BETTER BOUNDS FOR POSET DIMENSION AND BOXICITY

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ABSTRACT. The *dimension* of a poset P is the minimum number of total orders whose intersection is P. We prove that the dimension of every poset whose comparability graph has maximum degree Δ is at most $\Delta \log^{1+o(1)} \Delta$. This result improves on a 30-year old bound of Füredi and Kahn, and is within a $\log^{o(1)} \Delta$ factor of optimal. We prove this result via the notion of boxicity. The *boxicity* of a graph G is the minimum integer d such that G is the intersection graph of d-dimensional axis-aligned boxes. We prove that every graph with maximum degree Δ has boxicity at most $\Delta \log^{1+o(1)} \Delta$, which is also within a $\log^{o(1)} \Delta$ factor of optimal. We also show that the maximum boxicity of graphs with Euler genus g is $\Theta(\sqrt{g \log g})$, which solves an open problem of Esperet and Joret and is tight up to a constant factor.

1. INTRODUCTION

1.1. Poset Dimension and Degree.

The *dimension* of a poset P, denoted by $\dim(P)$, is the minimum number of total orders whose intersection is P. Let $\dim(\Delta)$ be the maximum dimension of a poset whose comparability graph has maximum degree at most Δ. Several bounds on $\dim(\Delta)$ have been proved in the literature. In unpublished work referenced in [25, 44], Rödl and Trotter proved the upper bound, $\dim(\Delta) \leqslant 2\Delta^2 + 2$. Füredi and Kahn [25] improved this result to

$$\dim(\Delta) \leqslant O(\Delta \log^2 \Delta). \tag{1}$$

On the other hand, Erdős, Kierstead, and Trotter [17] proved the lower bound,

$$\dim(\Delta) \geqslant \Omega(\Delta \log \Delta). \tag{2}$$

Both these proofs use probabilistic methods. The problem of narrowing the gap between (1) and (2) was described as "an important topic for further

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Date: December 19, 2018.

2010 *Mathematics Subject Classification.* Primary 05C62, 06A07.

Alex Scott is supported by a Leverhulme Trust Research Fellowship.

research" by Erdős et al. [17]; Trotter [44] speculated that the lower bound could be improved and wrote "a really new idea will be necessary to improve the upper bound—if this is at all possible"; and Wang [47, page 52] described the problem as "one of the most challenging (and probably quite difficult) problems in dimension theory."

Our first contribution is the following result, which is the first improvement to the Füredi–Kahn upper bound in 30 years, and shows that (2) is sharp to within a $(\log \Delta)^{o(1)}$ factor:

$$\dim(\Delta) = \Delta \log^{1+o(1)} \Delta. \tag{3}$$

A more precise result is given below (see Theorem 13).

1.2. **Boxicity and Degree.** We prove (3) via the notion of boxicity. The *boxicity* of a (finite undirected) graph G, denoted by $\mathrm{box}(G)$, is the minimum integer d, such that G is the intersection graph of boxes in $\mathbb{R}^d$. Here a *box* is a Cartesian product $I_1 \times I_2 \times \cdots \times I_d$, where $I_i \subseteq \mathbb{R}$ is an interval for each $i \in [d]$. So a graph G has boxicity at most d if and only if there is a set of d-dimensional boxes $\{B_v : v \in V(G)\}$ such that $B_v \cap B_w \neq \emptyset$ if and only if $vw \in E(G)$. Note that a graph has boxicity 1 if and only if it is an interval graph. It is easily seen that every graph has finite boxicity.

Let $\mathrm{box}(\Delta)$ be the maximum boxicity of a graph with maximum degree Δ. It is easily seen that $\mathrm{box}(2) = 2$, and Adiga and Chandran [2] proved that $\mathrm{box}(3) = 3$. Chandran, Francis, and Sivadasan [9] proved the first general upper bound of $\mathrm{box}(\Delta) \leqslant 2\Delta^2 + 2$, which was improved to $\Delta^2 + 2$ by Esperet [19]. A breakthrough was made by Adiga, Bhowmick, and Chandran [1] via the following connection to poset dimension.

Given a graph G, let P be the poset on $V(G) \times \{0, 1\}$ where $(u, i) \prec (v, j)$ if and only if $i = 0$ and $j = 1$, and $u = v$ or $uv \in E(G)$. Adiga et al. [1] proved that $\frac{1}{2}\dim(P) - 2 \leqslant \mathrm{box}(G) \leqslant 2\dim(P)$. The comparability graph of P has maximum degree $\Delta(G)+1$. Thus $\mathrm{box}(\Delta) \leqslant 2\dim(\Delta+1)$. Conversely, Adiga et al. [1] proved that if G is the comparability graph of a poset P, then $\dim(P) \leqslant 2\,\mathrm{box}(G)$, implying $\dim(\Delta) \leqslant 2\,\mathrm{box}(\Delta)$. This says that $\dim(\Delta) = \Theta(\mathrm{box}(\Delta))$. Thus Adiga et al. [1] concluded from (1) and (2) that

$$\Omega(\Delta \log \Delta) \leqslant \mathrm{box}(\Delta) \leqslant O(\Delta \log^2 \Delta). \tag{4}$$

We improve the upper bound, giving the following result, which is equivalent to (3):

$$\mathrm{box}(\Delta) = \Delta \log^{1+o(1)} \Delta. \tag{5}$$

Again, a more precise result is given below (see Theorem 12).

1.3. **Boxicity and Genus.** Now consider the boxicity of graphs embeddable in a given surface. Scheinerman [40] proved that every outerplanar graph has boxicity at most 2. Thomassen [43] proved that every planar graph has boxicity at most 3 (generalised to 'cubicity' by Felsner and Francis [24]).

Esperet and Joret [22] proved that every toroidal graph has boxicity at most 7, improved to 6 by Esperet [21]. The *Euler genus* of an orientable surface with h handles is $2h$. The *Euler genus* of a non-orientable surface with c cross-caps is c. The *Euler genus* of a graph G is the minimum Euler genus of a surface in which G embeds (with no crossings). Esperet and Joret [22] proved that every graph with Euler genus g has boxicity at most $5g + 3$. Esperet [20] improved this upper bound to $O(\sqrt{g} \log g)$ and also noted that there are graphs of Euler genus g with boxicity $\Omega(\sqrt{g \log g})$, which follows from the result of Erdős et al. [17] mentioned above. See [21] for more on the boxicity of graphs embedded in surfaces.

