SEPARATION DIMENSION AND DEGREE

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Abstract. The *separation dimension* of a graph G is the minimum positive integer d for which there is an embedding of G into $\mathbb{R}^d$, such that every pair of disjoint edges are separated by some axis-parallel hyperplane. We prove a conjecture of Alon et al. [*SIAM J. Discrete Math.* 2015] by showing that every graph with maximum degree Δ has separation dimension less than 20Δ, which is best possible up to a constant factor. We also prove that graphs with separation dimension 3 have bounded average degree and bounded chromatic number, partially resolving an open problem by Alon et al. [*J. Graph Theory* 2018].

1. Introduction

This paper studies the separation dimension of graphs and its relationship with maximum and average degree. For a graph G, a function $f : V(G) \to \mathbb{R}^d$ is *separating* if for all disjoint edges $vw, xy \in E(G)$ there is an axis-parallel hyperplane that separates the pair of points $\{f(v), f(w)\}$ from the pair $\{f(x), f(y)\}$. The *separation dimension* of a graph G is the minimum positive integer d for which there is a d-dimensional separating function for G; see [1–5, 12, 16] for recent work on the separation dimension of graphs.

This topic can also be thought of more combinatorially. Edges e and f in a graph G are *separated* in a linear ordering of $V(G)$ if both endpoints of e appear before both endpoints of f, or both endpoints of f appear before both endpoints of e. A *representation* of G is a non-empty set of linear orderings of $V(G)$. A *representation* $\mathcal{R}$ of G is *separating* if every pair of disjoint edges in G are separated in at least one ordering in $\mathcal{R}$. It is easily seen that the separation dimension of G equals the minimum size of a separating representation of G; see [1–4, 6].

A fundamental question is the relationship between separation dimension and maximum degree. Chandran et al. [6] proved that every graph with maximum degree Δ has separation dimension at most $2\Delta(\lceil\log_2 \log_2 \Delta\rceil + 3) + 1$. Alon et al. [1] improved this bound to $2^{9\log^*(\Delta)}\Delta$, and conjectured that a stronger $O(\Delta)$ bound should hold. We prove this conjecture.

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Theorem 1. *Every graph with maximum degree $\Delta \geqslant 1$ has separation dimension less than 20Δ.*

This linear bound is best possible up to a constant factor, since Alon et al. [1] proved that almost every Δ-regular graph has separation dimension at least $\frac{\Delta}{2}$. Theorem 1 is proved in Section 3.

Section 4 of this paper considers the following natural extremal question, first posed by Alon et al. [2]: What is the maximum average degree of an n-vertex graph with separation dimension s? Every graph with separation dimension at most 2 is planar, and thus has average degree less than 6. For $s \geqslant 3$, Alon et al. [2] proved the best known upper bound on the average degree of $O(\log^{s-2} n)$, and asked whether graphs with bounded separation dimension have bounded degeneracy (or equivalently, bounded average degree). We answer the first open case of this problem.

Theorem 2. *There is a constant c such that every graph with separation dimension 3 has average degree at most c.*

2. A Colouring Lemma

This section proves a straightforward lemma that shows how to colour a graph so that each vertex has few neighbours of each colour (Lemma 5). Several previous papers have proved similar results [6, 9–11, 14, 15]. The proof depends on the following two standard probabilistic tools. Let $[k] := \{1, 2, \dots, k\}$.

Lemma 3 (Lovász Local Lemma [8])**.** *Let $E_1, \dots, E_n$ be events in a probability space, each with probability at most p and mutually independent of all but at most D other events. If $4pD \leqslant 1$ then with positive probability, none of $E_1, \dots, E_n$ occur.*

Lemma 4 (Chernoff Bound [13])**.** *Let $X_1, \dots, X_n$ be independent random variables, where $X_i = 1$ with probability p and $X_i = 0$ with probability $1 - p$. Let $X := \sum_{i=1}^{n} X_i$. Then for $\delta > 0$,*

$$\mathbb{P}(X \geqslant (1+\delta)pn) \leqslant e^{-\delta^2 pn/3}.$$

Lemma 5. *For all positive integers k and Δ, for every graph G with maximum degree at most Δ, there is a partition $V_1, \dots, V_k$ of $V(G)$ such that for every vertex $v \in V(G)$ and integer $i \in [k]$,*

$$|N_G(v) \cap V_i| < d := \frac{\Delta}{k} + \sqrt{\frac{3\Delta \log(4k\Delta^2)}{k}}.$$

Proof. Independently and randomly colour each vertex with one of k colours. For each vertex $v \in V(G)$ and colour c, let $A_{v,c}$ be the event that at least d neighbours of v are all assigned colour c. Each event is mutually independent of all but at most $k\Delta^2$ other events.

