains* an ordered graph H if H is isomorphic to an induced subgraph H' of G, where the isomorphism carries the order on $V(H)$ to the inherited order on $V(H')$, and in this case we call H' a *copy* of H; and an ordered graph is H*-free* if it does not contain the ordered graph H.

One could ask for an analogue of 1.1 for ordered graphs, but it is false. That is a consequence of the following result of Fox [7]:

1.3 *Let H be the ordered graph with three vertices h_1, h_2, h_3 in this order, and with edges h_1h_2 and h_2h_3. For all sufficiently large n, there is an H-free ordered graph G with n vertices, such that there is no pure pair (Z_1, Z_2) in G with $|Z_1|, |Z_2| \geq n/\log(n)$.*

To deduce that 1.1 does not extend to ordered graphs, let T be an ordered forest such that both T and $\overline{T}$ contain H; for instance the ordered forest with four vertices h_1, h_2, h_3, h_4 in this order, in which h_1h_2 and h_2h_4 are edges. Then the graph G of 1.3 contains neither T nor its complement.

On the positive side, Pach and Tomon [10] proved an analogue of 1.2 for monotone paths. A *monotone path* is a path $x_1 \cdots x_k$ with vertices ordered $x_1 \leq \cdots \leq x_k$ (i.e. the path order agrees with the ordering of the graph). Pach and Tomon showed that the bound of 1.3 is in fact sharp for ordered paths (see Fox, Pach and Tóth [9] and Fox [7] for earlier work):

1.4 *Let P be a monotone path. There exists $\varepsilon > 0$ such that every P-free ordered graph G with at least two vertices has either*

- *a vertex with degree at least $\varepsilon|G|$; or*
- *an anticomplete pair (Z_1, Z_2) such that $|Z_1|, |Z_2| \geq \varepsilon|G|/\log(|G|)$.*

In this paper we prove an analogue of 1.2 that holds for all ordered forests. We show that excluding any ordered forest guarantees either a vertex of linear degree or an anticomplete pair of size $|G|^{1-o(1)}$.

1.5 *For every ordered forest T, and all $c > 0$, there exists $\varepsilon > 0$ such that every T-free ordered graph G with at least two vertices has either*

- *a vertex with degree at least $\varepsilon|G|$; or*
- *an anticomplete pair (Z_1, Z_2) such that $|Z_1|, |Z_2| \geq \varepsilon|G|^{1-c}$.*

As we will see in the next section, this implies that excluding an ordered forest and its complement gives a pure pair of size $|G|^{1-o(1)}$:

1.6 *For every ordered forest T, and all $c > 0$, there exists $\varepsilon > 0$ such that if G is an ordered graph with $|G| > 1$ that is both T-free and $\overline{T}$-free, then G contains a pure pair (Z_1, Z_2) with $|Z_1|, |Z_2| \geq \varepsilon|G|^{1-c}$.*

We note that this characterizes ordered forests and their complements, in that no other ordered graphs T have the property of 1.6. An easy random graph argument yields:

1.7 *For every ordered graph T such that T and $\overline{T}$ are not forests, there exists $c > 0$ such that for all $\varepsilon > 0$, there are infinitely many ordered graphs G not containing T or its complement, in which there is no pure pair (Z_1, Z_2) with $|Z_1|, |Z_2| \geq \varepsilon|G|^{1-c}$.*

Proof. Choose an integer g such that both T and $\overline{T}$ have a cycle of length at most g. Let $c < 1/g$, and let $\varepsilon > 0$. If we take a random graph G on n vertices where n is sufficiently large, in which every edge is present independently with probability $\frac{1}{2}n^{1-1/g}$, then with high probability, there will be a set X of at least $n/2$ vertices in which $G[X]$ has no cycle of length at most g (and so contains neither of $T, \overline{T}$) and has no pure pair Z_1, Z_2 with $|Z_1|, |Z_2| \geq \varepsilon|X|^{1-c}$. ∎

2 Reduction to the sparse case

The following very useful result is due to V. Rödl [12]:

2.1 *For every graph H and all $\varepsilon > 0$ there exists $\delta > 0$ such that for every H-free graph G, there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that in one of $G[X]$, $\overline{G}[X]$, every vertex in X has degree less than $\varepsilon|X|$.*

We will show that the same is true when G, H are ordered graphs, because of the following result of Rödl and Winkler [13]:

2.2 *For every ordered graph H, there exists a graph H' such that, for every ordering of $V(H')$, the resultant ordered graph contains H.*

We deduce a version of 2.1 for ordered graphs:

2.3 *For every ordered graph H and all $\varepsilon > 0$ there exists $\delta > 0$ such that for every H-free ordered graph G, there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that in one of $G[X]$, $\overline{G}[X]$, every vertex in X has degree less than $\varepsilon|X|$.*

Proof. Choose H' as in 2.2; and choose δ as in 2.1 with H replaced by H'. If G is an H-free ordered graph, then the underlying unordered graph is H'-free, and so the result holds by the choice of δ. ▮

Proof of 1.6, assuming 1.5. Let T be an ordered forest and $c > 0$. Let setting $\varepsilon = \varepsilon'$ satisfy 1.5. Now let δ satisfy 2.3 with H' replacing H, and let $\varepsilon = \varepsilon'\delta$. We claim that ε satisfies 1.6. To see this, let G be an an ordered graph with $|G| \geq 2$ that is T-free and $\overline{T}$-free. From the choice of δ, there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that in one of $G[X]$, $\overline{G}[X]$, every vertex in X has degree less than $\varepsilon'|X|$. Suppose that $|X| = 1$; then $\varepsilon|G| \leq \delta|G| \leq 1$, and any two vertices of G make a pure pair of singletons sets that satisfy the theorem. So we may assume that $|X| > 1$. By taking complements if necessary, we may assume that every vertex in X has degree in $G[X]$ less than $\varepsilon'|X|$. By 1.5 applied to $G[X]$, there is an anticomplete pair of subsets of X, both of cardinality at least

$$\varepsilon'|X|^{1-c} \geq \varepsilon'\delta^{1-c}|G|^{1-c} \geq \varepsilon'\delta|G|^{1-c} = \varepsilon|G|^{1-c}.$$

This proves 1.6. ▮

Actually there is a further small strengthening, the following (eliminating the multiplicative constant ε):

2.4 *For every ordered forest T, and all $c > 0$, there exists $\varepsilon > 0$ such that, if G is a T-free ordered graph with $|G| > 1/\varepsilon$, then either some vertex has degree at least $\varepsilon|G|$, or there are disjoint $Z_1, Z_2 \subseteq V(G)$ such that $|Z_1|, |Z_2| \geq |G|^{1-c}$ and Z_1 is anticomplete to Z_2.*

Proof of 1.5, assuming 2.4. Let T be an ordered forest and $c > 0$. Let ε be as in 2.4, and by reducing ε we may assume that $\varepsilon \leq 1/2$; we claim that it also satisfies 1.5. Let G be a T-free ordered graph with $|G| \geq 2$. If $|G| > 1/\varepsilon$, then the result follows from 2.4, so we may assume that $|G| \leq \varepsilon$. Any two nonadjacent vertices therefore make an anticomplete pair of singleton sets, both of cardinality at least $\varepsilon|G| \geq \varepsilon|G|^{1-c}$; so we may assume that G is complete. Since $|G| \geq 2$, each vertex has degree $|G| - 1 \geq |G|/2 \geq \varepsilon|G|$, and again the theorem holds. This proves 1.5. ▮

It remains to prove 2.4, and that occupies the remainder of the paper.

