# Packing random graphs and hypergraphs

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**Abstract**

We determine to within a constant factor the threshold for the property that two random $k$-uniform hypergraphs with edge probability $p$ have an edge-disjoint packing into the same vertex set. More generally, we allow the hypergraphs to have different densities. In the graph case, we prove a stronger result, on packing a random graph with a fixed graph.

## 1 Introduction

Let $G_1$ and $G_2$ be two $k$-uniform hypergraphs of order $n$. We say that $G_1$ and $G_2$ can be *packed* if they can be placed onto the same vertex set so that their edge sets are disjoint.

In the graph case, quite a lot is known. Bollobás and Eldridge [2] and Catlin [5] independently conjectured that if $(\Delta(G_1)+1)(\Delta(G_2)+1) \le n+1$ then $G_1$ and $G_2$ can be packed. Sauer and Spencer [12] proved that graphs $G_1$ and $G_2$ of order $n$ can be packed if $\Delta(G_1)\Delta(G_2) < n/2$. Let us note

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that the conjectured bound would be tight: suppose that $n = ab - 2$, and let $G_1 = (b-1)K_a \cup K_{a-2}$ (the vertex-disjoint union of $b-1$ complete graphs of order $a$ and a complete graph of order $a-2$) and $G_2 = (a-1)K_b \cup K_{b-2}$. Then $(\Delta(G_1)+1)(\Delta(G_2)+1) = n+2$, but $G_1$ and $G_2$ cannot be packed.

For fixed $k \geq 3$, the graph example given above is easy to generalize: suppose that $n = (a-1)(b-1)(k-1) + a + b - 3$. Let $G_1$ be the vertex-disjoint union of $b-1$ complete $k$-uniform graphs of order $(a-1)(k-1)+1$ and $a-2$ isolated vertices; let $G_2$ be the vertex-disjoint union of $a-1$ complete $k$-uniform graphs of order $(b-1)(k-1)+1$ and $b-2$ isolated vertices. Then $\Delta(G_1)\Delta(G_2) = \Theta(a^{k-1}b^{k-1}) = \Theta(n^{k-1})$, but $G_1$ and $G_2$ cannot be packed. For another family of examples, choose $r < k$ and fix an $r$-set $A \subset [n]$. Let $G_1$ have all edges containing $A$, and $G_2$ be an $(n,k,r)$-design (these are now known to exist for suitable $n$: see Keevash [9]). $G_1$ and $G_2$ cannot be packed, and we have $\Delta(G_1) = \Theta(n^{k-r})$ and $\Delta(G_2) = \Theta(n^{r-1})$, and so again $\Delta(G_1)\Delta(G_2) = \Theta(n^{k-1})$. On the positive side, much less is known. Teirlinck [13] (see Alon [1] for further results and discussion) showed that, for $n \geq 7$, any two Steiner triple systems $G_1$, $G_2$ can be packed: note that these satisfy $\Delta(G_1)\Delta(G_2) = \Theta(n^2)$. There are also some nice results when one of $G_1$ and $G_2$ has very small maximal degree: see Rödl, Ruciński and Taraz [11] and Conlon [6].

In this paper, we consider what happens when $G_1$ and $G_2$ are *random* hypergraphs. For integers $k, n$ and $p \in [0,1]$, we write $\mathcal{G}(n,k,p)$ for the random $k$-uniform hypergraph on $n$ vertices in which each possible edge is present indepedently with probability $p$; when $k = 2$, we write $\mathcal{G}(n,p) = \mathcal{G}(n,2,p)$. In the graph case, with $G_1 \in \mathcal{G}(n,p)$ and $G_2 \in \mathcal{G}(n,q)$, the extremal results mentioned above suggest that we should expect a condition of form $pqn \leq c$ (for suitable $c$) to be able to pack $G_1$ and $G_2$. More generally, for $k$-uniform hypergraphs, we might hope for a condition of form $pqn^{k-1} \leq c$, as this would give $\Delta(G_1)\Delta(G_2) = O(n^{k-1})$ with high probability (i.e. with probability $1 - o(1)$ as $n \to \infty$) provided $p$, $q$ are not extremely small (for instance $\min\{p,q\} \gg \log n / n^{k-1}$ is enough). In fact, we shall show here that it is possible to pack rather denser graphs: if $G_1$ and $G_2$ are both random then we can allow an additional factor $\log n$ in the product $pqn^{k-1}$, but not more. (We note that a similar phenomenon occurs when we try to minimize the overlap of two random hypergraphs: see Bollobás and Scott [3] and Ma, Naves and Sudakov [10].)

We will prove the following theorem.

**Theorem 1.** *Let $\delta \in (0,1)$. For every $k \geq 2$, there exists $\varepsilon > 0$ such that the following holds. Let $p = p(n)$ and $q = q(n)$ be positive reals such that*

- $\max\{p, q\} \leq 1 - \delta$
- $pq \leq \varepsilon \log n / n^{k-1}$.

