A note on intersecting hypergraphs with large cover number

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Abstract

We give a construction of r-partite r-uniform intersecting hypergraphs with cover number at least $r-4$ for all but finitely many r. This answers a question of Abu-Khazneh, Barát, Pokrovskiy and Szabó [2], and shows that a long-standing unsolved conjecture due to Ryser is close to being best possible for every value of r.

Keywords: partite hypergraph, intersecting, cover

1 Introduction

A hypergraph is said to be *r-partite* if it has a vertex partition $V_1 \cup \cdots \cup V_r$ such that each edge contains at most one vertex from each V_i. An old and well-studied conjecture of Ryser [12] asserts that every r-partite r-uniform hypergraph $\mathcal{H}$ satisfies $\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$, where $\tau(\mathcal{H})$ denotes the minimum size of a vertex cover and $\nu(\mathcal{H})$ denotes the maximum size of a set of pairwise disjoint edges in $\mathcal{H}$. In particular this would imply that every intersecting r-partite r-uniform hypergraph can be covered by $r-1$ vertices. Despite substantial work by many authors over many years, Ryser's Conjecture is known to be true in general only for $r = 2$ (König's Theorem) and $r = 3$ [3],

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and for intersecting hypergraphs only for $r \leq 5$ [13]. For more on the history of the problem see e.g. [8].

Ryser's Conjecture is tight for a given value of r if there is an r-partite r-uniform hypergraph $\mathcal{H}$ with $\tau(\mathcal{H}) \geq (r-1)\nu(\mathcal{H})$ (such hypergraphs are called r-*Ryser* hypergraphs in [2]). Because of the apparent difficulty of the problem in general, a significant amount of work has been done on constructing and understanding r-Ryser hypergraphs (e.g. [1, 4, 6, 7, 10]). For every prime power p there is a standard construction, based on the projective plane, that gives an intersecting r-Ryser hypergraph for $r = p + 1$. Very recently it was proved in [2] that intersecting r-Ryser hypergraphs exist also for every $r = p + 2$. Apart from these infinite families, the only other values of r for which the conjecture is known to be tight are $r = 7$ [4, 1], $r = 11$ [1] and $r = 12$ [6]. In [2] the authors ask whether there exists a constant K such that for every r there exists an intersecting r-partite r-uniform hypergraph $\mathcal{H}$ with $\tau(\mathcal{H}) \geq r - K$, thus showing that Ryser's problem is close to being best possible for every r. Here we answer this question in the affirmative, by showing in particular that we may take $K \leq 4$ for all sufficiently large integers r. This is our main result (Theorem 6), and it appears in Section 4. We also give a new construction for intersecting r-Ryser hypergraphs for special values of r in Section 5.

2 Basic construction

As in most known constructions for this problem (for example [2]), our construction will be based on finite projective planes and the corresponding affine planes. The affine plane $\mathrm{AG}(2, p)$ can be constructed by removing all points from a fixed line L of the projective plane $\mathrm{PG}(2, p)$. Thus lines in $\mathrm{AG}(2, p)$ each have p points, and they fall into $p + 1$ parallel classes C_i with $|C_i| = p$, and any two lines from different classes have exactly one vertex in common. We define a hypergraph $\mathcal{A}_p$ constructed from $\mathrm{AG}(2, p)$ by choosing an arbitrary point x and removing it, together with all lines that contain x. The remaining lines of $\mathrm{AG}(2, p)$ form the edges of $\mathcal{A}_p$.

Proposition 1. *The hypergraph $\mathcal{A}_p$ has the following properties.*

1. *$\mathcal{A}_p$ is $(p + 1)$-partite, with vertex classes $V_1, \ldots, V_{p+1}$ where $\{x\} \cup V_1, \ldots, \{x\} \cup V_{p+1}$ are the lines of $AG(2, p)$ containing x. Note $|V_i| = p - 1$ for each i.*

2. $\mathcal{A}_p$ *is* p*-uniform.*

3. *The edges of* $\mathcal{A}_p$ *fall into* $p+1$ *parallel classes* C_i *with* $|C_i| = p-1$ *and any two edges from different parallel classes have exactly one vertex in common.*

4. *Each edge of* C_i *is disjoint from* V_i*.*

Our aim is to construct an intersecting hypergraph based on $\mathcal{A}_p$, by adding a gadget for each parallel class C_i to make it intersect. To show that the cover number of the resulting construction is large we will make use of the following theorem of Jamison [9] and Brouwer and Schrijver [5].