3 Blockades

This section concerns graphs rather than ordered graphs. A *blockade* in a graph G is a family $(B_i : i \in I)$ of pairwise disjoint nonempty subsets of $V(G)$, where I is a set of integers. We call the sets B_i $(i \in I)$ its *blocks*, and $|I|$ its *length*. When $I = \{1, \ldots, k\}$ we sometimes write $(B_1, \ldots, B_k)$ for $(B_i : i \in I)$. It is convenient not to insist that all blocks have the same cardinality, but what matters is that the smallest block is not too small. If the smallest block has cardinality w, we call w the *width* and σ the *shrinkage* of the blockade, where $|G|^{1-\sigma} = w$. (If $I = \emptyset$, the width is $|G|$ and shrinkage is 0.)

Let $\mathcal{B} = (B_i : i \in I)$ be a blockade in a graph G, and let $B'_i \subseteq B_i$ for each $i \in I$, all nonempty; then $(B'_i : i \in I)$ is also a blockade, and we call it a *contraction* of $\mathcal{B}$. If $I' \subseteq I$, then $(B_i : i \in I')$ is also a blockade, called a *sub-blockade* of $\mathcal{B}$. A contraction of a sub-blockade of $\mathcal{B}$, or equivalently, a sub-blockade of a contraction, is called a *minor* of $\mathcal{B}$.

Let X, Y be disjoint nonempty subsets of $V(G)$. The *max-degree from X to Y* is defined to be the maximum over all $v \in X$ of the number of neighbours of v in Y. Let $(B_i : i \in I)$ be a blockade in a graph G, and for all distinct $i, j \in I$ let $d_{i,j}$ be the max-degree from B_i to B_j. Define $d_{i,i} = 0$ for all $i \in I$. We call $d_{i,j}$ $(i, j \in I)$ the *max-degree function* of the blockade. Let λ be the maximum of $d_{i,j}/|B_j|$, over all distinct $i, j \in I$; we call λ the *linkage* of $\mathcal{B}$. (If $|I| \leq 1$, the linkage is 0.)

We will prove in this section that if we are given a blockade in a graph G, with sufficiently large length and sufficiently small shrinkage and linkage, then it has a minor $(B_1, \ldots, B_k)$ of any prescribed length k, still with small (but slightly larger) shrinkage and linkage, where all the numbers $d_{i,j}/|B_j|$ are about the same, and for all distinct i, j, many of the vertices in B_i have about $d_{i,j}$ neighbours in B_j.

Let $\mathcal{B} = (B_i : i \in I)$ be a blockade in a graph G, with max-degree function $d_{i,j}$ $(i, j \in I)$. The product of the numbers $d_{i,j}$ for all distinct $i, j \in I$ is called the *max-degree product* of $\mathcal{B}$.

Let $0 < \phi, \mu \leq 1$. We say that $\mathcal{B}$ is *(ϕ, μ)-shrink-resistant* if for all distinct $h, j \in I$, and for all $X \subseteq B_h$ and $Y \subseteq B_j$ with $|X| \geq \mu|B_h|$ and $|Y| \geq \mu|B_j|$, the max-degree from X to Y is more than $d_{h,j}|G|^{-\phi}$. We begin with:

3.1 *Let $\mathcal{B} = (B_i : i \in I)$ be a blockade in a graph G, and let $0 < \phi, \mu \leq 1$. Let $\beta = \mu^{1+\frac{1}{\phi}|I|^2}$. Then either*

- *there exist distinct $h, j \in I$, and $B'_h \subseteq B_h$ and $B'_j \subseteq B_j$ with $|B'_h|/|B_h|, |B'_j|/|B_j| \geq \beta$, such that B'_h, B'_j are anticomplete; or*
- *there is a (ϕ, μ)-shrink-resistant contraction $(B'_i : i \in I)$ of $\mathcal{B}$, such that $|B'_i| \geq \beta|B_i|$ for each $i \in I$.*

Proof. Let $T = \lfloor \frac{1}{\phi}|I|^2 \rfloor$. Choose an integer t with $0 \leq t \leq T + 1$ maximum such that there is a contraction $\mathcal{B}' = (B'_i : i \in I)$ of $\mathcal{B}$ with

- $|B'_i| \geq \mu^t|B_i|$ for each $i \in I$; and
- max-degree product at most $|G|^{|I|^2 - \phi t}$.

(This is possible since we may take $t = 0$ and $\mathcal{B}' = \mathcal{B}$.) Let $d_{h,j}$ $(h, j \in I)$ be the max-degree function of $\mathcal{B}'$.

(1) *We may assume that $d_{h,j} \ge 1$ for all distinct $h, j \in I$, and so $t \le T$.*

If $d_{h,j} < 1$, then $d_{h,j} = 0$, since it is an integer. Thus B'_h, B'_j are anticomplete. Since $t \le T+1$ and hence $\mu^t \ge \mu^{T+1} \ge \beta$, it follows that $|B'_h|/|B_h|, |B'_j|/|B_j| \ge \beta$, and the first outcome of the theorem holds. Thus we may assume that $d_{h,j} \ge 1$ and similarly $d_{j,h} \ge 1$. Hence the max-degree product of $\mathcal{B}'$ is at least one, and since it is at most $|G|^{|I|^2-\phi t}$, it follows that $|I|^2 - \phi t \ge 0$, and so $t \le T$. This proves (1).

(2) *$(B'_i : i \in I)$ is (ϕ, μ)-shrink-resistant.*

Let $h, j \in I$ be distinct, and let $C_h \subseteq B'_h$ and $C_j \subseteq B'_j$, with $|C_h| \ge \mu|B'_h|$ and $|C_j| \ge \mu|B'_j|$. Let d be the max-degree from C_h to C_j. For all $i \in I$ with $i \ne h, j$ let $C_i = B'_i$. From the maximality of t, and since $t \le T$, it follows that the max-degree product of $(C_i : i \in I)$ is more than $|G|^{|I|^2-\phi(t+1)}$. Since it is at most $d/d_{h,j}$ times the max-degree product of $(B'_i : i \in I)$, and the latter is at most $|G|^{|I|^2-\phi t}$, it follows that $d/d_{h,j} > |G|^{-\phi}$. This proves (2).

Since $|B'_i| \ge \mu^t|B_i| \ge \beta|B_i|$ for each $i \in I$, the second outcome of the theorem holds. This proves 3.1. ∎

Let $(B_i : i \in I)$ be a blockade in a graph G, and let $0 < \tau, \phi, \mu \le 1$. We say that τ is a *(ϕ, μ)-band* for $(B_i : i \in I)$ if

- for all distinct $h, j \in I$, the max-degree from B_h to B_j is at most $\tau|B_j|$; and
- for all distinct $h, j \in I$, and all $X \subseteq B_h$ and $Y \subseteq B_j$ with $|X| \ge \mu|B_h|$ and $|Y| \ge \mu|B_j|$, the max-degree from X to Y is more than $\tau|G|^{-\phi}|B_j|$.

3.2 *Let $k \ge 0$ be an integer, and let $0 < \phi, \mu \le 1$ and $\phi \le 1/5$. Then there is an integer $K \ge k$ with the following property. Let G be a graph, and let $(B_i : i \in I)$ be a (ϕ, μ)-shrink-resistant blockade in G, of length at least K. Assume that $1 - \mu \ge |G|^{-\phi}$. Then there exists $I' \subseteq I$ with $|I'| = k$ such that $(B_i : i \in I')$ has a $(5\phi, \mu)$-band.*

Proof. From Ramsey's theorem, there is an integer $K \ge 1$ such that for every complete graph with vertex set I where $|I| \ge K$, and every colouring of its edges with $\lfloor 1/(2\phi) + 2 \rfloor$ colours, there exists $I' \subseteq I$ with $|I'| = k$ such that all edges with both ends in I' have the same colour.