We now prove that $\mathbb{P}(A_{v,c}) \leqslant (4k\Delta^2)^{-1}$. Since $\mathbb{P}(A_{v,c})$ is increasing with $\deg(v)$, we may assume that $\deg(v) = \Delta$. Say $w_1, \dots, w_\Delta$ are the neighbours of v. For $i \in [\Delta]$, let $X_i := 1$ if w_i is coloured c, otherwise let $X_i := 0$. Then $\mathbb{P}(X_i) = p := \frac{1}{k}$. Let

$X := \sum_{i=1}^{\Delta} X_i$. Then $A_{v,c}$ holds if and only if $X \geqslant d$. Let $\delta := \frac{dk}{\Delta} - 1$, so $d = (1+\delta)p\Delta$. Then $\mathbb{P}(A_{v,c}) = \mathbb{P}(X \geqslant d) = \mathbb{P}(X \geqslant (1+\delta)p\Delta)$. Now

$$\frac{\delta^2 p\Delta}{3} = \frac{1}{3}\left(\frac{dk}{\Delta} - 1\right)^2 p\Delta = \log(4k\Delta^2).$$

By Lemma 4 with $n = \Delta$,

$$\mathbb{P}(A_{v,c}) \leqslant e^{-\delta^2 p\Delta/3} = (4k\Delta^2)^{-1},$$

as claimed. By Lemma 3, with positive probability no event occurs, implying the desired partition exists. $\square$

3. Proof of Theorem 1

Our proof works by considering sets of orderings with stronger properties than separation. We start with a lemma about complete graphs.

Lemma 6. *Let G be the complete graph on n vertices including loops. Then for some integer $p \leqslant 10 \log n$, there are linear orderings $<_1, \dots, <_p$ of $V(G)$, such that:*

(1) every pair of disjoint edges $e, f \in E(G)$ are separated in some $<_i$, and

(2) for every vertex $v \in V(G)$ and distinct vertices $u, w \in V(G) \setminus \{v\}$, for some $i \in [p]$ we have $u <_i v <_i w$ or $w <_i v <_i u$.

Proof. Let $p := \lfloor 10 \log n \rfloor$. For $i \in [p]$, let $<_i$ be a random linear ordering of $V(G)$.

Let e and f be edges in G with no common endpoint. If neither e nor f are loops, then the probability that e and f are separated in $<_i$ is $\frac{1}{3}$. If e is a loop and f is a non-loop, then the probability that e and f are separated in $<_i$ is $\frac{2}{3}$. If both e and f are loops, then they are always separated in $<_i$. Thus the probability that e and f are not separated in $<_i$ is at most $\frac{2}{3}$. Hence the probability that (1) fails for e and f is at most $(\frac{2}{3})^p$.

Now consider a vertex $v \in V(G)$ and distinct vertices $u, w \in V(G) \setminus \{v\}$. For each $i \in [p]$ the probability that $u <_i v <_i w$ or $w <_i v <_i u$ is $\frac{1}{3}$. Hence the probability (2) fails for every $i \in [p]$ is at most $(\frac{2}{3})^p$.

By the union bound, the probability that both (1) and (2) fail is at most $\binom{|E(G)|}{2}(\frac{2}{3})^p + n\binom{n-2}{2}(\frac{2}{3})^p = (\binom{n(n+1)/2}{2} + n\binom{n-2}{2})(\frac{2}{3})^p < n^4(\frac{2}{3})^p < 1$. Thus there exists linear orderings $<_1, \dots, <_p$ such that (1) and (2) hold. $\square$

Note that we need $\Omega(\log n)$ orderings in Lemma 6 because of (2): if $p < \log_2(n-1) - 1$ then for any vertex v and any set of p orderings, there are distinct vertices x, y are on the same side of v in each of the orderings.

The following definition is a key to the proof of Theorem 1. A representation $<_1, \dots, <_p$ of a graph G is *strongly separating* if:

(a) for all disjoint edges $vw, xy \in E(G)$, for some ordering $<_i$, we have $v, w <_i x, y$ or $x, y <_i v, w$, and

(b) for every edge $vw \in E(G)$ and vertex $x \in V(G) \setminus \{v, w\}$, we have $x <_i v, w$ and $v, w <_j x$ for some $i, j \in [p]$.

We define the *strong separation dimension* of a graph G to be the minimum number of linear orderings in a strongly separating representation of G. Clearly the separation dimension of a graph is at most its strong separation dimension, and it will be helpful to work with the latter.