Theorem 2. $\tau(\mathrm{AG}(2,p)) = 2p-1$.

3 Near-extremal constructions

Let $\mathcal{J}$ be an r_0-partite r_0-uniform intersecting hypergraph with $r_0 \leq p$ and $\tau(\mathcal{J}) \geq r_0 - 1 - d_0 \geq 2$ for some $d_0 \geq 0$.

Set $r = p + r_0$. We construct an r-partite r-uniform hypergraph $\mathcal{H}_r$ as follows. Fix a copy of $\mathcal{A}_p$ with vertex classes $V_1, \ldots, V_{p+1}$ (as in Proposition 1). For each parallel class C_i of $\mathcal{A}_p$ place a copy $\mathcal{J}^i$ of $\mathcal{J}$ with one vertex class in V_i and the remaining $r_0 - 1$ classes in $V_{p+2} \ldots V_{p+r_0}$ in an arbitrary way, such that all $\mathcal{J}^i$ are disjoint from each other and from $\mathcal{A}_p$. Extend every edge e of C_i to $|\mathcal{J}|$ edges $e \cup f$ of $\mathcal{H}_r$ by appending each edge f of $\mathcal{J}^i$ to e. Thus the edge set of $\mathcal{H}_r$ is $\bigcup_{i=1}^{p+1}\{e \cup f : e \in C_i, f \in \mathcal{J}^i\}$.

Proposition 3. *The hypergraph* $\mathcal{H}_r$ *has the following properties.*

1. $\mathcal{H}_r$ *is an* r*-partite* r*-uniform intersecting hypergraph,*

2. $\tau(\mathcal{H}_r) \geq r - 1 - (d_0 + 1)$.

Proof. It follows immediately from the definitions that $\mathcal{H}_r$ is r-partite and r-uniform. To see that $\mathcal{H}_r$ is intersecting, let $e \cup f$ and $e' \cup f'$ be two edges of $\mathcal{H}_r$. If $e = e'$ or if e and e' are from different parallel classes of $\mathcal{A}_p$ then they intersect in $\mathcal{A}_p$, implying that $e \cup f$ and $e' \cup f'$ intersect in $\mathcal{H}_r$. If e and e' are from the same parallel class of $\mathcal{A}_p$ then f and f' are two (not necessarily distinct) edges from the same copy of the intersecting hypergraph $\mathcal{J}$, and therefore they intersect.

To estimate $\tau(\mathcal{H}_r)$, consider a minumum cover T.

Case 1: No vertex of T is in any $\mathcal{J}^i$.

In this case $T \cup \{x\}$ is a cover of the affine plane $\mathrm{AG}(2,p)$, which by Theorem 2 must have size at least $2p-1$. Hence $|T| \geq 2p-2 \geq p+r_0-2 = r-2 \geq r-1-(d_0+1)$.

To address the remaining cases, we claim that if T contains a vertex z of $\mathcal{J}^i$ then T contains a cover of $\mathcal{J}^i$. To see this, suppose on the contrary that some edge f of $\mathcal{J}^i$ is disjoint from T. Since T is a minimum cover there exists an edge $e \cup f'$ of $\mathcal{H}_r$ such that $T \cap (e \cup f') = \{z\}$, where $e \in C_i$ and $f' \in \mathcal{J}^i$. Then $T \cap e = \emptyset$. But then $e \cup f \in \mathcal{H}_r$ is disjoint from T, contradicting the fact that T is a cover. This verifies the claim.