The second contribution of this paper is to improve the upper bound to match the lower bound up to a constant factor (see Theorem 14). We conclude that the maximum boxicity of a graph with Euler genus g is

$$\Theta(\sqrt{g \log g}). \tag{6}$$

Furthermore, the implicit constant in (6) is not large: the upper bound in Theorem 14 is $(12 + o(1))\sqrt{g \log g}$.

1.4. **Boxicity and Layered Treewidth.** The third contribution of the paper is to prove a new upper bound on boxicity in terms of layered treewidth, which is a graph parameter recently introduced by Dujmović et al. [15] (see Section 5). This generalises the known bound in terms of treewidth, and leads to generalisations of known results for graphs embedded in surfaces where each edge is in a bounded number of crossings.

1.5. **Related Work.** The present paper can be considered to be part of a body of research connecting poset dimension and graph structure theory. Several recent papers [28–33, 35, 42, 46] show that structural properties of the cover graph of a poset lead to bounds on its dimension. Finally, we mention the following relationships between boxicity and chromatic number. Graphs with boxicity 1 (interval graphs) are perfect. Asplund and Grünbaum [4] proved that graphs with boxicity 2 are χ-bounded. But Burling [7] constructed triangle-free graphs with boxicity 3 and unbounded chromatic number.

2. Tools

Roberts [38] introduced boxicity and proved the following two fundamental results.

Lemma 1 ([38])**.** *For all graphs $G, G_1, \dots, G_r$ such that $G = G_1 \cap \dots \cap G_r$,*

$$\operatorname{box}(G) \leqslant \sum_{i=1}^{r} \operatorname{box}(G_i).$$

Note that Lemma 1 is proved trivially via the product construction.

Lemma 2 ([38])**.** *Every n-vertex graph has boxicity at most $\lfloor \frac{n}{2} \rfloor$.*

Note that Trotter [45] characterised those graphs for which equality holds in Lemma 2.

A graph is *k-degenerate* if every subgraph has a vertex of degree at most k. Note that 1-degenerate graphs (that is, forests) have boxicity at most 2, but 2-degenerate graphs have unbounded boxicity, since the 1-subdivision of K_n is 2-degenerate and has boxicity $\Theta(\log\log n)$ [22]. Adiga et al. [3] proved the following bound. Throughout this paper, all logarithms are natural unless otherwise indicated.

Lemma 3 ([3])**.** *Every k-degenerate graph on n vertices has boxicity at most $(k+2)\lceil 2e\log n\rceil$.*

The following lemma, due to Esperet [21], is the starting point for our work on embedded graphs.

Lemma 4 ([21])**.** *Every graph G with Euler genus g has a set X of at most $60g$ vertices such that $G-X$ has boxicity at most 5.*

Let $[n] := \{1,2,\dots,n\}$. For our purposes, a *permutation* of a set X is a bijection from X to $[|X|]$. A set $\{\pi_1,\dots,\pi_p\}$ of permutations of a set X is *r-suitable* if for every r-subset S of X and for every element $x\in S$, there is permutation π_i such that $\pi_i(x)<\pi_i(y)$ for all $y\in S\setminus\{x\}$. This definition was introduced by Dushnik [16]; see [8, 13, 41] for further results on suitable sets. Spencer [41] attributes the following result to Hajnal. We include the proof for completeness, and so that the dependence on k is absolutely clear (since Spencer assumed that k is fixed).

Lemma 5 ([41])**.** *For every $k\geqslant 2$ and $n\geqslant 10^4$ there is a k-suitable collection of permutations of size at most $k2^k\log\log n$.*

Proof. A sequence $S_1,\dots,S_r$ of subsets of $[s]$ is *t-scrambling* if for every set $I\subseteq[r]$ with $|I|\leqslant t$ and every $A\subseteq I$, we have

$$\bigcap_{i\in A} S_i \cap \bigcap_{j\in I\setminus A}([s]\setminus S_j)\neq\emptyset. \tag{7}$$

For $s\geqslant t\geqslant 1$, let $M(s,t)$ be the maximum cardinality of a t-scrambling family of subsets of $[s]$. Note that $M(s,t)$ is monotone increasing in s (and trivially $M(s,t)\leqslant 2^s$).

Claim. *Let $s\geqslant t\geqslant 1$. If m is a positive integer such that*

$$2^t\binom{m}{t}(1-2^{-t})^s<1, \tag{8}$$

then $M(s,t)\geqslant m$.

Proof of Claim. Choose subsets $S_1,\dots,S_m$ of $[s]$ independently and uniformly at random. For any t-set $I\subseteq[m]$ and any $A\subseteq I$, the probability that (7) is not satisfied is $(1-2^{-t})^s$. There are $\binom{m}{t}$ choices for I and then 2^t choices for A, so taking a union bound, the probability that there is some pair (I, A) such that (7) is not satisfed is at most the left hand side of (8) (note that it is enough to consider sets I of size exactly t). Since this is smaller than 1, we are done. □

Given n and k, choose s minimal so that $2^{M(s,k-1)} \geqslant n$. Let $M := M(s, k-1)$, let $S_1, \dots, S_M$ be a $(k-1)$-scrambling set of subsets of $[s]$, and let $Q_1, \dots, Q_n$ be distinct subsets of $[M]$. We define orders $<_1, \dots, <_s$ on $[n]$ as follows: for $a, b \in [n]$, let $j(a, b) = \min(Q_a \triangle Q_b)$; then $a <_i b$ if either

- $i \in S_{j(a,b)}$ and $j(a, b) \in Q_a$; or
- $i \notin S_{j(a,b)}$ and $j(a, b) \in Q_b$.

(Note that if $S_i = [M]$ then this gives the lex order on the Q_s, and if $S_i = \emptyset$ it is reverse lex.)