Lemma 7. *Every graph G with maximum degree Δ has strong separation dimension at most the separation dimension of G plus $2\Delta + 2$.*

Proof. Say G has separation dimension d. By Vizing's Theorem, there is a partition $E_1, \dots, E_{\Delta+1}$ of $E(G)$ into matchings. Starting from a separating representation of G in d dimensions, we now add two orderings $<_i$ and $<'_i$ for each $i \in [\Delta + 1]$. Say $E_i = \{v_1w_1, \dots, v_nw_n\}$. Let $<_i$ be $v_1, w_1, \dots, v_nw_n$ followed by $V(G) \setminus \{v_1, w_1, \dots, v_n, w_n\}$ in any ordering. Let $<'_i$ be the reverse of $<'_i$. Every edge vw of G is in some E_i. Since v and w are consecutive in $<_i$, for each vertex $x \in V(G) \setminus \{v, w\}$, we have $v, w <_i x$ and $x <'_i v, w$, or $v, w <'_i x$ and $x <_i v, w$. Hence we have a strongly separating representation of G with $d + 2\Delta + 2$ orderings in total. □

Lemma 8. *Let $G_1, \dots, G_k$ be the connected components of a graph G. For $a \in [k]$, let p_a be the strong separation dimension of G_a. Then G has strong separation dimension at most $\max\{p_1, \dots, p_k, 2\}$. Moreover, there is such a representation such that in each ordering, $V(G_1) < V(G_2) < \cdots < V(G_k)$ or $V(G_k) < V(G_{k-1}) < \cdots < V(G_1)$.*

Proof. Let $p := \max\{p_1, \dots, p_k, 2\}$. For $a \in [k]$, let $\{<^a_1, \dots, <^a_p\}$ be a strongly separating representation of G_a. For $j \in [p - 1]$, let $<_j$ be the ordering of $V(G)$ with $V(G_1) <_j \cdots <_j V(G_k)$, where $V(G_a)$ is internally ordered according to $<^a_j$, for $a \in [k]$. Finally, let $<_p$ be the ordering of $V(G)$ with $V(G_k) <_p \cdots <_p V(G_1)$, where $V(G_a)$ is internally ordered according to $<^a_p$, for $a \in [k]$. Thus $\{<_1, \dots, <_p\}$ is a representation of G, which we now show is strongly separating. Consider disjoint edges $vw, xy \in E(G)$. If vw and xy are in the same component, then (a) holds by assumption. Otherwise, vw and xy are in distinct components, implying that $v, w <_1 x, y$ or $x, y <_1 v, w$, and again (a) holds. Now consider an edge $vw \in E(G)$ and vertex $x \in V(G) \setminus \{v, w\}$. If vw and x are in the same component, then (b) holds by assumption. So we may assume that $vw \in E(G_a)$ and $x \in V(G_b)$ for distinct $a, b \in [k]$. If $a < b$ then $v, w <_1 x$ and $x <_p v, w$. If $b < a$ then $v, w <_p x$ and $x <_1 v, w$. Thus (b) holds, and $\{<_1, \dots, <_p\}$ is strongly separating. □

Note that every connected graph with at least three vertices has strong separation dimension at least 2, so Lemma 8 implies that for every graph G with at least three vertices in some component, the strong separation dimension of G equals the maximum strong separation dimension of the components of G.

For a graph G and disjoint sets $A, B \subseteq V(G)$, let $G[A, B]$ be the bipartite subgraph of G with vertex set $A \cup B$ and edge set $\{vw \in E(G) : v \in A, w \in B\}$.

Lemma 9. *Fix integers $s, t, k \geqslant 2$, where k is even. Let G be a graph, and let $V_1, \dots, V_k$ be a partition of $V(G)$, such that $G[V_i]$ has strong separation dimension at most s for each $i \in [k]$, and $G[V_i, V_j]$ has strong separation dimension at most t for all distinct $i, j \in [k]$. Then G has strong separation dimension at most $2s + (k - 1)t + 20 \log k$.*

Proof. Let $G_0 := \bigcup_{i=1}^{k} G[V_i]$. Let H be the complete graph with vertex set $[k]$. Let $E_1, \dots, E_{k-1}$ be a partition of $E(H)$ into perfect matchings, which exists since k is even. For $i \in [k-1]$, let $G_i := \bigcup_{ab \in E_i} G[V_a, V_b]$. Note that $V(G_i) = V(G)$ for $i \in [0, k-1]$, and that $G = G_0 \cup G_1 \cup \cdots \cup G_{k-1}$.