Case 2: For some $i \neq j$, the cover T intersects $\mathcal{J}^i$ but not $\mathcal{J}^j$.

Then by the claim T contains a cover of $\mathcal{J}^i$, which has size at least r_0-1-d_0. Since T has no vertices in $\mathcal{J}^j$ it must cover C_i within the vertex set of $\mathcal{A}_p$ which is disjoint from $\mathcal{J}^i$. Hence we get another $p-1$ vertices in T, for a total of $p-1+r_0-1-d_0 = r-1-(d_0+1)$ as required.

Case 3: T intersects $\mathcal{J}^i$ for every i.

Since the $\mathcal{J}^i$ are all disjoint we find by the claim that $|T| \geq \tau(\mathcal{J})(p+1) \geq 2p+2 > r-1-(d_0+1)$.

Therefore in all cases the statement holds. □

4 The main theorem

We begin with a construction of r-partite r-uniform hypergraphs when r has a special form.

Lemma 4. *Let $r = \sum_{i=1}^{k} p_i + 1$, where each p_i is a prime power and $p_i \geq \sum_{j<i} p_j + 1$ for each $i \geq 2$. Then there exists an r-partite r-uniform intersecting hypergraph $\mathcal{H}_r$ with $\tau(\mathcal{H}) \geq r-k$.*

Proof. We use induction on k. The case $k=1$ is dealt with by the standard example of the truncated projective plane (formed by removing one point from the projective plane, together with every line containing it): we obtain an example with $r = p+1$ classes and $\tau = p = r-1 \geq 2$.

Assume $k \geq 2$ and let $\mathcal{H}_s$ be a hypergraph with the claimed properties for $s = r - p_k$. Observe that the conditions guarantee $p_k \geq s$. Note also that

$\tau(\mathcal{H}_s) \geq 2$. Construct $\mathcal{H}_r$ as in Section 3, starting with the hypergraph $\mathcal{A}_{p_k}$ and using $\mathcal{J} = \mathcal{H}_s$. Then by the induction hypothesis $\tau(\mathcal{J}) \geq s - (k-1) = s - 1 - d_0$ where $d_0 = k - 2$. By Proposition 3 we obtain an intersecting hypergraph $\mathcal{H}_r$ for $r = s + p_k$ that is r-partite and r-uniform, that satisfies

$$\tau(\mathcal{H}_r) \geq r - 1 - (d_0 + 1) = r - k.$$

This completes the proof. □

In fact we will use this lemma below only when each p_i is prime and $k = 3$.

To show the existence of suitable primes we use the following classical result of Montgomery and Vaughan [11].

Theorem 5. *There exist Q, $\gamma > 0$ and N such that for all $n > N$, all but at most $Qn^{1-\gamma}$ even integers in the interval $(0, n)$ are the sum of two primes.*

We are now ready to prove our main theorem.

Theorem 6. *There exists M such that for every integer $r > M$*

- *if r is even then there exists an r-partite r-uniform intersecting hypergraph $\mathcal{H}$ with $\tau(\mathcal{H}) \geq r - 3$,*
- *if r is odd then there exists an r-partite r-uniform intersecting hypergraph $\mathcal{H}$ with $\tau(\mathcal{H}) \geq r - 4$.*

Proof. We note that the second claim follows immediately from the first, since we may construct an r-partite r-uniform intersecting hypergraph $\mathcal{H}'$ from an $(r-1)$-partite $(r-1)$-uniform intersecting hypergraph $\mathcal{H}$ by adding a new vertex class, and adding a new vertex in this class to every edge of $\mathcal{H}$. Then $\tau(\mathcal{H}') = \tau(\mathcal{H})$. Thus we may assume that r is even.