Let $(B_i : i \in I)$ be a (ϕ, μ)-shrink-resistant blockade in G, where $|I| \ge K$, with max-degree function $d_{i,j}$ $(i, j \in I)$.

(1) *$(d_{h,j}/|B_j|)|G|^{-2\phi} < d_{j,h}/|B_h|$ for all distinct $h, j \in I$.*

At least $(1-\mu)|B_h|$ vertices in B_h have more than $d_{h,j}|G|^{-\phi}$ neighbours in B_j, because otherwise there would be a set $X \subseteq B_h$ with $|X| \ge \mu|B_h|$ such that the max-degree from X to B_j is at most $d_{h,j}|G|^{-\phi}$, contradicting that $\mathcal{B}$ is (ϕ, μ)-shrink-resistant. So there are more than $(1-\mu)d_{h,j}|B_h| \cdot |G|^{-\phi}$ edges between B_h and B_j. But there are at most $d_{j,h}|B_j|$; and so $(d_{h,j}/|B_j|)(1-\mu)|G|^{-\phi} < d_{j,h}/|B_h|$. Since $1 - \mu \ge |G|^{-\phi}$, this proves (1).

From (1) it follows that for all $h, j \in I$ with $h < j$, there is an integer $t \geq 0$ such that

$$|G|^{-2t\phi} < d_{h,j}/|B_j|, d_{j,h}/|B_h| \leq |G|^{-2(t-2)\phi},$$

and we call t the *type* of the pair (h, j). We claim that for all such h, j, the type t of (h, j) satisfies $0 < t \leq 1/(4\phi) + 1$. Since $|G|^{-2t\phi} < d_{h,j}/|B_j| \leq 1$, it follows that $t > 0$. Since $d_{h,j} \geq 1$ (from the definition of (ϕ, μ)-shrink-resistant), and $|B_j| \leq |G|$, it follows that $1/|G| \leq d_{h,j}/|B_j| \leq |G|^{-2(t-2)\phi}$, and so $1 \leq |G|^{1-2(t-2)\phi}$, that is, $2(t-2)\phi \leq 1$. This proves our claim that $0 < t \leq 1/(2\phi) + 2$. Thus t is one of the integers $1, \ldots, \lfloor 1/(2\phi) + 2 \rfloor$.

From the choice of K, there exists $I' \subseteq I$ with $|I'| = k$ such that every pair (h, j) with $h < j$ and $h, j \in I'$ has the same type, t say. Let $\tau = |G|^{-2(t-2)\phi}$; then for all distinct $h, j \in I'$,

$$\tau|G|^{-4\phi} \leq d_{h,j}/|B_j| \leq \tau.$$

We claim that τ is a $(5\phi, \mu)$-band for $(B_i : i \in I')$. To show this, it remains to show that for all distinct $h, j \in I'$, and for all $X \subseteq B_h$ and $Y \subseteq B_j$ with $|X| \geq \mu|B_h|$ and $|Y| \geq \mu|B_j|$, the max-degree from X to Y is more than $\tau|G|^{-5\phi}|B_j|$. But $\mathcal{B}$ is (ϕ, μ)-shrink-resistant, and so the max-degree from X to Y is more than $d_{h,j}|G|^{-\phi}$; and since $d_{h,j} \geq \tau|G|^{-4\phi}|B_j|$, the claim follows. This proves 3.2. ∎

By combining 3.1 and 3.2, we deduce:

3.3 *Let $k \geq 0$ be an integer, and let $0 < \phi, \mu \leq 1$. Then there exists an integer $K > 0$ with the following property. Let $\mathcal{B} = (B_i : i \in I)$ be a blockade of length at least K in a graph G, where $|G|^{\phi/5} \geq 1/(1-\mu)$. Let $\beta = \mu^{1+\frac{5}{\phi}|I|^2}$. Then either*

- *there exist distinct $h, j \in I$, and $B'_h \subseteq B_h$ and $B'_j \subseteq B_j$ with $|B'_h|/|B_h|, |B'_j|/|B_j| \geq \beta$, such that B'_h, B'_j are anticomplete; or*
- *there exist $I' \subseteq I$ with $|I'| = k$, and a subset $B'_i \subseteq B_i$ for each $i \in I'$, such that $|B'_i| \geq \beta|B_i|$ for each $i \in I'$, and $(B'_i : i \in I')$ has a (ϕ, μ)-band.*

Proof. Let K satisfy 3.2 with ϕ replaced by $\phi/5$. Let G be a graph with $|G|^{\phi/5} \geq 1/(1-\mu)$. By 3.1, either

- there exist distinct $h, j \in I$, and $B'_h \subseteq B_h$ and $B'_j \subseteq B_j$ with $|B'_h|/|B_h|, |B'_j|/|B_j| \geq \beta$, such that B'_h, B'_j are anticomplete; or
- there is a $(\phi/5, \mu)$-shrink-resistant contraction $\mathcal{B}' = (B'_i : i \in I)$ of $\mathcal{B}$, such that $|B'_i| \geq \beta|B_i|$ for each $i \in I$.

In the first case the first outcome of the theorem holds. In the second case, by 3.2 applied to $\mathcal{B}'$ with ϕ replaced by $\phi/5$, the second outcome of the theorem holds. This proves 3.3. ∎

Consequently we have:

3.4 *Let $k \geq 0$ be an integer, and let $0 < c, \phi, \mu, \sigma, \Sigma, \Lambda \leq 1$ with $\sigma < \Sigma < c$. Then there exist $\lambda > 0$ and integers N and $K \geq 2$, with the following property. Let G be a graph with $|G| \geq N$, such that there do not exist $Z_i \subseteq V(G)$ with $|Z_i| \geq |G|^{1-c}$ for $i = 1, 2$, disjoint and anticomplete. Let $\mathcal{B} = (B_i : i \in I)$ be a blockade of length at least K in G, with shrinkage at most σ and linkage at most λ. Then there exist $I' \subseteq I$ with $|I'| = k$, and a subset $B'_i \subseteq B_i$ for each $i \in I'$, such that $(B'_i : i \in I')$ has shrinkage at most Σ, and has a (ϕ, μ)-band which is at most Λ.*

Proof. Let K satisfy 3.3, and let $\beta = \mu^{1+\frac{5}{\phi}K^2}$. Let $N \geq 0$ such that $N^{\Sigma-\sigma} \geq 1/\beta$, and $N^{\phi/5} \geq 1/(1-\mu)$. Let $\lambda = \beta\Lambda$. Let G be a graph with $|G| \geq N$, such that there do not exist $Z_i \subseteq V(G)$ with $|Z_i| \geq |G|^{1-c}$ for $i = 1, 2$, disjoint and anticomplete. Let $\mathcal{B} = (B_i : i \in I)$ be a blockade of length at least K in G, with shrinkage at most σ and linkage at most λ. If $h, j \in I$ are distinct, and $B'_h \subseteq B_h$ and $B'_j \subseteq B_j$ with $|B'_h|/|B_h|, |B'_j|/|B_j| \geq \beta$, then B'_h, B'_j are not anticomplete, since $|B'_h| \geq \beta|B_h| \geq \beta|G|^{1-\sigma} \geq |G|^{1-c}$ and similarly $|B'_j| \geq |G|^{1-c}$. Thus the first outcome of 3.3 does not hold. Since $|G|^{\phi/5} \geq 1/(1-\mu)$, the second outcome of 3.3 holds, that is, there exist $I' \subseteq I$ with $|I'| = k$, and a subset $B'_i \subseteq B_i$ for each $i \in I'$, such that $|B'_i| \geq \beta|B_i|$ for each $i \in I'$, and $(B'_i : i \in I')$ has a (ϕ, μ)-band. Since $\mathcal{B}$ has shrinkage at most σ, and $|B'_i| \geq \beta|B_i|$ for each $i \in I'$, it follows that $(B'_i : i \in I')$ has shrinkage at most Σ, because $\beta|G|^{1-\sigma} \geq |G|^{1-\Sigma}$. Also, since $\mathcal{B}$ has linkage at most λ, and $|B'_i| \geq \beta|B_i|$ for each $i \in I'$, it follows that $(B'_i : i \in I')$ has linkage at most $\lambda/\beta = \Lambda$, and therefore there is a (ϕ, μ)-band for $(B'_i : i \in I')$ that is at most Λ. This proves 3.4. ∎