*Let $G_1 \in \mathcal{G}(n,k,p)$ and $G_2 \in \mathcal{G}(n,k,q)$ be random $k$-uniform hypergraphs of order $n$. Then, with high probability, there is a packing of $G_1$ and $G_2$.*

Note that if $pq = \varepsilon \log n / n^{k-1}$ then with high probability $G_1$ and $G_2$ satisfy $\Delta(G_1)\Delta(G_2) = \Theta(n^{k-1} \log n)$.

The bound on $pq$ in Theorem 1 is easily seen to be sharp up to the constant. Indeed, if $G_1 \in \mathcal{G}(n,k,p)$ and $G_2 \in \mathcal{G}(n,k,q)$ then the probability that $G_1$ and $G_2$ can be packed is at most the expected number of packings

$$n!(1-pq)^{\binom{n}{k}} \leq \exp(n \log n - (1+o(1))pqn^k/k!)$$

which is $o(1)$ if $pq \geq \alpha \log n / n^{k-1}$ for any constant $\alpha > k!$. In particular, if we take $p = q$, then combining this bound with Theorem 1 shows that the threshold density for two random $k$-uniform hypergraphs to be unpackable is $\Theta(\sqrt{\log n / n^{k-1}})$.

In the case of graphs, we will in fact prove a much stronger result: it turns out that we can take just *one* of the two graphs to be random. Indeed, we prove the following.

**Theorem 2.** *For all $\gamma, K > 0$ and $\delta \in (0,1)$ there exists $\varepsilon > 0$ such that the following holds. Let $p = p(n)$ and $q = q(n)$ be positive reals such that*

- $p \leq 1 - \delta$
- $q \leq n^{-\gamma}$
- $pqn \leq \varepsilon \log n$.

*Let $G_1$ be a graph of order $n$ with maximal degree at most $qn$ and let $G_2 \in \mathcal{G}(n,p)$. Then with failure probability $O(n^{-K})$ there is a packing of $G_1$ and $G_2$.*

The rest of the paper is organized as follows. In Section 2 we prove Theorem 2, and in Section 3 we prove the extension to hypergraphs. We conclude in Section 4 with some open problems.

## 2 Packing random graphs

The aim of this section is to prove Theorem 2. We begin by noting a couple of standard facts.

We will use the following Chernoff-type inequalities. Let $X$ be a sum of Bernoulli random variables, and let $\mu = \mathbb{E}X$. Then for $t > 0$, we have

$$\mathbb{P}[X \le \mathbb{E}X - t] \le \exp(-t^2/2\mu) \tag{1}$$

and

$$\mathbb{P}[X \ge \mathbb{E}X + t] \le \exp(-t^2/(2\mu + 2t/3)) \tag{2}$$

(see, e.g., [8, Theorems 2.1 and 2.8] or [4, Chapter 2]). Inequality (2) is often called Bernstein's inequality.

It will also be useful to note a simple (and standard) fact about the binomial distribution, see e.g., [8, Corollary 2.4].

**Proposition 3.** *For every $K > 0$ there is $\delta > 0$ such that if $x > 0$ and $X \sim \mathrm{Bi}(n, p)$ is a binomial random variable with $np \le \delta x$ then $\mathbb{P}[X \ge x] \le e^{-Kx}$.*

*Proof.* This is standard; we include a proof for completeness. We have, assuming as we may that $x$ is an integer,

$$\mathbb{P}[X \ge x] \le \binom{n}{x} p^x \le \left(\frac{enp}{x}\right)^x \le (e\delta)^x$$

where we have used the standard bound $\binom{n}{k} \le (en/k)^k$ in the second line. The result follows by choosing $\delta = e^{-K-1}$. □

Our first lemma is the following, which shows that, if $\mathcal{A}$ is a large, sparse set system then a random set (of suitable size) is quite likely to be disjoint from some member of $\mathcal{A}$.

**Lemma 4.** *For all $\delta, \gamma \in (0, 1)$ there is $\varepsilon > 0$ such that the following holds for all sufficiently large $n$. Let $d = n^{1-\gamma}$, let $X$ be any set, and let $\mathcal{A}$ be a set sequence in $\mathcal{P}(X)$ such that:*

- $|\mathcal{A}| \ge n$
- *every element of $X$ belongs to at most $d$ sets from $\mathcal{A}$*
- *all sets in $\mathcal{A}$ have size at most $\varepsilon \log n$.*

*Let $B \subset X$ be a random set where each element of $X$ independently belongs to $B$ with probability $1-\delta$. Then $B$ is disjoint from at least $n^{1-\gamma/4}$ sets of $\mathcal{A}$, with failure probability $O(\exp(-n^{\gamma/3}))$.*

*Proof.* This can be proved in more than one way (an alternative proof pointed out by a referee runs an element exposure martingale on X and then applies the Hoeffding-Azuma inequality).

We may assume that $|\mathcal{A}| = n$. We choose a small $\varepsilon > 0$, and assume that $n$ is large. We ignore below insignificant roundings to integers.