Now $<_1, \dots, <_s$ is a k-suitable collection of orders on $[n]$. This is straightforward, but a little tricky: given a set B of k elements of $[n]$ and $b \in B$, let $I := \{\min(Q_h \triangle Q_b) : h \in B \setminus \{b\}\}$, let $A = I \cap Q_b$, and choose an element i of the intersection on the left hand side of (7). Consider the order $<_i$. It is enough to show that $b <_i h$ for each $h \in B \setminus \{b\}$. Given $h \in B \setminus \{b\}$, let $q = j(b, h) = \min Q_b \triangle Q_h$. If $q \in Q_b$ then $q \in A$ and so $i \in S_q$, and therefore $b <_i h$; if $q \notin Q_b$ then $q \notin A$ and so $i \notin S_q$, and again $b <_i h$.

How big is s? By the choice of s and monotonicity, $M(s-1, k-1) < \log_2 n$. The left hand side of (8) is less than

$$(2em/t)^t \exp(-s2^{-t}) = \left(\frac{2em \exp(-s/t2^t)}{t}\right)^t, \tag{9}$$

and so

$$M(s, t) \geqslant \frac{t}{2e} e^{s/t2^t},$$

as setting m equal to the right hand side of this expression leaves (9) less than 1. Thus, bounding $M(s-1, k-1)$, we have

$$\frac{k-1}{2e} \exp\left(\frac{s-1}{(k-1)2^{k-1}}\right) - 1 \leqslant M(s-1, k-1) \leqslant \log_2 n$$

and so

$$s \leqslant 1 + (k-1)2^{k-1} \log\left(\frac{2e}{k-1} \log_2(2n)\right),$$

which is at most $k2^k \log\log n$ for $k \geqslant 2$ and $n \geqslant 10^4$. □

We will also use the Lovász Local Lemma:

Lemma 6 ([18]). *Let $E_1, \dots, E_n$ be events in a probability space, each with probability at most p and mutually independent of all but at most D other events. If $4pD \leqslant 1$ then with positive probability, none of $E_1, \dots, E_n$ occur.*

For a graph G and set $X \subseteq V(G)$, the graph $G[X]$ with vertex set X and edge set $\{vw \in E(G) : v, w \in X\}$ is called the subgraph of G *induced* by X. Let $G\langle X\rangle$ be the graph obtained from G by adding an edge between every pair of non-adjacent vertices at least one of which is not in X.

Lemma 7. $\operatorname{box}(G\langle X\rangle) = \operatorname{box}(G[X])$.

Proof. Given a d-dimensional box-representation of $G\langle X\rangle$, delete the boxes representing the vertices in $V(G) \setminus X$ to obtain a d-dimensional box-representation of $G[X]$. Thus $\operatorname{box}(G[X]) \leqslant \operatorname{box}(G\langle X\rangle)$. Given a d-dimensional box-representation of $G[X]$, for every vertex x in $V(G) \setminus X$, add a box with interval $\mathbb{R}$ in every dimension (so that it meets all other boxes). We obtain a d-dimensional box-representation of $G\langle X\rangle$. Thus $\operatorname{box}(G\langle X\rangle) \leqslant \operatorname{box}(G[X])$. □

For a graph G and disjoint sets $X, Y \subseteq V(G)$, the graph $G[X,Y]$ with vertex set $X \cup Y$ and edge set $\{vw \in E(G) : v \in X, w \in Y\}$ is called the bipartite subgraph of G *induced* by X, Y. For non-adjacent vertices $v \in X$ and $w \in Y$, we say vw is a *non-edge* of $G[X,Y]$. Let $G\langle X, Y\rangle$ be the graph obtained from G by adding an edge between distinct vertices v and w whenever $v, w \in V(G) \setminus X$ or $v, w \in V(G) \setminus Y$.

3. Bounded Degree

The first ingredient in our proof is the following colouring result that bounds the number of monochromatic neighbours of each vertex. A very similar result was proved by Hind et al. [27]; they required the additional property that the colouring is proper, but had $k = \max\{(d+1)\Delta, e^3\Delta^{1+1/d}/d\}$, which is too much for our purposes.

Lemma 8. *For every graph G with maximum degree $\Delta > 0$ and for all integers $d \geqslant 1$ and $k \geqslant \frac{(4d+4)^{1/d}e}{d}\Delta^{1+1/d}$, there is a partition $V_1, \dots, V_k$ of $V(G)$, such that $|N_G(v) \cap V_i| \leqslant d$ for each $v \in V(G)$ and $i \in [k]$.*

Proof. Colour each vertex of G independently and randomly with one of k colours. Let $V_1, \dots, V_k$ be the corresponding colour classes. For each set S of exactly $d+1$ vertices in G, such that $S \subseteq N_G(v)$ for some vertex $v \in V(G)$, introduce an event which holds if only if $S \subseteq V_i$ for some $i \in [k]$. Each such event has probability $p := k^{-d}$. The colour on one vertex affects at most $\Delta\binom{\Delta-1}{d}$ events. Thus each event is mutually independent of all but at most D other events, where

$$D := (d+1)\Delta\binom{\Delta-1}{d} \leqslant (d+1)\Delta\left(\frac{e\Delta}{d}\right)^d = (d+1)\left(\frac{e}{d}\right)^d\Delta^{d+1}.$$

It follows that $4pD \leqslant 1$. By Lemma 6, with positive probability, no event occurs. Thus the desired partition exists. □

Note that an example in [27], due to Noga Alon, shows that the value of k in Lemma 8 is within a constant factor of optimal. Lemma 8 leads to our next lemma. A similar result was used by Füredi and Kahn [25] in their work on poset dimension.