4 Rainbow trees

If $\mathcal{B}$ is a blockade in a graph G, an induced subgraph H of G is *$\mathcal{B}$-rainbow* if every vertex of H belongs to some block of $\mathcal{B}$, and every block of $\mathcal{B}$ contains at most one vertex of H. If $A, B \subseteq V(G)$ are disjoint, we say A *covers* B if every vertex of B has a neighbour in A.

Here is the idea of the proof of 2.4. We extend the concept of "blockade" to ordered graphs; in an ordered graph, a blockade is a sequence of subsets of the vertex set, such that if $i < j$ then all vertices of B_i appear before the first vertex of B_j, in the ordering of the vertices of G. We will work by induction on $|V(T)|$; we will delete a leaf of T, find a copy of the remainder of T by induction, and then replace the leaf. To keep control of where this "remainder of T" appears, so that we can find a replacement for the missing leaf, we might try to prove a stronger statement:

- Let G be an ordered graph such that no two subsets of $V(G)$ of cardinality at least $|G|^{1-c}$ are disjoint and anticomplete, and every vertex has degree less than $\varepsilon|G|$, where $\varepsilon > 0$ is some sufficiently small constant. If $\mathcal{B}$ is a blockade in G of sufficient length and and sufficiently small shrinkage, then there is a $\mathcal{B}$-rainbow copy of T.

Unfortunately, to make the induction on $|V(T)|$ work, the sizes of the blocks of $\mathcal{B}$ need to be sublinear in $|G|$; and then the statement is not true, because for instance it might be that every block was complete to every other block. We have to restrict ourselves to blockades where, although the width might be sublinear in $|G|$, each vertex of each block is only adjacent to a small linear fraction of each other block, that is, the linkage is at most some fixed constant < 1. Given this, we can omit the condition that every vertex has degree less than $\varepsilon|G|$. So our goal is:

- Let G be an ordered graph such that no two subsets of $V(G)$ of cardinality at least $|G|^{1-c}$ are disjoint and anticomplete. If $\mathcal{B}$ is a blockade in G, of sufficient length and and sufficiently small shrinkage and linkage, then there is a $\mathcal{B}$-rainbow copy of T.

To prove this, we may assume that T is a tree, and that the result is true for all trees with fewer vertices. Now let T' be obtained from T by deleting a leaf v, and let u be the neighbour of v in T. We will find a rainbow copy of T' in some minor $\mathcal{B}'$ of $\mathcal{B}$, and then try and find a vertex that can play the role of the missing leaf. But this vertex not only has to be adjacent to the vertex of the copy of T' that corresponds to u, and nonadjacent to all the others, but also has to fit in the right place in the linear order of T. To arrange this, we will

- use 3.4 to construct a blockade $\mathcal{B}'$, a minor of $\mathcal{B}$, twice as long (plus one) as we need to find a rainbow copy of T', say $(B'_1, \ldots, B'_{2k+1})$;
- prove that for all even i and all odd j, there is a subset X of B'_j that covers B'_i and is anticomplete to $B_{i'}$ for all other even values of i;
- find a copy of T' that is $\mathcal{B}'$-rainbow, using only the blocks B'_i with i even (and to simplify notation, we assume that this actually is T', rather than just a copy of it);
- choose a block B'_j with j odd, in the right place in the order to provide the missing vertex v; let B'_i be the block that contains u; let $X \subseteq B'_j$ cover B'_i and be anticomplete to $B'_{i'}$ for all other even i', and choose $x \in X$ adjacent to u.

Then adding x to T' gives the copy of T we need. The second step above is the part that needs attention, and that is the goal of the next section.

5 Covering with leaves

Again, this section concerns graphs rather than ordered graphs.

5.1 *Let $k \geq 0$ be an integer, and let $0 < \tau, \phi, \mu \leq 1$, with $2k\mu \leq 1$, and $\phi \leq 1/2$, and $4k^2\tau \leq 1$. Let $\{0\}, H, I, J$ be pairwise disjoint sets of integers, with union of cardinality k. Let G be a graph, such that $|G|^\phi \geq 2(16k^2)^k$. Let $\mathcal{A} = (A_i : i \in \{0\} \cup H \cup I \cup J)$ be a blockade in G, such that:*

- *τ is a (ϕ, μ)-band for $(A_i : i \in \{0\} \cup I \cup J)$; and*
- *for each $h \in H$, and each $i \in \{0\} \cup I \cup J$, the max-degree from A_h to A_i is at most $\tau|A_i|$.*

Then for all $i \in \{0\} \cup H \cup I \cup J$ there exists $B_i \subseteq A_i$, such that:

- *$|B_0| \geq |G|^{-k\phi}|A_0|$, and $|B_i| \geq |A_i|/2$ for all $i \in H \cup I$, and $B_i = A_i$ for all $i \in J$;*
- *for all $j \in J$ there exists $C_j \subseteq A_j$ that covers B_0 and is anticomplete to all the sets B_i ($i \in H \cup I$);*
- *2τ is a $(2\phi, 2\mu)$-band for $(B_i : i \in I \cup J)$;*
- *for each $h \in H \cup \{0\}$, and each $i \in I \cup J$, the max-degree from B_h to B_i is at most $2\tau|B_i|$; and*
- *for each $h \in H$, the max-degree from B_0 to B_h is at most $4k\tau|B_h|$.*

Proof. We may assume that $k \geq 2$.

(1) *There exists $D_j \subseteq A_j$ with $|D_j| \geq |A_j|/2$ for each $j \in J \cup \{0\}$, such that:*

- *for each $j \in J \cup \{0\}$ and each $h \in H$, every vertex in D_j has fewer than $2k\tau|A_h|$ neighbours in A_h; and*
- *for each $j \in J$, every vertex in D_0 has more than $\tau|G|^{-\phi}|A_j|$ neighbours in D_j.*

For each $j \in J \cup \{0\}$ and each $h \in H$, let $Z_{j,h}$ be the set of vertices in A_j that have at least $2k\tau|A_h|$ neighbours in A_h; and for each $j \in J \cup \{0\}$, let

$$D_j = A_j \setminus \bigcup_{h \in H} Z_{j,h}.$$

Since every vertex in A_h has at most $\tau|A_j|$ neighbours in A_j, there are at most $\tau|A_h| \cdot |A_j|$ edges between A_h and A_j, and so $2k\tau|A_h| \cdot |Z_{j,h}| \leq \tau|A_h| \cdot |A_j|$, that is, $|Z_{j,h}| \leq |A_j|/(2k)$. For each $j \in J$, the union of the sets $Z_{j,h}$ (over all $h \in H$) has cardinality at most $|A_j|/2$, and so the first statement of (1) holds.