We begin by partitioning $\mathcal{A}$ into sets of pairwise disjoint elements. Let $G$ be the intersection graph of $\mathcal{A}$: so the vertices of $G$ are the elements of $\mathcal{A}$, and $G$ has edges $AA'$ whenever $A \cap A'$ is nonempty. Since every vertex belongs to at most $d$ sets from $\mathcal{A}$, and every set has size at most $\varepsilon \log n$, each set in $\mathcal{A}$ meets at most $\varepsilon d \log n$ other sets. Thus $G$ has maximal degree at most $\varepsilon d \log n$. It follows by a theorem of Hajnal and Szemerédi [7] that $G$ has a colouring with at most $\varepsilon d \log n + 1$ colours in which the sizes of distinct colour classes differ by at most 1. Thus we may partition $G$ into independent sets (and so $\mathcal{A}$ into collections of pairwise disjoint sets) of size at least $n/(\varepsilon d \log n + 1) \geq n^{\gamma/2}$.

Let $\mathcal{A}'$ be one of these collections of pairwise disjoint sets, and set $m = |\mathcal{A}'| \geq n^{\gamma/2}$. The random set $B$ is disjoint from each member of $\mathcal{A}'$ independently with probability at least $\delta^{\varepsilon \log n} = n^{-\varepsilon \log(1/\delta)} > n^{-0.01\gamma}$ provided we have chosen a sufficiently small $\varepsilon$; it follows that the probability that $B$ is disjoint from fewer than $m/n^{\gamma/4}$ sets in $\mathcal{A}'$ is at most

$$\begin{aligned}\binom{m}{m/n^{\gamma/4}}(1-n^{-0.01\gamma})^{m-m/n^{\gamma/4}} &\leq \left(\frac{em}{m/n^{\gamma/4}}\right)^{m/n^{\gamma/4}} \exp(-n^{-0.01\gamma}m/2) \\ &< e^{m\log n/n^{\gamma/4}} e^{-n^{-0.01\gamma}m/2} \\ &< e^{-n^{-0.01\gamma}m/4},\end{aligned}$$

provided $n$ is sufficiently large. There are $\varepsilon d \log n + 1 = o(n)$ colour classes, so with failure probability $o(ne^{-n^{-0.01\gamma}n^{\gamma/2}/4}) = O(e^{-n^{\gamma/3}})$, $B$ is disjoint from at least a fraction $n^{-\gamma/4}$ of the sets in each colour class, and hence is disjoint from at least $n^{1-\gamma/4}$ sets in $\mathcal{A}$. □

For positive integers $m, n$, and $p \in [0,1]$ we write $\mathcal{S}(n,m,p)$ for a random sequence $(S_i)_{i=1}^m$ of $m$ subsets of $[n]$, where the subsets are independent and each set independently contains each element of $[n]$ with probability $p$.

Equivalently, we could consider a random $m \times n$ matrix with entries 0 and 1, where each element independently takes value 1 with probability $p$. We shall refer to $S \in \mathcal{S}(n, m, p)$ as a *random set sequence*.

Given two random set sequences $\mathcal{A} \in \mathcal{S}(m, n, p)$ and $\mathcal{A}' \in \mathcal{S}(m, n, q)$, where $m \leq n$, it will be useful to pair up the sets from $\mathcal{A}$ and $\mathcal{A}'$ so that each pair is disjoint. For $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$, the probability that $A$ and $A'$ are disjoint is $(1 - pq)^n \leq \exp(-npq)$, so if $pq > 2 \log n/n$ it is likely that we do not have any disjoint pairs at all. However, if $pq < c \log n/n$, for small enough $c$, we will show that such a pairing is possible. In fact we will prove a much stronger result: we can take just one of the set systems to be random, provided the other satisfies certain sparsity conditions.

**Lemma 5.** *For all $K > 0$ and $\eta, \gamma, \delta \in (0, 1)$ there is $\varepsilon > 0$ such that the following holds for all sufficiently large $n$. Suppose that $p = p(n), q = q(n) \in [0, 1]$ satisfy $0 \leq p < 1 - \delta$ and $pq < \varepsilon \log n/n$. Let $m \in [n^\eta, n]$ be an integer and set $d = m^{1-\gamma}$, and suppose that $\mathcal{A} = (A_i)_{i=1}^m$ is a sequence of subsets of $[n]$ such that*

- *every $i \in [n]$ belongs to at most $d$ sets from $\mathcal{A}$*
- $\max_{A \in \mathcal{A}} |A| \leq qn$.