Since $s, t \geqslant 2$, by Lemma 8, G_0 has strong separation dimension at most s, and G_i has strong separation dimension at most t for each $i \in [k-1]$. This gives $s + (k-1)t$ orderings of $V(G)$. Moreover, by Lemma 8, for each of the s orderings of G_0, we have $V_1 < \cdots < V_k$ or $V_k < \cdots < V_1$. For each such ordering of G_0 of the form $V_1 < \cdots < V_k$, add the *extra* ordering $V_k < \cdots < V_1$ to the representation of G. And for each such ordering of G_0 of the form $V_k < \cdots < V_1$, add the *extra* ordering $V_1 < \cdots < V_k$ to the representation of G. In these extra orderings, each set V_i inherits its ordering from the original. (So the extra ordering is not simply the reverse of the original.) This gives $2s + (k-1)t$ orderings of $V(G)$.

For each $i \in [k]$, let $\overrightarrow{V_i}$ be an arbitrary linear ordering of V_i. Let $\overleftarrow{V_i}$ be the reverse ordering. Let H^+ be the complete graph on vertex set $[k]$ including loops. By Lemma 6, for some $p \leqslant 10 \log k$, there is a representation $\{<_1, \dots, <_p\}$ of H^+ such that:

(1) each pair of disjoint edges $e, f \in E(H^+)$ are separated in some $<_i$, and
(2) for every vertex $v \in V(H^+)$ and for all distinct vertices $u, w \in V(H^+) \setminus \{v\}$, for some $i \in [p]$ we have $u <_i v <_i w$ or $w <_i v <_i u$.

For each $i \in [p]$, introduce two orderings $<_i^+$ and $<_i^-$ of $V(G)$ constructed from $<_i$: in the first replace each vertex $i \in V(H^+)$ by $\overrightarrow{V_i}$, and in the second replace each vertex $i \in V(H^+)$ by $\overleftarrow{V_i}$. Together with the previous orderings, this gives a total of at most $2s + (k-1)t + 20 \log k$ orderings of $V(G)$.

We now check that each pair of disjoint edges vw and xy in G are separated in some ordering. Say $v \in V_i$, $w \in V_j$, $x \in V_a$ and $y \in V_b$.

If $i = j$ and $a = b$, then vw and xy are both in G_0, and are thus separated in some ordering arising from G_0. So we may assume that $i \neq j$ or $a \neq b$. Without loss of generality, $i \neq j$.

If $\{i, j\} = \{a, b\}$ then $ij \in E_\ell$ for some $\ell \in [k-1]$, implying vw and xy are both in G_ℓ, and are thus separated in some ordering arising from G_ℓ. So we may assume that $\{i, j\} \neq \{a, b\}$. Thus ij and ab are distinct edges of H^+, where ab is possibly a loop.

If $\{i, j\} \cap \{a, b\} = \emptyset$ then ij and ab are separated in some ordering $<_h$ arising from H^+, implying that vw and xy are also separated (in both $<_h^+$ and $<_h^-$). So we may assume that $\{i, j\} \cap \{a, b\} \neq \emptyset$. Without loss of generality, $i = a$.

First suppose that $a = b$ $(= i)$. Then $xy \in E(G_0)$ and $v \in V(G_0)$. Thus for some ordering $<_\alpha$ of G_0, we have $v <_\alpha x, y$. By construction, $V_j <_\alpha V_i$ or $V_i <_\alpha V_j$. If $V_j <_\alpha V_i$ then $w <_\alpha v <_\alpha x, y$. Otherwise, $V_i <_\alpha V_j$. Then in the extra ordering associated with $<_\alpha$, we have $w < v < x, y$. In both cases, vw and xy are separated.

So we may assume that $a \neq b$. Thus $j \neq b$, as otherwise $\{i, j\} = \{a, b\}$. By property (2) above, for some $r \in [p]$ we have $j <_r i <_r b$ or $b <_r i <_r j$. Without loss of generality, $j <_r i <_r b$. Since $v < x$ in $\overrightarrow{V_i}$ or in $\overleftarrow{V_i}$, in one of $<_r^+$ and $<_r^-$, we have $w < v < x < y$, implying vw and xy are separated.