For each $j \in J$, let Z_j be the set of vertices in A_0 that have at most $\tau|G|^{-\phi}|A_j|$ neighbours in D_j. Since τ is a (ϕ, μ)-band for $(A_i : i \in I \cup J \cup \{0\})$, and $|D_j| \geq |A_j|/2 \geq \mu|A_j|$, it follows that $|Z_j| \leq \mu|A_0| \leq |A_0|/(2k)$. Thus the union of the sets $Z_{0,h}$ ($h \in H$) and the sets Z_j ($j \in J$) has cardinality at most $|A_0|/2$, since $|H \cup J| \leq k$. This proves the second statement of (1), and so proves (1).

(2) *Let $Y \subseteq D_0$ and $j \in J$. Then there exists $Y' \subseteq Y$ with $|Y'| \geq |G|^{-\phi}|Y|/(16k^2)$, and a subset $C_j \subseteq D_j$, such that C_j covers Y', and for each $h \in H \cup I$, at most $|A_h|/(2k)$ vertices in A_h have a neighbour in C_j.*

We may assume that $Y \neq \emptyset$. Every vertex in Y belongs to D_0, and hence has more than $\tau|G|^{-\phi}|A_j|$ neighbours in D_j. Choose $X \subseteq D_j$ maximal such that

- $|X| \leq 1/(4k^2\tau)$; and
- $|Y'| \geq (\tau/2)|G|^{-\phi}|X| \cdot |Y|$, where Y' is the set of vertices in Y that have a neighbour in X.

For each $i \in I$, since $|X| \leq 1/(4k^2\tau)$ and every vertex in X has at most $\tau|A_i|$ neighbours in A_i, it follows that at most $|A_i|/(4k^2) \leq |A_i|/(2k)$ vertices in A_i have a neighbour in X. For each $h \in H$, since every vertex in D_j has at most $2k\tau|A_h|$ neighbours in A_h, it follows that at most

$$2k\tau|A_h| \cdot |X| \leq 2k\tau|A_h|/(4k^2\tau) = |A_h|/(2k)$$

vertices in A_h have a neighbour in X. Thus if $|Y'| \geq |Y|/2$ then (2) holds, since $1/2 \geq |G|^{-\phi}/(16k^2)$; so we may assume that $|Y'| < |Y|/2$, and hence $|Y \setminus Y'| \geq |Y|/2$. Every vertex in $Y \setminus Y'$ has at least $\tau|G|^{-\phi}|A_j|$ neighbours in D_j, and none of these neighbours is in X since $Y \setminus Y'$ is anticomplete to X. Thus there exists $v \in D_j \setminus X$ with at least

$$\tau|G|^{-\phi}|A_j|\frac{|Y|/2}{|D_j \setminus X|} \geq (\tau|Y|/2)|G|^{-\phi}$$

neighbours in $Y \setminus Y'$ (since $|A_j| \geq |D_j \setminus X|$). From the maximality of X, replacing X by $X \cup \{v\}$ violates one of the two bullets in the definition of X. The second is satisfied, and so the first is violated, that is, $|X| + 1 > 1/(4k^2\tau)$. Consequently $X \neq \emptyset$, and so $2|X| \geq |X| + 1 > 1/(4k^2\tau)$, and therefore $|X| > 1/(8k^2\tau)$. Since $|Y'| \geq (\tau/2)|G|^{-\phi}|X|\cdot|Y|$, it follows that $|Y'| > \tau|G|^{-\phi}|Y|/(16k^2\tau) = |G|^{-\phi}|Y|/(16k^2)$. This proves (2).

By $|J|$ applications of (2), one for each $j \in J$, applied initially with $Y = D_0$, we obtain that there exists $B_0 \subseteq D_0$ with $|B_0| \geq |G|^{-|J|\phi}(16k^2)^{-|J|}|D_0|$, and for each $j \in J$ there exists a subset $C_j \subseteq D_j$, such that C_j covers B_0, and for each $h \in H \cup I$, at most $|A_h|/(2k)$ vertices in A_h have a neighbour in C_j.

Since $|J| < k$, and $(16k^2)^{-|J|} \geq 2|G|^{-\phi}$, we obtain $|B_0| \geq 2|G|^{-k\phi}|D_0| \geq |G|^{-k\phi}|A_0|$. For each $h \in H \cup I$, let B_h be the set of vertices in A_h with no neighbours in any of the sets C_j $(j \in J)$. Then $|B_h| \geq |A_h|/2$.

The conclusion of 5.1 has five bullets, and we have shown that the first two hold. For the third bullet, we need the following:

(3) *Let $B_i = A_i$ for $i \in J$; then 2τ is a $(2\phi, 2\mu)$-band for $(B_i : i \in I \cup J)$, and $(B_i : i \in I \cup J)$ has linkage at most 2τ.*

Since $\tau \leq 1/(4k^2) \leq 1/2$, certainly $2\tau \leq 1$. We claim that 2τ is a $(2\phi, 2\mu)$-band for $(B_i : i \in I \cup J)$. To show this, let $i, j \in I \cup J$ be distinct; we must show that:

- the max-degree from B_i to B_j is at most $2\tau|B_j|$; and
- for all $X \subseteq B_i$ and $Y \subseteq B_j$ with $|X| \geq 2\mu|B_i|$ and $|Y| \geq 2\mu|B_j|$, the max-degree from X to Y is more than $2\tau|G|^{-2\phi}|B_j|$.

Since the max-degree from A_i to A_j is at most $\tau|A_i|$, and $|B_j| \geq |A_j|/2$, it follows that the max-degree from B_i to B_j is at most $\tau|A_j| \leq 2\tau|B_j|$. Now let $X \subseteq B_i$ and $Y \subseteq B_j$ with $|X| \geq 2\mu|B_i|$ and $|Y| \geq 2\mu|B_j|$. Thus $|X| \geq \mu|A_i|$ and $|Y| \geq \mu|A_j|$; and since τ is a (ϕ, μ)-band for $(A_i : i \in I \cup J)$, it follows that the max-degree from X to Y is more than $\tau|G|^{-\phi}|A_j| \geq 2\tau|G|^{-2\phi}|B_j|$ since $|A_j| \geq |B_j|$ and $|G|^{-\phi} \leq 1/2$. This proves that 2τ is a $(2\phi, 2\mu)$-band for $(B_i : i \in I \cup J)$.

Next we must show that $(B_i : i \in I \cup J)$ has linkage at most 2τ. Let $i, j \in I \cup J$ be distinct. The linkage of $(A_i : i \in I \cup J)$ is at most τ, and so the max-degree from A_i to A_j is at most $\tau|A_j|$. Thus the max-degree from B_i to B_j is at most $\tau|A_j| \leq 2\tau|B_j|$. This proves (3).

Consequently the third bullet of the conclusion of the theorem is satisfied. The fourth holds, since for $h \in H \cup \{0\}$, and $i \in I \cup J$, every vertex in B_h has at most $\tau|A_i| \leq 2\tau|B_i|$ neighbours in B_i. And the fifth holds since every vertex in B_0 belongs to D_0, and so has at most $2k\tau|A_h| \leq 4k\tau|B_h|$ neighbours in A_h. This proves 5.1. ∎

In order to use 5.1 we need the following definition. Let $k \geq 0$ be an integer, and let H, I, J be disjoint sets of integers, with union of cardinality k. Let $0 < \tau, \phi, \mu, \lambda \leq 1$. Let $\mathcal{B} = (B_i : i \in H \cup I \cup J)$ be a blockade in a graph G. Suppose that:

- $(B_h : h \in H)$ has width at least w and linkage at most λ;
- $(B_i : i \in I)$ has width at least W;
- for each $h \in H$ and $j \in J$, there exists $X \subseteq B_j$ such that X covers B_h and is anticomplete to B_i for all $i \in (H \cup I) \setminus \{h\}$;

- τ is a (ϕ, μ)-band for $(B_i : i \in I \cup J)$;
- for each $h \in H$, and each $i \in I \cup J$, the max-degree from B_h to B_i is at most $\tau|B_i|$.