*Let $\mathcal{B} = (B_i)_{i=1}^m \in S(n, m, p)$ be a random set sequence, and let $H$ be the bipartite graph with vertex classes $\mathcal{A}$ and $\mathcal{B}$, where we join $A_i$ to $B_j$ if $A_i \cap B_j = \emptyset$. Then, with failure probability $O(n^{-K})$, $H$ has minimal degree at least $m^{1-\gamma/4}$; furthermore, $H$ has a perfect matching.*

*Proof.* Let $\varepsilon, \varepsilon' > 0$ be fixed, small quantities (with $\varepsilon \ll \varepsilon'$) that we shall choose later. We generate $\mathcal{B}$ in two steps: we first choose a random set sequence $\mathcal{B}' = (B_i')_{i=1}^m \in S(n, m, (1 + \delta)p)$, and then obtain $\mathcal{B}$ from $\mathcal{B}'$ by deleting each element from each set $B_i'$ independently with probability $\delta' = \delta/(1 + \delta)$.

Note first that for any $i, j$, the distribution of the intersection $|A_i \cap B_j'|$ is stochastically dominated by a binomial $\mathrm{Bi}(nq, p(1 + \delta))$. So for fixed $\varepsilon' > 0$, it follows from Proposition 3 that we have $|A_i \cap B_j'| < \varepsilon' \log m$ for all $i$ and $j$, with failure probability $O(n^{-K})$, provided $\varepsilon$ is small enough in terms of $\varepsilon'$. We may therefore assume from now on that this event occurs, and condition on the choice of $\mathcal{B}'$ (so $\mathcal{B}'$ is fixed and $\mathcal{B}$ is still random).

Now consider the bipartite graph $H$. We need to prove that $H$ has a perfect matching. We shall apply Hall's condition to $\mathcal{B}$, so it is enough to show that for every subset $S \subset \mathcal{B}$ we have $|\Gamma_H(S)| \geq |S|$.

Consider $B'_i \in \mathcal{B}'$, and let $\mathcal{A}'_i = (A_j \cap B'_i)_{j=1}^m$ be the restriction of $\mathcal{A}$ to $B'_i$. Then every vertex belongs to at most $d$ sets from $\mathcal{A}'_i$ and $\max_j |A_j \cap B'_i| < \varepsilon' \log m$, so provided $\varepsilon'$ is sufficiently small we can apply Lemma 4 to deduce that with failure probability $O(e^{-m^{\gamma/3}})$ the set $B_i$ is disjoint from at least $m^{1-\gamma/4}$ sets from $\mathcal{A}'_i$. This occurs independently for each $i$ (recall that we are conditioning on $\mathcal{B}'$), so with failure probability $O(me^{-m^{\gamma/3}}) = O(n^{-K})$ every vertex in $\mathcal{B}$ has degree at least $m^{1-\gamma/4}$ in $H$, and so Hall's condition holds for every $S \subset \mathcal{B}$ with $|S| < m^{1-\gamma/4}$.

Now consider an element $A_i \in \mathcal{A}$. Each $B'_j$ meets $A_i$ in at most $\varepsilon' \log m$ vertices, and so each $B_j$ independently is disjoint from $A_i$ with probability at least $(\delta')^{\varepsilon' \log m} > m^{-\gamma/6}$, provided $\varepsilon'$ is sufficiently small. The number of $B_j$ disjoint from $A_i$ is thus a binomial with expectation at least $m^{1-\gamma/6}$ and so, by (1), is at least $m^{1-\gamma/6}/2 > m^{1-\gamma/4}$, with failure probability $O(e^{-m^{1-\gamma/6}/8})$. So with failure probability $O(me^{-m^{1-\gamma/6}/8}) = O(n^{-K})$ every vertex in $\mathcal{A}$ has degree at least $m^{1-\gamma/4}$ in $H$, and so Hall's condition holds for every $S \subset \mathcal{B}$ with $|S| > m - m^{1-\gamma/4}$.

We have now shown that $H$ has minimal degree at least $m^{1-\gamma/4}$. All that remains is to verify Hall's condition for sets $S \subset \mathcal{B}$ of size between $m^{1-\gamma/4}$ and $m - m^{1-\gamma/4}$. Let $t \in [m^{1-\gamma/4}, m - m^{1-\gamma/4}]$: we shall bound the probability that there is any subset of $\mathcal{B}$ of size $t$ with $t$ or fewer neighbours in $\mathcal{A}$. Suppose that $S \subset \mathcal{B}$ has size $t$ and $T \subset \mathcal{A}$ has size $m-t$. For any fixed $B'_i \in S$, the set sequence $\mathcal{A}' = (A \cap B'_i)_{A \in T}$ has $\max_{A' \in \mathcal{A}'} |A'| \le \varepsilon' \log m$ and every vertex belongs to at most $d$ sets from $\mathcal{A}'$, where $d = m^{1-\gamma} \le |\mathcal{A}'|^{1-\gamma/4}$. So by Lemma 4, the probability that $B_i$ intersects every set in $T$ is at most $\exp(-(m-t)^{\gamma/12})$. Thus the probability that (in the graph $H$) $S$ has no neighbours in $T$ is at most $\exp(-t \cdot (m-t)^{\gamma/12})$. Since there are at most $n^{2t} = \exp(2t \log n)$ choices for the pair $(S, T)$, we deduce that the probability that there is any set $S$ of size $t$ with at most $t$ neighbours is bounded by $\exp(2t \log n) \exp(-t \cdot (m-t)^{\gamma/12}) = O(n^{-(K+1)})$, uniformly in $t$. Summing over $t$, we see that Hall's condition holds with failure probability $O(n^{-K})$.