It remains to show that for every edge $vw \in E(G)$ and vertex $x \in V(G) \setminus \{v, w\}$, we have $x < v, w$ in some ordering and $v, w < x$ in another ordering. Since $vw \in E(G_i)$ for some $i \in [0, k-1]$, and $x \in V(G_i)$, this property holds by assumption. □

We now prove Theorem 1, which says that every graph with maximum degree Δ has separation dimension less than 20Δ. Recall that Chandran et al. [6] proved the upper bound of $2\Delta(\lceil \log_2 \log_2 \Delta \rceil + 3) + 1$, which is less than 20Δ if $\Delta \leqslant 2^{17}$. So it suffices to assume that $\Delta \geqslant 2^{17}$. In this case, to enable an inductive proof, we prove the following strengthening.

Lemma 10. *For $\Delta \geqslant 2^{17}$, every graph with maximum degree at most Δ has strong separation dimension at most $20\Delta(1 - \Delta^{-1/5})$.*

Proof. We proceed by induction on Δ. In the base case, suppose that $2^{17} \leqslant \Delta \leqslant 2^{32}$. Let G be a graph with maximum degree Δ. By Lemma 7 and the result of Chandran et al. [6] mentioned above, the strong separation dimension of G is at most

$$2\Delta(\lceil \log_2 \log_2 \Delta \rceil + 4) + 3 = 18\Delta + 3 \leqslant 20\Delta(1 - \Delta^{-1/5}).$$

So we may assume that $\Delta > 2^{32}$. Let G be a graph with maximum degree Δ. Let k be the largest even integer at most $\Delta^{1/4}$. Let

$$d := (1 + k^{-1})\,\frac{\Delta}{k}.$$

By Lemma 5, there is a partition $V_1, \dots, V_k$ of $V(G)$ such that for every vertex $v \in V(G)$ and integer $i \in [k]$,

$$|N_G(v) \cap V_i| < \frac{\Delta}{k} + \sqrt{\frac{3\Delta \log(4k\Delta^2)}{k}} < d,$$

where the final inequality holds since $k \leqslant \Delta^{1/4}$ and $\Delta > 2^{32}$. Thus $G[V_i]$ and $G[V_i, V_j]$ have maximum degree at most d for all distinct $i, j \in [k]$.

Now $d \geqslant \frac{\Delta}{k} \geqslant \Delta^{3/4} \geqslant 2^{24}$ and $d < \Delta$. By induction, $G[V_i]$ and $G[V_i, V_j]$ both have strong separation dimension at most $20d(1 - d^{-1/5})$ for all distinct $i, j \in [k]$. Since $20d(1 - d^{-1/5}) \geqslant 2$, by Lemma 9, G has strong separation dimension at most $20(k+1)d(1-d^{-1/5}) + 20 \log k$, which is at most $20(k+2)d(1-d^{-1/5})$. All that remains is to prove that

$$(k + 2)d(1 - d^{-1/5}) \leqslant \Delta(1 - \Delta^{-1/5}). \tag{1}$$

Suppose for the sake of contradiction that (1) does not hold. Substituting for d and since $k + 4 \geqslant (k + 2)(1 + k^{-1})$,

$$(k + 4)\,\frac{\Delta}{k}\,(1 - d^{-1/5}) \geqslant (k + 2)(1 + k^{-1})\,\frac{\Delta}{k}\,(1 - d^{-1/5}) > \Delta(1 - \Delta^{-1/5}).$$

Thus

$$(1 + 4k^{-1})(1 - d^{-1/5}) > 1 - \Delta^{-1/5}.$$

Hence

$$4k^{-1} + \Delta^{-1/5} > (1 + 4k^{-1})d^{-1/5} > d^{-1/5}.$$

Since $k \geqslant \frac{4}{5}\Delta^{1/4}$ and $d < \frac{3}{2}\Delta^{3/4}$,

$$5\Delta^{-1/4} + \Delta^{-1/5} > \left(\frac{3}{2}\Delta^{3/4}\right)^{-1/5},$$

which is a contradiction since $\Delta > 2^{32}$. Hence (1) holds, which completes the proof. $\square$

4. Proof of Theorem 2

This section shows that graphs with separation dimension 3 have bounded average degree. Much of the proof works in any dimension, so we present it in general. We include proofs of the following two folklore lemmas for completeness.