In these circumstances we say that $\mathcal{B}$ is *leaf-covered with partition* (H, I, J) *and parameters*

$$w, W, \lambda, \phi, \mu, \tau.$$

From 5.1 we deduce:

5.2 *Let $k \geq 0$ be an integer, and let $0 < \tau, \phi, \lambda, \mu \leq 1$, with $2k\mu \leq 1$, and $\phi \leq 1/2$, and $4k^2\tau \leq 1$, and $\lambda \geq 2k\tau$. Let G be a graph, such that $|G|^\phi \geq 2(16k^2)^k$. Let $\mathcal{A}$ be a blockade in G that is leaf-covered with partition (H, I, J) and parameters*

$$w, W, \lambda, \phi, \mu, \tau,$$

where $|G|^{-k\phi}W \geq w/4$. Suppose that $g \in I$. Then there is a contraction $\mathcal{B} = (B_i : i \in H \cup I \cup J)$ of $\mathcal{A}$, such that $\mathcal{B}$ is leaf-covered with partition $(H \cup \{g\}, I \setminus \{g\}, J)$ and parameters

$$w/4, W/2, 4\lambda, 2\phi, 2\mu, 2\tau.$$

Proof. We may assume that $g = 0$. Thus $H, I \setminus \{0\}, J, \{0\}$ are pairwise disjoint with union of cardinality k. By 5.1 with I replaced by $I \setminus \{0\}$, for all $i \in H \cup I \cup J$ there exists $A'_i \subseteq A_i$, such that:

- $|A'_0| \geq |G|^{-k\phi}|A_0|$, and $|A'_i| \geq |A_i|/2$ for all $i \in (H \cup I) \setminus \{0\}$, and $A'_j = A_j$ for all $j \in J$;
- for all $j \in J$ there exists $C_j \subseteq A_j$ that covers A'_0 and is anticomplete to all the sets A'_i $(i \in (H \cup I) \setminus \{0\})$;
- 2τ is a $(2\phi, 2\mu)$-band for $(A'_i : i \in (I \cup J) \setminus \{0\})$;
- for each $h \in H \cup \{0\}$, and each $i \in (I \cup J) \setminus \{0\}$, the max-degree from A'_h to A'_i is at most $2\tau|A'_i|$; and
- for each $h \in H$, the max-degree from A'_0 to A'_h is at most $4k\tau|A'_h|$.

Let $B_i = A'_i$ for each $i \in I \cup J \cup \{0\}$. For each $h \in H$, let B_h be the set of vertices in A'_h that have at most $8k\tau|B_0|$ neighbours in B_0. Since there are at most $4k\tau|A'_h| \cdot |B_0|$ edges between A'_h and B_0, it follows that $|B_h| \geq |A'_h|/2$. Since the linkage of $(A_h : h \in H)$ is at most λ, the linkage of $(B_h : h \in H)$ is therefore at most 2λ. We claim that $(B_i : i \in H \cup I \cup J)$ is leaf-covered with partition $(H \cup \{0\}, I \setminus \{0\}, J)$ and parameters

$$w/4, W/2, 4\lambda, 2\phi, 2\mu, 2\tau.$$

To show this, we must check that:

- $(B_h : h \in H \cup \{0\})$ has width at least $w/4$ and linkage at most 2λ;
- $(B_i : i \in I \setminus \{0\})$ has width at least $W/2$;

- for each $h \in H \cup \{0\}$ and $j \in J$, there exists $X \subseteq B_j$ such that X covers B_h and is anticomplete to B_i for all $i \in (H \cup I) \setminus \{h\}$;
- 2τ is a $(2\phi, 2\mu)$-band for $(B_i : i \in (I \cup J) \setminus \{0\})$; and
- for each $h \in H \cup \{0\}$, and each $i \in (I \cup J) \setminus \{0\}$, the max-degree from A_h to A_i is at most $2\tau|A_i|$.

For the first bullet: for $h \in H$, $|B_h| \geq |A'_h|/2 \geq |A_h|/4 \geq w/4$, and $|B_0| \geq |G|^{-k\phi}|A_0| \geq |G|^{-k\phi}W \geq w/4$, so $(B_h : h \in H \cup \{0\})$ has width at least $w/4$. Since $(A_h : h \in H)$ has linkage at most λ, it follows that $(B_h : h \in H)$ has linkage at most 4λ (because $|B_h| \geq |A_h|/4$ for each $h \in H$). For each $h \in H$, every vertex in B_0 has at most $4k\tau|A'_h|$ neighbours in A'_h, and hence at most $8k\tau|B_h| \leq 4\lambda|B_0|$ neighbours in B_h; and every vertex in B_h has at most $8k\tau|B_0| \leq 4\lambda|B_0|$ neighbours in B_0. Thus the linkage of $(B_h : h \in H \cup \{0\})$ is at most 4λ. This proves that the first bullet holds.

The second bullet holds since $|B_i| \geq |A_i|/2 \geq W/2$ for each $i \in I \setminus \{0\}$. The third bullet holds, since if $h \in H$ the statement is true by hypothesis, and if $h = 0$ then the statement is true because we may take $X = C_j$. The fourth bullet holds by the application of 5.1. Finally, the fifth bullet holds since if $v \in A_h$ where $h \in H \cup \{0\}$, and $i \in (I \cup J) \setminus \{0\}$, then v has at most $\tau|A_i| \leq 2\tau|B_i|$ neighbours in A_i. This proves 5.2. ∎

By $|H|$ applications of 5.2, taking g to be each element of H in turn, we deduce:

5.3 *Let $k \geq 0$ be an integer, and let I, J be disjoint sets of integers with union of cardinality k. Let $0 < \tau, \phi, \mu \leq 1$, with $k2^k\mu \leq 1$, and $\phi 2^k \leq 1$, and $k^2 2^{k+1}\tau \leq 1$. Let G be a graph with $|G|^\phi \geq 2(16k^2)^k$. Let $\mathcal{A} = (A_i : i \in I \cup J)$ be a blockade in G with a (ϕ, μ)-band τ. Let W be the width of $(A_i : i \in I)$. For all $H \subseteq I$, there is a contraction $\mathcal{B} = (B_i : i \in I \cup J)$ of $\mathcal{A}$, such that $\mathcal{B}$ is leaf-covered with partition $(H, I \setminus H, J)$ and parameters*

$$4^{-|H|}|G|^{-k2^{k-1}\phi}W,\ 2^{-|H|}W,\ 4^{|H|}k\tau,\ 2^{|H|}\phi,\ 2^{|H|}\mu,\ 2^{|H|}\tau.$$

Proof. We proceed by induction on $|H|$. The result is true when $H = \emptyset$, since τ is a (ϕ, μ)-band for $\mathcal{A} = (A_i : i \in I \cup J)$. Thus we assume that $H \neq \emptyset$. Choose $g \in H$. From the inductive hypothesis, there is a contraction $\mathcal{B}' = (B'_i : i \in I \cup J)$ of $\mathcal{A}$, such that $\mathcal{B}'$ is leaf-covered with partition $(H \setminus \{g\}, I \setminus (H \setminus \{g\}), J)$ and parameters

$$w' = 4^{1-|H|}|G|^{-k2^{k-1}\phi}W,\ W' = 2^{1-|H|}W,\ \lambda' = 4^{|H|-1}k\tau,\ \phi' = 2^{|H|-1}\phi,\ \mu' = 2^{|H|-1}\mu,\ \tau' = 2^{|H|-1}\tau.$$