We conclude by noting that we can choose first $\varepsilon'$ and then $\varepsilon$ sufficiently small for the estimates above to hold. □

We are now ready to prove Theorem 2.

*Proof of Theorem 2.* Let $\eta = \gamma/2$, $t = \lceil (K+2)/\eta \rceil$, and let $G_1$ have vertex set $V$ and $G_2$ have vertex set $W$. We begin by finding a partition of $V$ into sets $V_1, V_2, \ldots$ of size $\Theta(n^\eta)$ such that:

- $V_i$ is an independent set in $G_1$ for every $i$,
- Every vertex in $V$ has fewer than $t$ neighbours in each set $V_j$.

Indeed, we first colour $V$ randomly with $n^{1-\eta}$ colours, giving each vertex a colour selected uniformly at random and independently. It follows from (1) and (2) that, with failure probability $O(n^{-K})$, every colour class has size $(1+o(1))n^{\eta}$. Consider a vertex $v \in V$, say with degree $d$. Then by assumption $d \le qn \le n^{1-\gamma}$. So the probability that $v$ has a set of $t$ neighbours, all with the same colour, is at most

$$\binom{d}{t}(1/n^{1-\eta})^{t-1} \le d^t n^{\eta t - t + 1} \le n^{1+t\eta - t\gamma} = n^{1-t\eta} = O(n^{-(K+1)}).$$

It follows that, with failure probability $O(n^{-K})$, no vertex has $t$ neighbours of the same colour. Each colour class now induces a subgraph with maximum degree less than $t$, so we can apply the Hajnal-Szemerédi Theorem [7] to each class, splitting it into $O(t)$ independent sets of (almost) the same size. The vertex classes are now independent, have size $\Theta(n^{\eta})$, and no vertex has $t$ neighbours in any other class.

Reordering if necessary, we may assume that $|V_1| \ge |V_2| \ge \cdots$. Now let $W = W_1 \cup W_2 \cup \cdots$ be an arbitrary partition of $W$ (chosen before revealing $G_2$) such that $|W_i| = |V_i|$ for every $i$. We construct a bijection between $V$ and $W$ that defines a packing (i.e., does not map any edge of $G_1$ to an edge of $G_2$) by constructing suitable bijections between $V_i$ and $W_i$ for $i = 1, 2, \dots$.

For $i = 1$, we choose an arbitrary bijection between $V_1$ and $W_1$. (Recall that $V_1$ is independent.) For $i > 1$, we set $S_i = \bigcup_{j<i} V_j$ and $T_i = \bigcup_{j<i} W_j$, and suppose that we have found a bijection $\varphi_i : S_i \to T_i$. The neighbourhoods of vertices in $V_i$ and $W_i$ define set sequences $\mathcal{A} = (\Gamma(v) \cap S_i)_{v \in V_i}$ in $S_i$ and $\mathcal{B} = (\Gamma(v) \cap T_i)_{v \in W_i}$ in $T_i$, and the bijection $\varphi_i$ allows us to identify $S_i$ and $T_i$. We now check that these two set sequences satisfy the conditions of Lemma 5, which we will then apply to obtain a bijection between $V_i$ and $W_i$. Let

$$\widetilde{n} = |S_i| = |T_i| = \Theta(in^{\eta}),$$
$$\widetilde{m} = |V_i| = |W_i| = \Theta(n^{\eta}),$$

and note that $|\mathcal{A}| = |\mathcal{B}| = \widetilde{m}$ and $\widetilde{m} \in [\widetilde{n}^{\eta/2}, \widetilde{n}]$. By construction of the partition $(V_j)_{j \ge 1}$, no vertex belongs to $t$ sets from $\mathcal{A}$, as each vertex in $S_i$ has fewer than $t$ neighbours in $V_i$. Let $\widetilde{q} = \max_{A \in \mathcal{A}} |A|/\widetilde{n}$. Each set in $\mathcal{A}$

has size at most $qn$ and so $\widetilde{q} \leq qn/\widetilde{n} = O(qn^{1-\eta}/i)$. The set sequence $\mathcal{B}$ is random with $\mathcal{B} \in S(\widetilde{n}, \widetilde{m}, p)$, and depends only on the edges between $W_i$ and $T_i$. Furthermore,

$$p\widetilde{q}\widetilde{n} \leq p \cdot (qn/\widetilde{n}) \cdot \widetilde{n} = pqn \leq \varepsilon \log n = O(\varepsilon \log \widetilde{n}).$$

We can therefore apply Lemma 5, to deduce that if $\varepsilon$ is sufficiently small then with failure probability $O(n^{-(K+1)})$ there is a bijection between the two set sequences for which the corresponding pairs are disjoint; this corresponds to a bijection between $V_i$ and $W_i$ so that there are no common edges in the bipartite graphs between $(V_i, S_i)$ and $(W_i, T_i)$ where $S_i$ and $T_i$ are identified by $\varphi_i$. Extending $\varphi_i$ with this bijection, we obtain a bijection $\varphi_{i+1} : S_{i+1} \to T_{i+1}$.