Corollary 9. *For every graph G with maximum degree $\Delta \geqslant 2$ and for all integers $d \geqslant 100 \log \Delta$ and $k \geqslant \frac{3\Delta}{d}$, there is a partition $V_1, \dots, V_k$ of $V(G)$, such that $|N_G(v) \cap V_i| \leqslant d$ for each $v \in V(G)$ and $i \in [k]$.*

Proof. Since $d \geqslant 100 \log \Delta \geqslant 69$, we have $(4d+4)^{1/d} e \leqslant 2.95$ and $d \geqslant \log^{-1}(\frac{3}{2.95}) \log \Delta$ and $\Delta^{1/d} \leqslant \frac{3}{2.95}$. Thus $\frac{(4d+4)^{1/d}e}{d} \Delta^{1+1/d} \leqslant \frac{2.95}{d} \Delta^{1+1/d} \leqslant \frac{3\Delta}{d} \leqslant k$. The result follows from Lemma 8. □

The next lemma is a key new idea in our proof. Its proof is a straightforward application of the Lovász Local Lemma.

Lemma 10. *Let G be a bipartite graph with bipartition $\{A, B\}$, where vertices in A have degree at most d and vertices in B have degree at most Δ. Let r, t, ℓ be positive integers such that*

$$\ell \geqslant e \left(\frac{ed}{r+1} \right)^{1+1/r} \quad \text{and } t \geqslant \log(4d\Delta).$$

Then there exist t colourings $c_1, \dots, c_t$ of B, each with ℓ colours, such that for each vertex $v \in A$, for some colouring c_i, each colour is assigned to at most r neighbours of v under c_i.

Proof. For $i \in [t]$ and for each vertex $w \in B$, let $c_i(w)$ be a random colour in $[\ell]$. Let X_v be the event that for each $i \in [t]$, some set of $r+1$ neighbours of v are monochromatic under c_i. The probability that there is a monochromatic set of at least $r+1$ neighbours of v under c_i is

$$\binom{\deg(v)}{r+1} \ell^{-r} \leqslant \binom{d}{r+1} \ell^{-r} \leqslant \left(\frac{ed}{r+1} \right)^{r+1} \ell^{-r} \leqslant e^{-1}.$$

Thus $\mathbb{P}(X_v) \leqslant e^{-t}$. Observe that X_v is mutually independent of all but at most $d\Delta$ other events. By assumption, $4e^{-t} d\Delta \leqslant 1$. By Lemma 6, with positive probability no event X_v occurs. Therefore, the desired colourings exists. □

Lemma 11. *Let G be a bipartite graph with bipartition $\{A, B\}$, where vertices in A have degree at most d and vertices in B have degree at most Δ, for some $\Delta \geqslant d \geqslant 2$. Let $G' = G\langle A, B \rangle$ be the graph obtained from G by adding a clique on A and a clique on B. Then, as $d \to \infty$,*

$$\operatorname{box}(G') \leqslant (60 + o(1))\, d \,\log(d\Delta) \,\log\log(\Delta)\, (2e)^{\sqrt{\log d}}.$$

Proof. Let $r := \lceil \sqrt{\log d} \rceil$ and $\ell := \lceil e \left(\frac{ed}{r+1} \right)^{1+1/r} \rceil$ and $t := \lceil \log(4d\Delta) \rceil$. As $d \to \infty$, we may assume that d is large.

By Lemma 10, there exist t colourings $c_1, \dots, c_t$ of B, each with ℓ colours, such that for each vertex $v \in A$, for some colouring c_j, each colour is assigned to at most r neighbours of v. Let $\{A_j : j \in [t]\}$ be a partition of A such that for each $v \in A_j$, at most r neighbours of v are assigned the same colour under c_j. Assume that $[\ell]$ is the set of colours used by each c_j.

Our aim is to construct a box representation for each of the graphs $G\langle A_i, B\rangle$, and then take their intersection using Lemma 1. In light of Lemma 7, it is enough to concentrate on the subgraphs $G[A_i \cup B]$. To handle $G[A_i \cup B]$, we further decompose B according to $c_1, \dots, c_t$ as follows. For each $j \in [t]$ and each colour $\alpha \in [\ell]$, let $B_{j,\alpha} := \{w \in B : c_j(w) = \alpha\}$. Let $G_{j,\alpha} := G\langle A_j, B_{j,\alpha}\rangle$. Note that $G' = \bigcap_{j,\alpha} G_{j,\alpha}$.

We now bound the boxicity of $G_{j,\alpha}$. Let H be the graph with vertex set $B_{j,\alpha}$, where distinct vertices $x, y \in B_{j,\alpha}$ are adjacent in H whenever x and y have a common neighbour in A_j. Since each vertex in A_j has at most r neighbours in $B_{j,\alpha}$, the graph H has maximum degree at most $r\Delta$. Thus $\chi(H) \leqslant h := r\Delta + 1$. Let $X_1, \dots, X_h$ be the colour classes in a proper colouring of H. For $q \in [h]$, let $\overrightarrow{X_q}$ denote an arbitrary linear ordering of X_q. Let $\overleftarrow{X_q}$ be the reverse of $\overrightarrow{X_q}$. Since we may assume that d is large, Lemma 5 shows that there exists a set of $(r+1)$-suitable permutations $\pi_1, \dots, \pi_p$ of $[h]$ for some $p \leqslant (r+1)2^{r+1} \log\log(h)$.

For each $a \in [p]$, we introduce two 2-dimensional representations of $G_{j,\alpha}$. Let σ_a be the ordering $\overrightarrow{X_{\pi_a(1)}}, \overrightarrow{X_{\pi_a(2)}}, \dots, \overrightarrow{X_{\pi_a(h)}}$ of $B_{j,\alpha}$. Similarly, let σ'_a be the ordering $\overleftarrow{X_{\pi_a(1)}}, \overleftarrow{X_{\pi_a(2)}}, \dots, \overleftarrow{X_{\pi_a(h)}}$ of $B_{j,\alpha}$. For each vertex x in B, say x is the b_x-th vertex in σ_a and x is the b'_x-th vertex in σ'_a. Then represent x by the box with corners $(-\infty, +\infty)$ and $(2b_x, 2b_x)$. For each vertex $v \in A_j$, if v has no neighbours in B, then represent v by the point $(2|B|, -2|B|)$; otherwise, if x is the leftmost neighbour of v in σ_a and y is the rightmost neighbour of v in σ_a, then represent v by the box with corners $(\infty, -\infty)$ and $(2b_x - 1, 2b_y + 1)$, as illustrated in Figure 1. Now, in two new dimensions introduce the following representation. Represent each x in $B_{j,\alpha}$ by the box with corners $(-\infty, +\infty)$ and $(2b'_x, 2b'_x)$. For each vertex $v \in A_j$, if v has no neighbours in B, then represent v by the point $(2|B|, -2|B|)$; otherwise, if x is the leftmost neighbour of v in σ'_a and y is the rightmost neighbour of v in σ'_a, then represent v by the box with corners $(\infty, -\infty)$ and $(2b'_x - 1, 2b'_y + 1)$. In each of these four dimensions, add every vertex in $V(G) \setminus (B_{j,\alpha} \cup A_j)$ with interval $\mathbb{R}$. Observe that A and B are both cliques in this representation.