Since

$$\begin{aligned}
2k\mu' = 2k2^{|H|-1}\mu &\leq 2k2^{k-1}\mu \leq 1\\
\phi' = 2^{|H|-1}\phi &\leq 2^{k-1}\phi \leq 1/2\\
4k^2\tau' = 4k^2 2^{|H|-1}\tau &\leq 4k^2 2^{k-1}\tau \leq 1\\
\lambda' = 4^{|H|-1}k\tau &\geq 2^{|H|-1}k\tau = k\tau'\\
|G|^{\phi'} = |G|^{2^{|H|-1}\phi} &\geq |G|^\phi \geq 2(16k^2)^k\\
|G|^{-k\phi'}W' = |G|^{-k2^{|H|-1}\phi}2^{1-|H|}W &\geq |G|^{-k2^{k-1}\phi}4^{-|H|}W \geq 4^{-|H|}W|G|^{-k2^{k-1}\phi} = w'/4
\end{aligned}$$

it follows from 5.2 that there is a contraction $\mathcal{B} = (B_i : i \in I \cup J)$ of $\mathcal{B}'$, such that $\mathcal{B}$ is leaf-covered with partition $(H \cup \{g\}, I \setminus (H \cup \{g\}), J)$ and parameters

$$w'/4, W'/2, 4\lambda', 2\phi', 2\mu', 2\tau'.$$

But then $\mathcal{B}$ is the required contraction of $\mathcal{A}$. This proves 5.3. ∎

We will only apply 5.3 when $H = I$, and in that case it becomes much simpler, so much so that it is worth stating separately, in the following:

5.4 *Let $k \geq 0$ be an integer, and let H, J be disjoint sets of integers with union of cardinality k. Let $0 < \tau, \phi, \mu \leq 1$, with $k2^k\mu \leq 1$, and $\phi 2^k \leq 1$, and $k^2 2^{k+1}\tau \leq 1$. Let G be a graph with $|G|^\phi \geq 2(16k^2)^k$. Let $\mathcal{A} = (A_i : i \in H \cup J)$ be a blockade in G, with a (ϕ, μ)-band τ. Let W be the width of $(A_i : i \in H)$. For each $h \in H$ there exists $B_h \subseteq A_h$, such that:*

- *$(B_h : h \in H)$ has width at least $4^{-|H|}|G|^{-k2^{k-1}\phi}W$ and linkage at most $4^{|H|}k\tau$; and*
- *for all $h \in H$ and all $j \in J$ there exists $X \subseteq A_j$ that covers B_h and is anticomplete to B_i for all $i \in H \setminus \{h\}$.*

Proof. By 5.3, taking $I = H$, there is a contraction $\mathcal{B} = (B_i : i \in H \cup J)$ of $\mathcal{A}$, such that $\mathcal{B}$ is leaf-covered with partition $(H, \emptyset, J)$ and parameters

$$4^{-|H|}|G|^{-k2^{k-1}\phi}W,\ 2^{-|H|}W,\ 4^{|H|}k\tau,\ 2^{|H|}\phi,\ 2^{|H|}\mu,\ 2^{|H|}\tau.$$

It follows that $(B_h : h \in H)$ has width at least $4^{-|H|}|G|^{-k2^{k-1}\phi}W$ and linkage at most $4^{|H|}k\tau$ (and the other four parameters are irrelevant). This proves 5.4. ∎

By combining 3.4 and 5.4, we obtain:

5.5 *Let $k \geq 0$ be an integer, and let $0 < c, \sigma, \sigma', \lambda' \leq 1$ with $\sigma < \sigma' < c$. Then there exist $\lambda > 0$ and integers $K, N > 0$ with the following property. Let G be a graph with $|G| \geq N$, such that there do not exist $Z_i \subseteq V(G)$ with $|Z_i| \geq |G|^{1-c}$ for $i = 1, 2$, disjoint and anticomplete. Let $\mathcal{A} = (A_i : i \in I)$ be a blockade of length at least K in G, with shrinkage at most σ and linkage at most λ. Then there exist $I' \subseteq I$ with $|I'| = k$, such that for every partition (H, J) of I', there exists $B_h \subseteq A_h$ for each $h \in H$, where*

- *$(B_h : h \in H)$ has shrinkage at most σ' and linkage at most λ'; and*
- *for all $h \in H$ and all $j \in J$ there exists $X \subseteq A_j$ that covers B_h and is anticomplete to B_i for all $i \in H \setminus \{h\}$.*

Proof. Choose Σ with $\sigma < \Sigma < \sigma'$. Let ϕ satisfy $(k2^{k-1} + 1)\phi = \sigma' - \Sigma$. Let $\mu = 2^{-k}/k$, and $\Lambda = \lambda' 4^{-k}/k$. Choose $\lambda > 0$ and integers N_1 and $K \geq 2$ such that 3.4 is satisfied with N replaced by N_1. Choose $N \geq N_1$ such that $N^\phi \geq 2(16k^2)^k$. We claim that λ, K, N satisfy the theorem.

Let G be a graph with $|G| \geq N$, such that there do not exist $Z_i \subseteq V(G)$ with $|Z_i| \geq |G|^{1-c}$ for $i = 1, 2$, disjoint and anticomplete. Let $\mathcal{A} = (A_i : i \in I)$ be a blockade of length at least K in G, with shrinkage at most σ and linkage at most λ. Since $\phi 2^k \leq 1$, and $k^2 2^{k+1}\Lambda \leq k^2 2^{k+1} 4^{-k}/k \leq 1$, it follows from 3.4 that there exist $I' \subseteq I$ with $|I'| = k$, and a subset $A'_i \subseteq A_i$ for each $i \in I'$, such that $\mathcal{A}' = (A'_i : i \in I')$ has a (ϕ, μ)-band $\tau \leq \Lambda$, and has shrinkage at most Σ. Let $W = |G|^{1-\Sigma}$; thus $(A'_i : i \in I')$ has width at least W. Since $|G|^\phi \geq 2(16k^2)^k$, 5.4 implies that for every partition (H, J) of I', there exists $B_h \subseteq A'_h$ for each $h \in H$, such that:

- $(B_h : h \in H)$ has width at least $4^{-|H|}|G|^{-k2^{k-1}\phi}W$ and linkage at most $4^{|H|}k\tau \leq 4^k k\Lambda = \lambda'$; and
- for all $h \in H$ and all $j \in J$ there exists $X \subseteq A'_j$ that covers B_h and is anticomplete to B_i for all $i \in H \setminus \{h\}$.

Since $4^{|H|} \geq 4^{-k} \geq |G|^{-\phi}$, it follows that

$$4^{-|H|}|G|^{-k2^{k-1}\phi}W \geq |G|^{-(1+k2^{k-1})\phi}|G|^{1-\Sigma} = |G|^{1-\sigma'},$$

and so $(B_h : h \in H)$ has shrinkage at most σ'. This proves 5.5. ∎

6 The proof of the main theorem

In this section we use these lemmas to prove 2.4. Let G be an ordered graph, and let H be the unordered graph obtained from G by omitting the ordering. A *blockade* in G is a blockade $(B_i : i \in I)$ in H, such that for all $i, j \in I$ with $i < j$, every vertex of B_i is earlier than each vertex of B_j in the ordering of G. Width, shrinkage and so on are defined as for blockades in unordered graphs.