It follows that, with failure probability $O(n^{-K})$, we succeed at every step and construct the desired bijection. □

Finally in this section, we note that Theorem 2 can be used to pack several random graphs.

**Corollary 6.** *Let $\gamma, K > 0$, let $\delta \in (0,1)$, and let $t$ be a positive integer. Then there exists $\varepsilon > 0$ such that the following holds. Let $p_0(n), \ldots, p_t(n)$ satisfy*

- $\max_i p_i \leq 1 - \delta$
- $p_0 \leq n^{-\gamma}$
- $\max_{i<j} p_i p_j n \leq \varepsilon \log n.$

*Let $G_0$ be a graph of order $n$ with maximal degree at most $p_0 n$ and, for $i = 2, \ldots, t$, let $G_i \in \mathcal{G}(n, p_i)$. Then with failure probability $O(n^{-K})$ there is a packing of $G_0, \ldots, G_t$.*

*Proof.* We may assume that $p_1 \leq \cdots \leq p_t$. Thus, by the second and third conditions above, we have $\sum_{i=0}^{t-1} p_i = O(n^{-\min\{\gamma,1/3\}})$. We first pack $G_0$ and $G_1$, then add in the remaining graphs one at a time, applying Theorem 2 at each stage. Thus at the $i$th stage we have packed $G_0, \ldots, G_i$ to obtain a graph $H_i$: it follows easily from Proposition 3 that with high probability the maximum degree condition of Theorem 2 is satisfied by $H_i$ (with a slightly smaller $\gamma$). Provided $\varepsilon$ is sufficiently small, we get that with failure probability $O(n^{-K})$ we can pack $H_i$ with $G_{i+1}$. □

## 3 Packing hypergraphs

In this section, we will prove Theorem 1.

*Proof of Theorem 1.* Note that the case $k = 2$ follows immediately from Theorem 2, so we can assume $k \geq 3$. Let $\eta = 1/5$, $t = 15k$, and let $\varepsilon, \varepsilon' > 0$ be small constants and $K, K'$ large constants; we will choose $\varepsilon, \varepsilon'$ and $K, K'$ later. (In fact, we will first choose $\varepsilon'$; $K'$ will be determined by $\varepsilon'$; we then choose $K$ and finally $\varepsilon$.) We may assume that $q \leq p$, and so in particular $q = O(\sqrt{\log n/n^{k-1}}) < n^{-1/2}$ (for large $n$). We may also assume that $q \geq \varepsilon \log n/n^{k-1}$ , or increase to this value.

Our argument will follow a similar strategy to Theorem 2, but there are some additional complications. It will be helpful to reveal the edges of $G_1$ and $G_2$ in several steps. This time we let $V$ be the vertex set of $G_2$ and $W$ the vertex set of $G_1$.

We first generate a partition of $V$ into sets $V_1, V_2, \ldots$ by colouring $V$ randomly with $n^{1-\eta}$ colours, giving each vertex a colour selected uniformly at random and independently. It follows from (1) and (2) that, with failure probability $o(1)$, every colour class $V_i$ has size $(1+o(1))n^\eta$, so we may assume that this holds. Reordering if necessary, we may assume that $|V_1| \geq |V_2| \geq \cdots$. Let $W = W_1 \cup W_2 \cup \cdots$ be a random partition of $W$ such that $|W_i| = |V_i|$ for every $i$. For $i \geq 1$, we set $S_i = \bigcup_{j<i} V_j$ and $T_i = \bigcup_{j<i} W_j$ (note that $S_1 = T_1 = \emptyset$; also $S_L = V$ and $T_L = W$, where $L = n^{1-\eta} + 1$).

As before, we will construct a bijection between $V$ and $W$ by constructing bijections between $V_i$ and $W_i$ for $i = 1, 2, \ldots$. However, we need to be a little more careful than in the graph case, as there are more ways for edges to intersect the classes $V_i$ and $W_i$. For $j = 1, \ldots, k$, and any $i$, we say that an edge is *of type $j$ for $V_i$ or $W_i$* if it has $j$ vertices in $V_i$ or $W_i$, and the remaining $k - j$ vertices in $S_i$ or $T_i$.