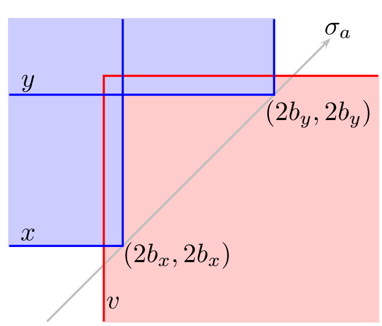

FIGURE 1. Representation of $G_{j,\alpha}$ with respect to σ_a.

By construction, for every edge vw of $G_{j,\alpha}$ the boxes of v and w intersect in every dimension. Now consider a non-edge zv of $G_{j,\alpha}$ with $z \in B_{j,\alpha}$ and $v \in A_j$. Let C be the set of integers $q \in [h]$ such that some neighbour of v is in X_q. Thus $|C| \leqslant r$. Say z is in $X_{q'}$. First suppose that $q' \notin C$. Since $|C \cup \{q'\}| \leqslant r+1$, for some permutation π_a, we have $\pi_a(q') < \pi_a(q)$ for each $q \in C$. Let x be the leftmost neighbour of v in σ_a. Thus $b_z < b_x$, and in the first 2-dimensional representation corresponding to π_a, the right-hand-side of the box representing z is to the left of the left-hand-side of the box representing v, as illustrated in Figure 2(a). Thus the boxes representing v and z do not intersect. Now assume that $q' \in C$. By construction, there is exactly one neighbour x of v in $X_{q'}$. Since $|C| \leqslant r$, for some permutation π_a, we have $\pi_a(q') \leqslant \pi_a(q)$ for each $q \in C$. If $z < x$ in $\overrightarrow{X_q}$, then $b_z < b_x$, and as argued above and illustrated in Figure 2(b), the boxes representing v and z do not intersect. Otherwise, $z < x$ in $\overleftarrow{X_q}$. Then $b'_z < b'_x$, and in the second 2-dimensional representation corresponding to π_a, the right-hand-side of the box representing z is to the left of the left-hand-side of the box representing v. Hence the boxes representing v and z do not intersect. Therefore $\mathrm{box}(G_{j,\alpha}) \leqslant 4p$.

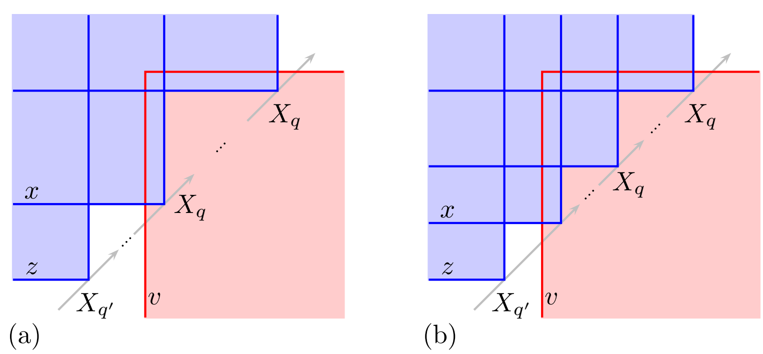

FIGURE 2. Proof for the representation of $G_{j,\alpha}$.

By Lemma 1,

$$\begin{aligned}\mathrm{box}(G') &\leqslant 4t\ell p \\ &\leqslant 4\lceil \log(4d\Delta)\rceil \left\lceil e\left(\frac{ed}{r+1}\right)^{1+1/r}\right\rceil (r+1)2^{r+1} \log\log(r\Delta+1).\end{aligned}$$

Since $\log(4d\Delta) \leqslant (1+o(1))\log(d\Delta)$ and $e^{1+1/r} \leqslant (1+o(1))e$ and $\log(r\Delta + 1) \leqslant (1+o(1))\log(\Delta)$,

$$\begin{aligned}\operatorname{box}(G') &\leqslant (8e^2+o(1))\log(d\Delta)\left(\frac{d}{r+1}\right)^{1+1/r}(r+1)2^r\,\log\log(\Delta)\\ &\leqslant (60+o(1))\log(d\Delta)\,\log\log(\Delta)\,d^{1+1/r}\,\frac{2^r}{(r+1)^{1/r}}\\ &\leqslant (60+o(1))\,d\,\log(d\Delta)\,\log\log(\Delta)\,\left(d^{1/r}\,2^r\right)\\ &\leqslant (60+o(1))\,d\,\log(d\Delta)\,\log\log(\Delta)\,(2e)^{\sqrt{\log d}}.\end{aligned}$$

□

We now prove our first main result.