We will prove:

6.1 *Let $0 < c \leq 1$. For every ordered tree T and all σ with $0 < \sigma < c$, there exist λ with $0 < \lambda \leq 1$, and integers $K, N \geq 0$, with the following property. Let G be an ordered graph with $|G| \geq N$ such that there do not exist disjoint $Z_1, Z_2 \subseteq V(G)$, where $|Z_1|, |Z_2| \geq |G|^{1-c}$ and Z_1 is anticomplete to Z_2. Let $\mathcal{A}$ be a blockade in G of length K, with shrinkage at most σ and linkage at most λ. Then there is an $\mathcal{A}$-rainbow copy of T.*

Proof. We proceed by induction on $|V(T)|$, and we may assume that $|V(T)| \geq 2$. Let v be a leaf of T, and let $T' = T \setminus \{v\}$. Choose σ' with $\sigma < \sigma' < c$. From the inductive hypothesis, there exist λ', K', N' so that 6.1 holds with T, σ, λ, K, N replaced by $T', \sigma', \lambda', K', N'$ respectively. Define $k = 2K' + 1$. Choose λ, K, N such that 5.5 is satisfied. We claim that λ, K, N satisfy 6.1.

Let G be an ordered graph with $|G| \geq N$ such that there do not exist disjoint $Z_1, Z_2 \subseteq V(G)$, where $|Z_1|, |Z_2| \geq |G|^{1-c}$ and Z_1 is anticomplete to Z_2. Let $\mathcal{A} = (A_i : i \in I)$ be a blockade in G of length K, with shrinkage at most σ and linkage at most λ. From the choice of λ, K, N, there exist $I' \subseteq I$ with $|I'| = k$, such that for every partition (H, J) of I', there exists $B_h \subseteq A_h$ for each $h \in H$, where

- $(B_h : h \in H)$ has shrinkage at most σ' and linkage at most λ'; and
- for all $h \in H$ and all $j \in J$ there exists $X \subseteq A_j$ that covers B_h and is anticomplete to B_i for all $i \in H \setminus \{h\}$.

Let $I' = \{i_1, \ldots, i_k\}$ where $i_1 < i_2 < \cdots < i_k$; let $H = \{i_2, i_4, i_6, \ldots, i_{2K'}\}$ and $J = \{i_1, i_3, i_5, \ldots, i_{2K'+1}\}$ (this is well-defined since $k = 2K'+1$), and choose $B_h \subseteq A_h$ for each $h \in H$, satisfying the two bullets above. Let $\mathcal{B} = (B_h : h \in H)$. It follows from the choice of $N', K', \sigma', \lambda'$ that there is a $\mathcal{B}$-rainbow copy of T', and to simplify notation, we assume that this $\mathcal{B}$-rainbow copy of T' is T' itself. We recall that v is a leaf of T, and $T' = T \setminus \{v\}$. Let the linear order of the vertices of T be $(v_1, \ldots, v_n)$, where $v = v_t$. Since the vertices of T' appear in the boxes of $\mathcal{B}$ in the correct order, and J interleaves H, there exists $j \in J$ such that for all $i \in \{1, \ldots, n\} \setminus \{t\}$:

- if $i < t$ then $v_i \in B_h$ for some $h \in H$ with $h < j$;
- if $i > t$ then $v_i \in B_h$ for some $h \in H$ with $h > j$.

Let u be the neighbour of v in T, and let $u \in B_h$. From the first set of bullets in this proof, there exists $X \subseteq A_j$ that covers B_h and is anticomplete to B_i for all $i \in H \setminus \{h\}$. Choose $v' \in X$ adjacent to u. Then adding v' to T' gives a $\mathcal{B}$-rainbow, and hence $\mathcal{A}$-rainbow, copy of T. This proves 6.1. ▮

Finally we deduce 2.4, which we restate (and which we have already shown to imply 1.6):

6.2 *For every ordered forest T, and all $c > 0$, there exists $\varepsilon > 0$ such that, if G is an ordered graph with $|G| > 1/\varepsilon$, and every vertex has degree less than $\varepsilon|G|$, and there do not exists disjoint anticomplete sets Z_1, Z_2 with $|Z_1|, |Z_2| \geq |G|^{1-c}$, then G contains T.*

Proof. By adding vertices and edges to T if necessary, we may assume that T is an ordered tree. Let $\sigma = c/2$, and let λ, K, N satisfy 6.1. Choose $M \geq \max(N, K)$ such that $M^\sigma \geq 2K$, and let $\varepsilon = \min(1/M, \lambda/(2K))$. Let G be an ordered graph with $|G| > 1/\varepsilon$, such that every vertex has degree less than $\varepsilon|G|$, and there do not exist disjoint anticomplete sets Z_1, Z_2 with $|Z_1|, |Z_2| \geq |G|^{1-c}$.

Since $|G| \geq K$ there is a blockade $\mathcal{B}$ in G of length K and width $W \geq \lfloor |G|/K \rfloor \geq |G|/(2K)$. Hence $W \geq |G|^{1-\sigma}$, because $|G| \geq M$ and so $|G|/(2K) \geq |G|^{1-\sigma}$; and therefore $\mathcal{B}$ has shrinkage at most σ. Since every vertex has degree less than $\varepsilon|G|$, it follows that $\mathcal{B}$ has linkage at most $\varepsilon|G|/W \leq 2K\varepsilon \leq \lambda$. But then from 6.1 there is a copy of T in G. This proves 6.2 and hence proves 2.4. ▮

7 Conclusion

Let us say that a class $\mathcal{G}$ of graphs or ordered graphs has the *Erdős-Hajnal property* if there exists $c > 0$ such that every $G \in \mathcal{G}$ satisfies $\alpha(G)\omega(G) \geq |G|^c$. The Erdős-Hajnal conjecture [5, 6] asserts that, for every H, the class of H-free graphs has the Erdős-Hajnal property:

7.1 Conjecture: *For every graph H, there exists $c > 0$ such that every H-free graph G satisfies*

$$\alpha(G)\omega(G) \geq |G|^c.$$

Alon, Pach and Solymosi [2] showed that the Erdős-Hajnal conjecture is equivalent to the following statement for ordered graphs.

7.2 Conjecture: *For every ordered graph H, there exists $c > 0$ such that every H-free ordered graph G satisfies*

$$\alpha(G)\omega(G) \geq |G|^c.$$

The Erdős-Hajnal conjecture (7.1; or equivalently 7.2) has only been proved for a very small family of graphs. For example, it remains open for most forests; indeed, it is open even for the five-vertex path. However, 1.1 allows us to say something if we exclude both a forest and its complement. For an ideal $\mathcal{G}$ of graphs, the strong Erdős-Hajnal property implies the Erdős-Hajnal property (see [1, 8]); and the same follows straightforwardly for ideals of ordered graphs. Thus 1.1 implies the following:

7.3 *For every forest F, the class of graphs that are both F-free and $\overline{F}$-free has the Erdős-Hajnal property.*

For *ordered* forests, however, the situatation is different: 1.4 and 1.5 are not strong enough to deduce the Erdős-Hajnal property; and we know from 1.3 that excluding an ordered forest and its complement is not in general sufficient to obtain the strong Erdős-Hajnal property. Nevertheless, Pach and Tomon [11] recently showed the following:

7.4 *Let P be a monotone path. The class of ordered graphs that are both P-free and $\overline{P}$-free has the Erdős-Hajnal property.*

It would be interesting to extend this to other ordered trees. For example, what about extending 7.4 to all ordered paths?

Finally, what bounds could we hope for when excluding an ordered tree from a sparse n-vertex graph? Pach and Tomon gave a bound of order $n/\log n$ when excluding a monotone path (see 1.4), while 1.5 gives a bound of form $n^{1-o(1)}$ for general ordered trees. It would be interesting to determine the behaviour more precisely for a general ordered tree. For example, for a fixed ordered tree T, does a bound of form $n/(\log n)^K$ hold for some constant K that depends on T? What if T is an ordered path?

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