We now reveal all type 1 edges in $G_2$. For a $(k-1)$-set $A \subset S_i$, the probability that $V_i$ contains $t$ vertices $v$ such that $A \cup \{v\}$ is an edge of $G_2$ is at most

$$\binom{2n^\eta}{t} q^t = O(n^{\eta t - t/2}) = o(n^{-k}).$$

It follows that, with high probability, for every integer $i$ and every $(k-1)$-set $A \subset S_i$, $V_i$ contains fewer than $t$ vertices that can be added to $A$ to obtain an edge of $G_2$. In other words, each $(k-1)$-set in $S_i$ is contained in fewer than $t$ type 1 edges for $V_i$.

For each vertex $v \in V_i$, we define the *type 1 neighbourhood of* $v$ to be the $(k-1)$-uniform hypergraph on $S_i$ with edge set

$$\{A \subset S_i : A \cup \{v\} \text{ is a type 1 edge for } V_i\};$$

similarly, for vertices in $W_i$, the type 1 neighbourhood is a $(k-1)$-uniform hypergraph on $T_i$.

At the first step of the partitioning process, we take a random bijection between $V_1$ and $W_1$. The expected number of common edges is at most $pqn^{k\eta} = o(1)$, and so with high probability there are no common edges.

Now consider a later stage of the partitioning process: suppose we have constructed a bijection $\varphi_i : S_i \to T_i$ and wish to extend this to a bijection $\varphi_i : S_{i+1} \to T_{i+1}$. In constructing our bijection, we will only consider edges of type 1 and 2; we will consider edges of type 3 at the end of the argument.

We first consider type 1 edges in $V_i$ and $W_i$. For each $v \in V_i$, we consider the type 1 neighbourhood of $v$ as a subset of $S_i^{(k-1)}$ (rather than as a $k$-uniform hypergraph on $S_{i+1}$). The collection of type 1 neighbourhoods of vertices in $V_i$ then defines a set sequence $\mathcal{A}$ of subsets of $S_i^{(k-1)}$; similarly, the collection of type 1 neighbourhoods of vertices in $W_i$ defines a set sequence $\mathcal{B}$ of subsets of $T_i^{(k-1)}$; and the bijection $\varphi_i$ allows us to identify $S_i^{(k-1)}$ and $T_i^{(k-1)}$. As in the proof of Theorem 2, we wish to apply Lemma 5, so we need to check that its conditions are satisfied.

Let

$$\widetilde{n} = |S_i^{(k-1)}| = |T_i^{(k-1)}| = \Theta(i^{k-1}n^{\eta(k-1)}),$$
$$\tilde{m} = |V_i| = |W_i| = (1+o(1))n^{\eta},$$

and note that $|\mathcal{A}| = |\mathcal{B}| = \tilde{m}$ and $\tilde{m} \in [\tilde{n}^{\eta/k}, \tilde{n}]$.

By construction of the partition $(V_j)_{j\geq 1}$, no element of $S_i^{(k-1)}$ is contained in $t$ sets from $\mathcal{A}$, as each $(k-1)$-set $A \subset S_i$ is contained in fewer than $t$ type 1 edges for $V_i$. The size of each set in $\mathcal{A}$ has distribution $\mathrm{Bi}(\widetilde{n}, q)$. Choose a small $\varepsilon' > 0$, let $K' = 2/(\eta\varepsilon')$, and then choose a large $K$. Let $\widetilde{q} = \max\{Kq, \varepsilon'(\log \widetilde{n})/\widetilde{n}\}$. It follows from Proposition 3 that, provided $K$ is large enough (depending on $K'$), every set in $\mathcal{A}$ has size at most $\widetilde{n}\widetilde{q}$, with failure probability at most

$$\widetilde{m}e^{-K'\widetilde{n}\widetilde{q}} \leq ne^{-K'\varepsilon' \log \widetilde{n}} \leq n^{1-K'\varepsilon'\eta} = o(1/n).$$

Furthermore, since $\widetilde{n} \le n^{k-1}$, by choosing $\varepsilon$ small enough we get

$$pKq \le K\varepsilon \frac{\log n^{k-1}}{n^{k-1}} \le \varepsilon' \frac{\log \widetilde{n}}{\widetilde{n}}$$

and hence $p\widetilde{q} \le \varepsilon'(\log \widetilde{n})/\widetilde{n}$. We can therefore apply Lemma 5, to deduce that if $\varepsilon'$ is sufficiently small then with failure probability $O(n^{-2})$ we get the following:

- a bijection $\varphi^* : V_i \to W_i$ such that the corresponding pairs in the two set sequences are disjoint. This corresponds to a bijection between $V_i$ and $W_i$ so that there are no collisions between type 1 edges for $V_i$ and $W_i$. Also:
- for all distinct $u, v \in V_i$ and $x, y \in W_i$, a bijection

  $$\varphi^{**} : V_i \setminus \{u, v\} \to W_i \setminus \{x, y\}$$

  such that there are no collisions of type 1 edges for $V_i$ and $W_i$, except possibly for edges containing $u$, $v$, $x$ or $y$.