Lemma 11. *Every graph with average degree at least $2d$ contains a subgraph with minimum degree at least d.*

Proof. Deleting a vertex of degree less than d maintains the property that the average degree is at least $2d$. Thus, repeatedly deleting vertices of degree less than d produces a subgraph with average degree at least $2d$ and minimum degree at least d. $\square$

Lemma 12. *Every graph with minimum degree at least $2d$ contains a bipartite spanning subgraph with minimum degree at least d.*

Proof. For a partition A, B of $V(G)$, let $e(A, B)$ be the number of edges between A and B. Let A, B be a partition of $V(G)$ maximising $e(A, B)$. If some vertex v in A has fewer than d neighbours in B, then v has more than d neighbours in A, implying that $e(A \setminus \{v\}, B \cup \{v\}) > e(A, B)$, which contradicts the choice of A, B. Thus each vertex in A has at least d neighbours in B, and by symmetry, every vertex in B has at least d neighbours in A. The result follows. $\square$

Let G be a bipartite graph with bipartition (A, B). A representation $\{<_1, \dots, <_d\}$ of G is *consistent* if for every edge $vw \in E(G)$ with $v \in A$ and $w \in B$, we have $v <_i w$ for all $i \in [d]$. A representation $\{<_1, \dots, <_d\}$ of G is *A-homogeneous* if there are integers $a_1, \dots, a_d \in \{-1, +1\}$, such that for every vertex $v \in A$, there is a linear ordering $<_v$ of $N_G(v)$, with the property that for $i \in [d]$,

- if $a_i = 1$ then $N_G(v)$ is ordered in $<_i$ according to $<_v$, and
- if $a_i = -1$ then $N_G(v)$ is ordered in $<_i$ according to $<'_v$,

where $<'_v$ is the reverse of $<_v$. The definition of *B-homogeneous* is analogous.

Lemma 13. *Suppose that for some positive integers d and t, there is a graph G with average degree at least $2^{d+2}(2^{d+1}t)^{2^{d-1}}$ and separation dimension at most d. Then there is a bipartite subgraph G' of G with bipartition (A', B'), with minimum degree at least t, such that G' has a d-dimensional consistent separating representation that is A'-homogeneous or B'-homogeneous.*

Proof. Let $\{<_1, \dots, <_d\}$ be a separating representation of G. By Lemma 12, G contains a bipartite spanning subgraph G_1 with average degree at least $2^{d+1}(2^{d+1}t)^{2^{d-1}}$. Then $\{<_1, \dots, <_d\}$ is a separating representation of G_1. Let (A_1, B_1) be the bipartition of G_1.

For each edge $vw \in E(G_1)$ with $v \in A_1$ and $w \in B_1$, let $f(vw) = (f_1(vw), \dots, f_d(vw))$, where $f_i(vw) := 1$ if $v <_i w$, and $f_i(vw) := -1$ if $w <_i v$ (for $i \in [d]$). Since f takes at most 2^d values, there is a set $E_2 \subseteq E(G_1)$ with $f(vw) = f(xy)$ for all $vw, xy \in E_2$, and $|E_2| \geqslant |E(G_1)|/2^d$. Let G_2 be the spanning subgraph of G_1 with edge set E_2. Thus G_2 has average degree at least $2(2^{d+1}t)^{2^{d-1}}$. For $i \in [d]$, if $f_i(vw) = -1$ for $vw \in E_2$, then replace $<_i$ by $<'_i$. Thus $\{<_1, \dots, <_d\}$ is a consistent separating representation of G_2. This property is maintained for all subgraphs of G_2.

By Lemma 11, G_2 contains a subgraph G_3 with minimum degree at least $(2^{d+1}t)^{2^{d-1}}$. Let $A_3 := A_2 \cap V(G_3)$ and $B_3 := B_2 \cap V(G_3)$. Thus (A_3, B_3) is a bipartition of G_3. Without loss of generality, $|A_3| \geqslant |B_3|$.

For each vertex $v \in A_3$, by the Erdős-Szekeres Theorem [7] applied $d-1$ times, there is a subset M_v of $N_{G_3}(v)$ that is monotone with respect to $<_1$ in each ordering $<_2, \dots, <_d$, and
$$|M_v| \geqslant (\deg_{G_3}(v))^{1/2^{d-1}} \geqslant 2^{d+1}t.$$
Let $g(v) = (g_2(v), \dots, g_d(v))$, where $g_i(v) := 1$ if M_v is forward in $<_i$, and $g_i(v) := -1$ if M_v is backward in $<_i$, for $i \in [2, d]$. Since g takes at most 2^{d-1} values, there is a subset A_4 of A_3 such that $g(v) = g(x)$ for all $v, x \in A_4$, and $|A_4| \geqslant |A_3|/2^{d-1}$. Let $a_1 := 1$ and for $i \in [2, d]$, let $a_i := g_i(v)$ for $v \in A_4$. For $v \in A_4$, let $<_v$ be the ordering of M_v in $<_1$. Let $B_4 := \bigcup_{v \in A_4} M_v$. Let G_4 be the bipartite subgraph with bipartition (A_4, B_4), where $E(G_4) := \{vw : v \in A_4, w \in M_v\}$. By construction, $\{<_1, \dots, <_d\}$ is an A_4-homogeneous consistent separating representation of G_4. This property is maintained for all subgraphs of G_4.