Theorem 12. *For every graph G with maximum degree Δ, as $\Delta \to \infty$,*

$$\operatorname{box}(G) \leqslant (180+o(1))\,\Delta\log(\Delta)\,(2e)^{\sqrt{\log\log\Delta}}\,\log\log\Delta.$$

Proof. Let $d := \lceil 100\log\Delta\rceil$ and $k := \lceil \frac{3\Delta}{d}\rceil$. By Corollary 9, there is a partition $V_1,\dots,V_k$ of $V(G)$, such that $|N_G(v)\cap V_i| \leqslant d$ for each $v\in V(G)$ and $i\in[k]$. Note that

$$\operatorname{box}(G) = \bigcap_i G\langle V_i\rangle \cap G\langle V_i, V(G)\setminus V_i\rangle.$$

Since $G[V_i]$ has maximum degree at most d, by the result of Esperet [19], the graph $G[V_i]$ has boxicity at most d^2+2. By Lemma 7,

$$\operatorname{box}(G\langle V_i\rangle) \leqslant d^2+2.$$

Let $G_i := G[V_i, V(G)\setminus V_i]$. Every vertex in $V(G)\setminus V_i$ has degree at most d in G_i. Let G_i' be obtained from G_i by adding a clique on V_i and a clique on $V(G)\setminus V_i$. By Lemmas 7 and 11 and since $\log(d\Delta) \leqslant (1+o(1))\log\Delta$,

$$\operatorname{box}(G\langle V_i, V(G)\setminus V_i\rangle) \leqslant \operatorname{box}(G_i') \leqslant (60+o(1))\,d\,\log(\Delta)\,\log\log(\Delta)\,(2e)^{\sqrt{\log d}}.$$

Applying Lemma 1 again,

$$\begin{aligned}\operatorname{box}(G) &\leqslant k(d^2+2) + (60+o(1))\,k\,d\,\log(\Delta)\,\log\log(\Delta)\,(2e)^{\sqrt{\log d}}\\ &\leqslant (9+o(1))(\Delta\log\Delta) + (180+o(1))\,\Delta\log(\Delta)\,\log\log(\Delta)\,(2e)^{\sqrt{\log\log\Delta}}\\ &\leqslant (180+o(1)\,\Delta\log(\Delta)\,\log\log(\Delta)\,(2e)^{\sqrt{\log\log\Delta}}.\end{aligned}$$

□

Since $(2e)^{\sqrt{\log\log\Delta}}\log\log\Delta \leqslant \log^{o(1)}\Delta$, Theorem 12 implies (5). More precisely,

$$\operatorname{box}(\Delta) = \Delta(\log\Delta)e^{O(\sqrt{\log\log\Delta})}.$$

Theorem 12 and the result of Adiga et al. [1] mentioned in Section 1 imply the following quantitative version of (3).

Theorem 13. *For every poset P whose comparability graph has maximum degree Δ, as $\Delta\to\infty$,*

$$\dim(P) \leqslant (360+o(1))\,\Delta\log(\Delta)\,(2e)^{\sqrt{\log\log\Delta}}\,\log\log\Delta.$$

Again, with (2), this gives

$$\dim(\Delta) = \Delta(\log \Delta)e^{O(\sqrt{\log\log \Delta})}.$$

4. Euler Genus

We now prove our second main result.

Theorem 14. *For every graph G with Euler genus g, as $g \to \infty$.*

$$\operatorname{box}(G) \leqslant (12 + o(1))\sqrt{g \log g}.$$

Proof. By Lemma 4, G contains a set X of at most $60g$ vertices such that $\operatorname{box}(G - X) \leqslant 5$. First suppose that $|X| < 10^4$. Deleting one vertex reduces boxicity by at most 1. Thus $\operatorname{box}(G) \leqslant \operatorname{box}(G - X) + |X| \leqslant 10^4 + 5$, and we are done since $g \to \infty$. Now assume that $|X| \geqslant 10^4$.

Let $G_1 := G\langle V(G) \setminus X\rangle$. Let Y be the set of vertices in $G - X$ with exactly one or exactly two neighbours in X. Let $G_2 := \langle X, Y\rangle$. By Lemma 5, there is a 3-suitable set of permutations $\pi_1, \dots, \pi_p$ of X for some $p \leqslant 24 \log\log |X|$. For each π_i we introduce two dimensions, as illustrated in Figure 3. Represent each vertex $w \in X$ by the box with corners $(-\infty, +\infty)$ and $(2\pi_i(w), 2\pi_i(w))$. For each vertex $v \in Y$, if x and y are respectively the leftmost and rightmost neighbours of v in π_i, then represent v by the box with corners $(2\pi_i(x) - 1, 2\pi_i(y) + 1)$ and $(+\infty, -\infty)$. Observe that X and Y are both cliques in this representation. If $vw \in E(G)$ and $v \in Y$ and $w \in X$, then the box representing v intersects the box representing w. Consider a non-edge vz with $v \in Y$ and $z \in X$. Since $\pi_1, \dots, \pi_p$ is 3-suitable and $\deg_X(v) \leqslant 2$, for some i, we have $\pi_i(z) < \pi(x)$ for each $x \in N_G(v)$. Thus, for the 2-dimensional representation defined with respect to π_i, the boxes representing v and z do not intersect.

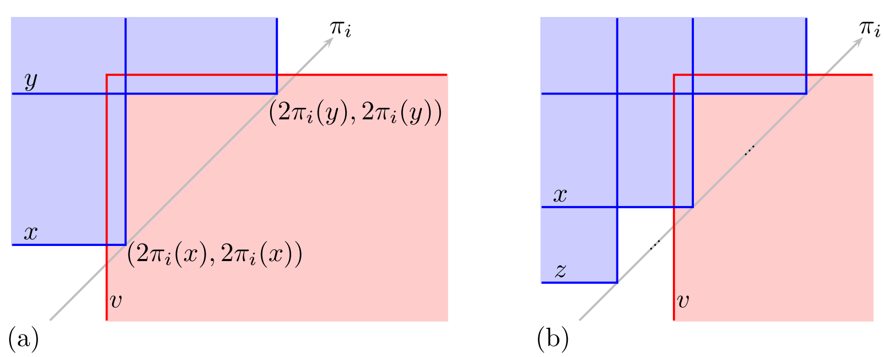

Figure 3. Representation of G_2 with respect to π_i.

Add each vertex in $G - (X \cup Y)$ to every dimension with interval $\mathbb{R}$. We obtain a box representation of G_2. Thus $\operatorname{box}(G_2) \leqslant 48 \log\log |X| \leqslant 48 \log\log(1000g)$.