The mapping $\varphi^*$ deals with collisions between type 1 edges. However, we must also consider type 2 edges for $V_i$ and $W_i$. We do not reveal type 2 edges at this stage, but only the number of collisions between type 2 edges created by the mapping $\varphi^*$. There are at most $n^{k-2+2\eta}$ type 2 edges for $V_i$ and $W_i$, and so the probability that $\varphi^*$ maps any type 2 edge for $V_i$ in $G_2$ to a type 2 edge in $G_1$ is at most $pqn^{k-2+2\eta} \le \log n/n^{1-2\eta}$; the probability that there are at least two collisions is $O(\log^2 n/n^{2-2\eta}) = o(1/n)$ (which is small enough to ignore). If there are no collisions, then we use $\varphi^*$ to extend $\varphi_i$.

This leaves the case when there is one collision between type 2 edges. We reveal the edge where this occurs: say $A \cup \{u, v\}$ maps to $A \cup \{x, y\}$ under $\varphi^*$. We thus condition on the existence of these two edges in $G_2$ and $G_1$, and on this being the only collision. We shall show the existence of another mapping $\varphi^{**}$ from $V_i$ to $W_i$ that avoids collisions for both type 1 and type 2 edges with probability at least $1 - O(\log n/\sqrt{n})$. Then the probability that we get collisions for both $\varphi^*$ and $\varphi^{**}$ is $O((\log n/n^{1-2\eta}) \cdot \log n/\sqrt{n})$, which is $o(1/n)$.

Let $D = \lceil 6(\log \widetilde{n})/\delta \rceil$. We choose distinct vertices $x_1, \ldots, x_D, y_1 \ldots, y_D$ in $W_i$ such that the type 1 neighbourhood of $u$ is edge-disjoint from the type

1 neighbourhoods of $x_1, \dots, x_D$, and the type 1 neighbourhood of $v$ is edge-disjoint from the type 1 neighbourhoods of $y_1, \dots, y_D$ (the existence of these vertices follows from the minimal degree condition on $H$ in Lemma 5).

We reveal the edges $A \cup \{x_\ell, y_\ell\}$ for each $\ell \leq D$: since $p \leq 1-\delta$, it follows that with probability $1 - o(1/n)$ there is some $\ell$ such that $A \cup \{x_\ell, y_\ell\}$ is not present in $G_1$. We then use the appropriate mapping $\varphi^{**}$ from $V_i \setminus \{u, v\}$ to $W_i \setminus \{x_\ell, y_\ell\}$ that we found above, and extend it by setting $\varphi^{**}(u) = x_\ell$ and $\varphi^{**}(v) = y_\ell$ so that we have a mapping from $V_i$ to $W_i$. The mapping $\varphi^{**}$ does not cause any collision of type 1 edges. Finally, we reexamine the type 2 edges for collisions. We have ensured that $A \cup \{u, v\}$ does not collide with anything; the probability of a collision involving any edge of form $A \cup \{x_j, y_j\}$ is at most $qD = O(\log n/\sqrt{n})$; and the probability of any other collision is at most $\log n/n^{1-2\eta} = O(1/\sqrt{n})$, as before. (More formally: we have conditioned on the edges $A \cup \{x_j, y_j\}$, on the event that a particular pair of type 2 edges collide, and the event that no other collisions occur. If we resample all type 2 edges that are not in the colliding pair or of form $A \cup \{x_j, y_j\}$, the number of collisions under $\varphi^{**}$ stochastically dominates the number before resampling, giving the same bound.) So the probability that $\varphi^{**}$ yields a collision is $O(\log n/\sqrt{n})$, as required.

It follows that, with probability $1 - o(1/n)$, we are able to find a good bijection between $V_i$ and $W_i$, and extend $\varphi_i$ to $\varphi_{i+1}$. Continuing in this way, we find a bijection from $V$ to $W$ in which there are no collisions between type 1 or 2 edges for any $V_i$, $W_i$.

Finally, we reveal all edges of type 3 or more. There are at most $n^{k-2+2\eta}$ possible edges of type 3 or more, and so the probability that any of these is an edge in both hypergraphs is at most $pqn^{k-2+2\eta} = o(1)$. The algorithm therefore succeeds with probability $1 - o(1)$. □

# 4 Conclusion

We conclude by mentioning a few open questions.

- The bound in Theorem 1 is sharp to within a constant factor. It is natural to expect that there is some $c = c(k) > 0$ such that almost surely a pair of random $k$-uniform hypergraphs $G_1, G_2 \in \mathcal{G}(n, k, p)$ are packable if $p < (c - \varepsilon)\sqrt{\log n/n^{k-1}}$ and are unpackable if $p > (c + \varepsilon)\sqrt{\log n/n^{k-1}}$. Is this correct? If so, what is the value of $c$?

- What happens with the results above if we take $G_1 = G_2$? We would expect this to make no difference.

- All our examples of unpackable $k$-uniform hypergraphs $G_1$, $G_2$ have $\Delta(G_1)\Delta(G_2) = \Omega(n^{k-1})$. What is the correct bound here?

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