Note that every vertex in A_4 has degree at least $2^{d+1}t$ in G_4, and that
$$|V(G_4)| = |A_4| + |B_4| \leqslant |A_4| + |B_3| \leqslant |A_4| + |A_3| \leqslant (1 + 2^{d-1})|A_4| \leqslant 2^d|A_4|.$$
Hence G_4 has average degree
$$\frac{2|E(G_4)|}{|V(G_4)|} \geqslant \frac{2^{d+1}t|A_4|}{2^d|A_4|} = 2t.$$
By Lemma 11, G_4 contains a subgraph G_5 with minimum degree at least t. Let $A_5 := A_4 \cap V(G_5)$. Then $\{<_1, \dots, <_d\}$ is an A_5-homogeneous consistent separating representation of G_5. $\square$

We now prove Theorem 2.

Lemma 14. *Every graph with separation dimension 3 has average degree less than* 2^{29}.

Proof. Suppose for the sake of contradiction that there is a graph with separation dimension 3 and average degree at least $2^{29} = 2^{3+2}(2^{3+1}4)^{2^{3-1}}$. By Lemma 13, without loss of generality (possibly exchanging the roles of A and B), there is a bipartite graph G with bipartition (A, B), with minimum degree at least 4, such that G has a 3-dimensional A-homogeneous consistent separating representation $\{<_1, <_2, <_3\}$. Thus there are integers $a_1, a_2, a_3 \in \{-1, +1\}$, such that for every vertex $v \in A$, there is a linear ordering $<_v$ of $N_G(v)$, with the property that for $i \in [3]$,

- if $a_i = 1$ then $N_G(v)$ is ordered in $<_i$ according to $<_v$, and
- if $a_i = -1$ then $N_G(v)$ is ordered in $<_i$ according to $<'_v$.

By symmetry (since we may reverse all orders $<_v$), we may assume that at least two of a_1, a_2, a_3 are $+1$. Reordering leaves two cases: $a_1 = a_2 = a_3 = 1$, or $a_1 = a_2 = 1$ and $a_3 = -1$.

Case 1. $a_1 = a_2 = a_3 = 1$: Let v be a vertex in A. Let b, c be neighbours of v with $b <_v c$. Since $a_1 = a_2 = a_3 = 1$, we have $v <_i b <_i c$ for each $i \in [3]$. Let x be a neighbour of b other than v (which exists since G has minimum degree at least 3). Then vc and bx are separated in no ordering, which is a contradiction.

Case 2. $a_1 = a_2 = 1$ and $a_3 = -1$: For each vertex $v \in A$, *mark* the rightmost edge incident with v according to the ordering $<_v$ of $N_G(v)$. Since G has at least $2|V(G)|$ edges and at most $|V(G)|$ edges are marked, G contains a cycle C of unmarked edges. As shown above, C is not a 4-cycle. So $|C| \geqslant 6$.

Let v be the leftmost vertex in C in $<_1$. Let b and c be the neighbours of v in C. Without loss of generality, $b <_v c$. Since $a_1 = a_2 = 1$ and $a_3 = -1$, we have that $v <_1 b <_1 c$ and $v <_2 b <_2 c$ and $v <_3 c <_3 b$. Let w be the neighbour of b in C, such that $w \neq v$. Note that $v, w \in A$ and $b, c \in B$. Since b is between v and c in $<_1$ and $<_2$, the edges vc and wb are not separated in $<_1$ and $<_2$. Thus vc and wb are separated in $<_3$, implying $v <_3 c <_3 w <_3 b$ by consistency. By the choice of v and by consistency, $v <_1 w <_1 b <_1 c$. And by consistency, $v <_2 w <_2 b$ or $w <_2 v <_2 b$.