Let Z be the set of vertices in $G - X$ with at least three neighbours in X. Let $G_3 := G\langle X \cup Z\rangle$. Observe that $G = G_1 \cap G_2 \cap G_3$.

To bound $\text{box}(G_3)$, we first bound $\text{box}(H)$, where $H := G[X \cup Z]$. The number of edges in $G[X, Z]$ is least $3|Z|$ and at most $2(|X| + |Z| + g - 2)$ by Euler's formula. Thus $|Z| < 2(|X| + g)$, implying $|X \cup Z| < 3002g$. Let $n := |V(H)| < 3002g$. Let $v_1, \dots, v_n$ be an ordering of $V(H)$, where v_i has minimum degree in $H[\{v_i, \dots, v_n\}]$. Define $k := 7 + \lceil \sqrt{g/\log g}\rceil$. Let i be minimum such that v_i has degree at least k in $H[\{v_i, \dots, v_n\}]$. If i is defined, then let $A := \{v_1, \dots, v_{i-1}\}$ and $B := \{v_i, \dots, v_n\}$, otherwise let $A := V(H)$ and $B := \emptyset$.

Observe that $H = H\langle A\rangle \cap H\langle B\rangle \cap H\langle A, B\rangle$.

By construction, $H[A]$ is k-degenerate and has at most n vertices. By Lemma 3, $\text{box}(H[A]) \leqslant (k + 2)\lceil 2e \log n\rceil$. By Lemma 7, $H\langle A\rangle \leqslant (k + 2)\lceil 2e \log n\rceil$.

By construction, $H[B]$ has minimum degree at least k. The number of edges in $H[B]$ is at least $\frac{1}{2}k|B|$ and at most $3(|B| + g - 2)$, implying $(\frac{k}{2} - 3)|B| < 3g$. By Lemma 2, $\text{box}(H[B]) \leqslant \frac{|B|}{2} < \frac{3g}{k-6}$. By Lemma 7, $H\langle B\rangle < \frac{3g}{k-6}$.

Now consider $H[A, B]$. By construction, every vertex in A has degree at most k in $H[A, B]$. A permutation σ of B *catches* a non-edge vw of $H[A, B]$ with $v \in A$ and $w \in B$ if there are edges vx, vy in $H[A, B]$, such that w is between x and y in σ. Let $t := \lceil \frac{3}{2}(k + 1) \log n\rceil$. Let $\sigma_1, \dots, \sigma_t$ be random permutations of B. For each non-edge vw of $H[A, B]$, the probability that σ_i catches vw equals $1 - \frac{2}{\deg(v)+1} \leqslant e^{-2/(k+1)}$. Thus, the probability that every σ_i catches vw is at most $e^{-2t/(k+1)} \leqslant n^{-3}$. Since the number of non-edges is at most n^2, by the union bound, the probability that for some non-edge vw, every σ_i catches vw is at most $n^{-1} < 1$. Hence, with positive probability, for every non-edge vw, some σ_i does not catch vw. Therefore, there exists permutations $\sigma_1, \dots, \sigma_t$ of B, such that for every non-edge vw, some σ_i does not catch vw.

For each permutation σ_i we introduce two dimensions. Represent each vertex $w \in B$ by the box with corners $(-\infty, +\infty)$ and $(2\sigma_i(w), 2\sigma_i(w))$. For each vertex $v \in A$, if v has no neighbours in B then represent v by the point $(2|B|, -2|B|)$; otherwise, if x and y are respectively the leftmost and rightmost neighbours of v in σ_i, then represent v by the box with corners $(2\sigma_i(x) - 1, 2\sigma_i(y) + 1)$ and $(+\infty, -\infty)$. Observe that A and B are both cliques in this representation. If $vw \in E(G)$ and $v \in A$ and $w \in B$, then the box representing v intersects the box representing w. For a non-edge vw with $v \in A$ and $w \in B$, the box representing v intersects the box representing w if and only if σ_i catches vw. Since for every non-edge vw, some σ_i does not catch vw, the boxes representing v and w do not intersect. Thus $\text{box}(H\langle A, B\rangle) \leqslant 2t \leqslant 2 + 3(k + 1) \log n$.

By Lemma 1,

$$\begin{aligned}\operatorname{box}(H) &\leqslant \operatorname{box}(H\langle A\rangle) + \operatorname{box}(H\langle B\rangle) + \operatorname{box}(H\langle A,B\rangle)\\ &\leqslant (k+2)\lceil 2e\log n\rceil + \frac{3g}{k-6} + 2 + 3(k+1)\log n\\ &\leqslant (9k+15)\log(3002g) + 3\sqrt{g\log g}\\ &\leqslant 12\sqrt{g\log g} + O(\sqrt{g/\log g}).\end{aligned}$$

Applying Lemma 1 again,

$$\begin{aligned}\operatorname{box}(G) &\leqslant \operatorname{box}(G_1) + \operatorname{box}(G_2) + \operatorname{box}(G_3)\\ &\leqslant 42 + 48\log\log(1000g) + 12\sqrt{g\log g} + O(\sqrt{g/\log g})\\ &\leqslant 12\sqrt{g\log g} + O(\sqrt{g/\log g}).\end{aligned}$$ □

As noted in Section 1, when combined with the lower bound proved by Esperet [20], this shows that the maximum possible boxicity of a graph with Euler genus g is $\Theta(\sqrt{g\log g})$.