Our aim is to write $r = p_1 + p_2 + p_3 + 1$, where p_1, p_2, p_3 satisfy $p_2 > p_1$ and $p_3 > p_2 + p_1$, as in Lemma 4. Let Q, N and γ be as in Theorem 5. For an interval I we write pI for the number of primes in I, and for a real number x we let $p(x)$ denote $p[1, x]$. The Prime Number Theorem tells us that $p(x) = (1 + o(1))x/\log x$. Therefore there exists $M \geq N$ such that for all $t \geq M$ we have

$$p(3t/4) + p(t/8) - p(t/2) - p(t/4) - Q \cdot (t/2)^{1-\gamma} \geq 1.$$

Let $r > M$ be an even integer. Set $t = r - 1$. Let $w = p(\frac{t}{2}, \frac{3t}{4}] = p(\frac{3t}{4}) - p(\frac{t}{2})$, so there are w choices for a prime p_3 in the interval $(\frac{t}{2}, \frac{3t}{4}]$. Thus there are w integers in the interval $[\frac{t}{4}, \frac{t}{2})$ of the form $t - p_3$ where p_3 is prime.

Now we show that one of these w integers can be written as $p_1 + p_2$ for distinct primes p_1 and p_2. Let us call such an integer *good*. By Theorem 5 there are at most $z = Q \cdot (\frac{t}{2})^{1-\gamma}$ integers in $[\frac{t}{4}, \frac{t}{2})$ that are not the sum of two primes. The number y of integers in $[\frac{t}{4}, \frac{t}{2})$ of the form $2p$ where p is prime is $p[\frac{t}{8}, \frac{t}{4}) = p(\frac{t}{8}, \frac{t}{4}]$ since neither $t/4$ nor $t/2$ is an integer. Thus $y = p(\frac{t}{4}) - p(\frac{t}{8})$. Thus the number of good integers is at least

$$w - z - y = p(3t/4) + p(t/8) - p(t/2) - p(t/4) - Q \cdot (t/2)^{1-\gamma} \geq 1.$$

Therefore a good integer exists and we can write $r - 1 = t = p_1 + p_2 + p_3$ where $p_1 < p_2$ and $p_3 \geq \frac{t+1}{2} = 1 + \frac{t-1}{2} \geq 1 + p_1 + p_2$. Therefore by Lemma 4 there exists an r-partite r-uniform intersecting hypergraph $\mathcal{H}_r$ with $\tau(\mathcal{H}) \geq r - 3$ as required. $\square$

5 Extremal constructions

Here we give another construction based on the hypergraph $\mathcal{A}_p$ of an r-partite r-uniform intersecting hypergraph with cover number exactly $r-1$. It exists whenever $r = 2p - 1$ and both p and $p - 1$ are prime powers.

Our construction gives a tight example for Ryser's conjecture for a few previously unknown values of r. Note that if p, $p - 1$ are both prime powers then one of p, $p-1$ must be a power of 2. If $p = 2^{i-1}+1$ then $r = 2^i+1$: since $r - 1$ is a prime power, there is already an extremal construction for this r. However, if $p = 2^{i-1}$ and $p - 1$ is also prime, then we obtain a construction for $r = 2p - 1 = 2^i - 1$. The construction gives a previously unknown value of r if neither of $r - 1 = 2^i - 2$ and $r - 2 = 2^i - 3$ is a prime power. For example, this holds when i is any of 8, 18, 32, 62, 90, 108, 128, 522, 608, 1280, 2204, 2282, 3218, 4254, 4424, 9690, 9942, 11214, 19938. We remark that the examples on this list all satisfy $r = 2p - 1$ where $p - 1 = 2^{i-1} - 1$ is a Mersenne prime (and recall that it is unknown whether there are infinitely many Mersenne primes).

We now describe the construction. We repeat the general idea of Section 3. Let p be a prime power such that $p - 1$ is also a prime power. This time we begin with the hypergraph $\mathcal{J}$ formed from $\mathrm{AG}(2, p-1)$ by removing the lines of one parallel class C_p and declaring them to be the classes of a vertex partition into $p-1$ vertex classes, each of size $p-1$. Then $\mathcal{J}$ is an r_0-partite r_0-uniform hypergraph with $r_0 = p-1$, with $p-1$ parallel classes of edges, each containing $p-1$ vertices, and any two edges from different parallel classes intersect. To bound $\tau(\mathcal{J})$, observe that if T is a cover of $\mathcal{J}$ then adding one vertex from each line in C_p gives a cover of $\mathrm{AG}(2, p-1)$. Hence by Theorem 2 we know that $|T| + p - 1 \geq 2(p-1) - 1$, which implies $|T| \geq p-2$.