Let b' be the rightmost neighbour of w in $<_w$. Thus wb' is marked. Since w is between v and b in $<_1$ and $<_3$, the edges vb and wb' are not separated in $<_1$ and $<_3$. Thus vb and wb' are separated in $<_2$. Since $a_2 = +1$ and b' is the rightmost neighbour of w in $<_w$, we have $b <_2 b'$. Thus $v <_2 w <_2 b <_2 b'$ or $w <_2 v <_2 b <_2 b'$. In both cases, vb and wb' are not separated in $<_2$, which is a contradiction. □

Alon et al. [2] state that it is open whether graphs with bounded separation dimension have bounded chromatic number. Since separation dimension is non-decreasing under taking subgraphs, Lemma 14 implies:

Corollary 15. *Every graph with separation dimension 3 is 2^{29}-colourable.*

Recall that Alon et al. [2] proved that every n-vertex graph with separation dimension $s \geqslant 2$ has average degree $O(\log^{s-2} n)$. Their proof is by induction on s. Applying Theorem 2 in the base case leads to the following result:

Corollary 16. *For $s \geqslant 3$, every n-vertex graph with separation dimension s has average degree $O(\log^{s-3} n)$.*

For each $s \geqslant 4$, it remains open whether graphs of separation dimension at most s satisfy analogues of Lemma 14 and Corollary 15.

References

[1] Noga Alon, Manu Basavaraju, L. Sunil Chandran, Rogers Mathew, and Deepak Rajendraprasad. Separation dimension of bounded degree graphs. *SIAM J. Discrete Math.*, 29(1):59–64, 2015. doi: 10.1137/140973013. MR: 3295685.

[2] Noga Alon, Manu Basavaraju, L. Sunil Chandran, Rogers Mathew, and Deepak Rajendraprasad. Separation dimension and sparsity. *J. Graph Theory*, 89(1):14–25, 2018. doi: 10.1002/jgt.22236.

[3] Manu Basavaraju, L. Sunil Chandran, Martin Charles Golumbic, Rogers Mathew, and Deepak Rajendraprasad. Boxicity and separation dimension. In *Graph-theoretic concepts in computer science*, vol. 8747 of *Lecture Notes in Comput. Sci.*, pp. 81–92. Springer, Cham, 2014. doi: 10.1007/978-3-319-12340-0_7. MR: 3295860.

[4] Manu Basavaraju, L. Sunil Chandran, Martin Charles Golumbic, Rogers Mathew, and Deepak Rajendraprasad. Separation dimension of graphs and hypergraphs. *Algorithmica*, 75(1):187–204, 2016. doi: 10.1007/s00453-015-0050-6. MR: 3492062.

[5] Arpitha P. Bharathi, Minati De, and Abhiruk Lahiri. Circular separation dimension of a subclass of planar graphs. *Discrete Math. Theor. Comput. Sci.*, 19(3):#8, 2017. https://dmtcs.episciences.org/4031/.

[6] L. Sunil Chandran, Rogers Mathew, and Naveen Sivadasan. Boxicity of line graphs. *Discrete Math.*, 311(21):2359–2367, 2011. doi: 10.1016/j.disc.2011.06.005. MR: 2832135.

[7] Paul Erdős and George Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935. http://www.numdam.org/item?id=CM_1935__2__463_0.

[8] Paul Erdős and László Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and Finite Sets*, vol. 10 of *Colloq. Math. Soc. János Bolyai*, pp. 609–627. North-Holland, 1975. https://www.renyi.hu/~p_erdos/1975-34.pdf. MR: 0382050.

[9] Zoltán Füredi and Jeff Kahn. On the dimensions of ordered sets of bounded degree. *Order*, 3(1):15–20, 1986. doi: 10.1007/BF00403406. MR: 850394.

[10] Hugh Hind, Michael Molloy, and Bruce Reed. Colouring a graph frugally. *Combinatorica*, 17(4):469–482, 1997. doi: 10.1007/BF01195001. MR: 1645682.

[11] Ross J. Kang and Tobias Müller. Frugal, acyclic and star colourings of graphs. *Discrete Appl. Math.*, 159(16):1806–1814, 2011. doi: 10.1016/j.dam.2010.05.008.

[12] Sarah J. Loeb and Douglas B. West. Fractional and circular separation dimension of graphs. *European J. Combin.*, 69:19–35, 2018. doi: 10.1016/j.ejc.2017.09.001.

[13] Michael Mitzenmacher and Eli Upfal. *Probability and computing.* Cambridge University Press, 2005. doi: 10.1017/CBO9780511813603.

[14] Michael Molloy and Bruce Reed. Asymptotically optimal frugal colouring. *J. Combin. Theory Ser. B*, 100(2):226–246, 2010. doi: 10.1016/j.jctb.2009.07.002.

[15] Alex Scott and David R. Wood. Better bounds for poset dimension and boxicity, 2018. arXiv: 1804.03271.

[16] Emile Ziedan, Deepak Rajendraprasad, Rogers Mathew, Martin Charles Golumbic, and Jérémie Dusart. The induced separation dimension of a graph. *Algorithmica*, 80(10):2834–2848, 2018. doi: 10.1007/s00453-017-0353-x.