5. Layered Treewidth

A *tree decomposition* of a graph G is a set $(B_x : x \in V(T)$ of non-empty sets $B_x \subseteq V(G)$ (called *bags*) indexed by the nodes of a tree T, such that for each vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty (connected) subtree of T, and for each edge $vw \in E(G)$ there is a node $x \in V(T)$ such that $v, w \in B_x$. The *width* of a tree decomposition $(B_x : x \in V(T))$ is $\max\{|B_x| - 1 : x \in V(T)\}$. The *treewidth* of a graph G, denoted by $\operatorname{tw}(G)$, is the minimum width of a tree decomposition of G. Treewidth is a key parameter in algorithmic and structural graph theory (see [6, 26, 37] for surveys). Chandran and Sivadasan [10] proved:

Theorem 15 ([10])**.** *For every graph G,*

$$\operatorname{box}(G) \leqslant \operatorname{tw}(G) + 2.$$

A *layering* of a graph G is a partition $(V_1, V_2, \dots, V_n)$ of $V(G)$ such that for every edge $vw \in E(G)$, for some $i \in [n-1]$, both v and w are in $V_i \cup V_{i+1}$. For example, if r is a vertex of a connected graph G and $V_i := \{v \in V(G) : \operatorname{dist}(r, v) = i\}$ for $i \geqslant 0$, then $(V_0, V_1, \dots)$ is a layering of G. The layered tree-width of a graph G is the minimum integer k such that there is a tree decomposition $(B_x : x \in V(T))$ and a layering $(V_1, V_2, \dots, V_n)$ of G, such that $|B_x \cap V_i| \leqslant k$ for each node $x \in V(T)$ and for each layer V_i. Of course, $\operatorname{ltw}(G) \leqslant \operatorname{tw}(G)+1$ and often $\operatorname{ltw}(G)$ is much less than $\operatorname{tw}(G)$. For example, Dujmović et al. [15] proved that every planar graph has layered treewidth at most 3, whereas the $n \times n$ planar grid has treewidth n. Thus the following result provides a qualitative generalisation of Theorem 15.

Theorem 16. *For every graph G,*

$$\operatorname{box}(G) \leqslant 6\operatorname{ltw}(G) + 4.$$

Proof. Consider a tree decomposition $(B_x : x \in V(T))$ and a layering $(V_1, V_2, \dots, V_n)$ of G, such that $|B_x \cap V_i| \leqslant \operatorname{ltw}(G)$ for each node $x \in V(T)$ and for each layer V_i. Note that $(B_x \cap (V_i \cup V_{i+1}) : x \in V(T))$ is a tree-decomposition of $G[V_i \cup V_{i+1}]$ with bags of size at most $2\operatorname{ltw}(G)$. Thus $\operatorname{tw}(G[V_i \cup V_{i+1}]) \leqslant 2\operatorname{ltw}(G) - 1$. For $i \in \{0, 1, 2\}$, let

$$G_i := \bigcup_{j \equiv i \pmod 3} G[V_j \cup V_{j+1}].$$

Each component of G_i is contained in $V_j \cup V_{j+1}$ for some $j \equiv i \pmod 3$. The treewith of a graph equals the maximum treewidth of its connected components. Thus $\operatorname{tw}(G_i) \leqslant 2\operatorname{ltw}(G) - 1$, and $\operatorname{box}(G_i) \leqslant 2\operatorname{ltw}(G) + 1$ by Theorem 15. Use three disjoint sets of $2\operatorname{ltw}(G) + 1$ dimensions for each G_i, and add each vertex not in G_i to the dimensions used by G_i with interval $\mathbb{R}$. Finally, add one more dimension, where the interval for each vertex $v \in V_i$ is $[i, i + 1]$. For adjacent vertices in G, the corresponding boxes intersect in every dimension. Consider non-adjacent vertices v and w in G. Say $v \in V_a$ and $w \in V_b$. If $|a - b| \geqslant 2$ then in the final dimension, the intervals for v and w are disjoint, as desired. If $|a - b| = 1$, then vw is a non-edge in some G_i, and thus the intervals for v and w are disjoint in some dimension corresponding to G_i. Hence we have a $3(2\operatorname{ltw}(G) + 1) + 1$-dimensional box representation of G. □

The following two examples illustrate the generality of Theorem 16. A graph is (g, k)-*planar* if it has a drawing in a surface of Euler genus at most g with at most k crossings per edge; see [34, 36, 39] for example. Dujmović et al. [14] proved that every (g, k)-planar graph has layered treewidth at most $(4g + 6)(k + 1)$. Theorem 16 then implies that every (g, k)-planar graph has boxicity at most $6(4g + 6)(k + 1) + 4$. Map graphs provide a second example. Start with a graph G_0 embedded in a surface of Euler genus g, with each face labelled a 'nation' or a 'lake', where each vertex of G_0 is incident with at most d nations. Let G be the graph whose vertices are the nations of G_0, where two vertices are adjacent in G if the corresponding faces in G_0 share a vertex. Then G is called a (g, d)-*map graph*; see [11, 12] for example. Dujmović et al. [14] proved that every (g, d)-map graph has layered treewidth at most $(2g + 3)(2d + 1)$. Theorem 16 then implies that every (g, d)-map graph has boxicity at most $6(2g + 3)(2d + 1) + 4$. By definition, a graph is $(g, 0)$-planar if and only if it has Euler genus at most g. Similarly, it is easily seen that a graph is a $(g, 3)$-map graph if and only if it has Euler genus at most g (see [14]). Thus these results provide qualitative generalisations of the fact that graphs with Euler genus g have boxicity $O(g)$, as proved by Esperet and Joret [22]. As discussed above, Esperet [20] improved this upper bound to $O(\sqrt{g}\log g)$ and Theorem 14 improves it further to $O(\sqrt{g \log g})$. On the other hand, Theorem 16 is within a constant factor of optimal, since Chandran and Sivadasan [10] constructed a family of graphs G with

$\mathrm{box}(G) \geqslant (1 - o(1)) \,\mathrm{tw}(G)$. See [5] for more examples of graph classes with bounded layered treewidth, for which Theorem 16 is applicable.

6. Open Problems

We conclude with a few open problems.

- What is the maximum boxicity of graphs with maximum degree 4?
- What is the maximum boxicity of k-degenerate graphs with maximum degree Δ?
- What is the maximum boxicity of graphs with treewidth k? Chandran and Sivadasan [10] proved lower and upper bounds of $k - 2\sqrt{k}$ and $k + 2$ respectively.
- What is the maximum boxicity of graphs with no K_t minor? The best known upper bound is $O(t^2 \log t)$ due to Esperet and Wiechert [23]. A lower bound of $\Omega(t\sqrt{\log t})$ follows from results of Esperet [20].

Acknowledgement. Thanks to Louis Esperet for useful feedback.

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