Set $r = p + r_0 = 2p - 1$. We construct an r-partite r-uniform hypergraph $\mathcal{G}_r$ as follows. Fix a copy of $\mathcal{A}_p$ with vertex classes $V_1, \ldots, V_{p+1}$. For each parallel class C_i of $\mathcal{A}_p$ place a copy $\mathcal{J}^i$ of $\mathcal{J}$ with one vertex class in V_i and the remaining $r_0 - 1$ classes in $V_{p+2} \ldots V_{p+r_0}$ in an arbitrary way, such that all $\mathcal{J}^i$ are disjoint from each other and from $\mathcal{A}_p$. Take an arbitrary matching between the set C_i and the set of parallel classes of $\mathcal{J}^i$ and extend every edge e of C_i to $p-1$ edges $e \cup f$ of $\mathcal{G}_r$ by appending to e each edge f of the parallel class of $\mathcal{J}$ matched to e.

Theorem 7. *The hypergraph $\mathcal{G}_r$ has the following properties.*

1. *$\mathcal{G}_r$ is an r-partite r-uniform intersecting hypergraph,*
2. *$\tau(\mathcal{G}_r) \geq r - 1$.*

Proof. It follows immediately from the definitions that $\mathcal{G}_r$ is r-partite and r-uniform. To see that $\mathcal{G}_r$ is intersecting, let $e \cup f$ and $e' \cup f'$ be two edges of $\mathcal{G}_r$. If $e = e'$ or if e and e' are from different parallel classes of $\mathcal{A}_p$ then they intersect in $\mathcal{A}_p$, implying that $e \cup f$ and $e' \cup f'$ intersect in $\mathcal{G}_r$. If e and e' are from the same parallel class of $\mathcal{A}_p$ then f and f' are two edges from distinct parallel classes of $\mathcal{J}$, and therefore they intersect.

To estimate $\tau(\mathcal{G}_r)$, consider a minumum cover T. If no vertex of T is in any $\mathcal{J}^i$, then as before $T \cup \{x\}$ is a cover of the affine plane $\mathrm{AG}(2, p)$, where x is the vertex deleted from $\mathrm{AG}(2, p)$ in the construction of $\mathcal{A}_p$. By Theorem 2, this must have size at least $2p - 1$. Hence $|T| \geq 2p - 2 = r - 1$.

To conclude the proof we show that there exists a minimum cover T that is disjoint from all $\mathcal{J}^i$. To see this, suppose that T contains a vertex z of $\mathcal{J}^i$. Since T is a minimum cover there exists an edge $e \cup f'$ of $\mathcal{G}_r$ such that $T \cap (e \cup f') = \{z\}$, where $e \in C_i$ and $f' \in \mathcal{J}^i$. Then $T \cap e = \emptyset$. But the $p-1$ edges $e \cup g$ for all g in a parallel class of $\mathcal{J}^i$ are edges of $\mathcal{G}_r$, and therefore T contains $p-1$ vertices of $\mathcal{J}^i$ to cover them. But then exactly the same set of edges of $\mathcal{G}_r$ could be covered with $p-1$ vertices of $\mathcal{A}_p$, one from each edge in

C_i. Repeating this argument shows the existence of a minimum T disjoint from all $\mathcal{J}^i$, thus completing the proof. $\square$

We remark in closing that except for a few sporadic small examples, all constructions of intersecting r-partite hypergraphs $\mathcal{H}$ with $\tau(\mathcal{H})$ close to r seem to be based in some way on finite projective planes, and hence depend on the existence of these special structures. It would be interesting either to find a different type of construction, or to show that near-extremal constructions must contain large pieces from a projective plane.

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