SEPARATION DIMENSION AND DEGREE

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Abstract. The separation dimension of a graph $G$ is the minimum positive integer $d$ for which there is an embedding of $G$ into $\mathbb{R}^d$, such that every pair of disjoint edges are separated by some axis-parallel hyperplane. We prove a conjecture of Alon et al. [SIAM J. Discrete Math. 2015] by showing that every graph with maximum degree $\Delta$ has separation dimension less than $20\Delta$, which is best possible up to a constant factor. We also prove that graphs with separation dimension 3 have bounded average degree and bounded chromatic number, partially resolving an open problem by Alon et al. [J. Graph Theory 2018].

1. Introduction

This paper studies the separation dimension of graphs and its relationship with maximum and average degree. For a graph $G$, a function $f : V(G) \rightarrow \mathbb{R}^d$ is separating if for all disjoint edges $vw, xy \in E(G)$ there is an axis-parallel hyperplane that separates the pair of points $\{f(v), f(w)\}$ from the pair $\{f(x), f(y)\}$. The separation dimension of a graph $G$ is the minimum positive integer $d$ for which there is a $d$-dimensional separating function for $G$; see [1–5, 12, 16] for recent work on the separation dimension of graphs.

This topic can also be thought of more combinatorially. Edges $e$ and $f$ in a graph $G$ are separated in a linear ordering of $V(G)$ if both endpoints of $e$ appear before both endpoints of $f$, or both endpoints of $f$ appear before both endpoints of $e$. A representation of $G$ is a non-empty set of linear orderings of $V(G)$. A representation $\mathcal{R}$ of $G$ is separating if every pair of disjoint edges in $G$ are separated in at least one ordering in $\mathcal{R}$. It is easily seen that the separation dimension of $G$ equals the minimum size of a separating representation of $G$; see [1–4, 6].

A fundamental question is the relationship between separation dimension and maximum degree. Chandran et al. [6] proved that every graph with maximum degree $\Delta$ has separation dimension at most $2\Delta(\lceil \log_2 \log_2 \Delta \rceil + 3) + 1$. Alon et al. [1] improved this bound to $2^{9 \log^* (\Delta)} \Delta$, and conjectured that a stronger $O(\Delta)$ bound should hold. We prove this conjecture.

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Theorem 1. Every graph with maximum degree $\Delta \geq 1$ has separation dimension less than $20\Delta$.

This linear bound is best possible up to a constant factor, since Alon et al. [1] proved that almost every $\Delta$-regular graph has separation dimension at least $\frac{\Delta}{2}$. Theorem 1 is proved in Section 3.

Section 4 of this paper considers the following natural extremal question, first posed by Alon et al. [2]: What is the maximum average degree of an $n$-vertex graph with separation dimension $s$? Every graph with separation dimension at most 2 is planar, and thus has average degree less than 6. For $s \geq 3$, Alon et al. [2] proved the best known upper bound on the average degree of $O(\log^{s-2} n)$, and asked whether graphs with bounded separation dimension have bounded degeneracy (or equivalently, bounded average degree). We answer the first open case of this problem.

Theorem 2. There is a constant $c$ such that every graph with separation dimension 3 has average degree at most $c$.

2. A Colouring Lemma

This section proves a straightforward lemma that shows how to colour a graph so that each vertex has few neighbours of each colour (Lemma 5). Several previous papers have proved similar results [6, 9–11, 14, 15]. The proof depends on the following two standard probabilistic tools. Let $[k] := \{1, 2, \ldots, k\}$.

Lemma 3 (Lovász Local Lemma [8]). Let $E_1, \ldots, E_n$ be events in a probability space, each with probability at most $p$ and mutually independent of all but at most $D$ other events. If $4pD \leq 1$ then with positive probability, none of $E_1, \ldots, E_n$ occur.

Lemma 4 (Chernoff Bound [13]). Let $X_1, \ldots, X_n$ be independent random variables, where $X_i = 1$ with probability $p$ and $X_i = 0$ with probability $1 - p$. Let $X := \sum_{i=1}^n X_i$. Then for $\delta > 0$,

$$P(X \geq (1 + \delta)pn) \leq e^{-\delta^2 pn/3}.$$ 

Lemma 5. For all positive integers $k$ and $\Delta$, for every graph $G$ with maximum degree at most $\Delta$, there is a partition $V_1, \ldots, V_k$ of $V(G)$ such that for every vertex $v \in V(G)$ and integer $i \in [k]$,

$$|N_G(v) \cap V_i| < d := \frac{\Delta}{k} + \sqrt{\frac{3\Delta \log(4k\Delta^2)}{k}}.$$ 

Proof. Independently and randomly colour each vertex with one of $k$ colours. For each vertex $v \in V(G)$ and colour $c$, let $A_{v,c}$ be the event that at least $d$ neighbours of $v$ are all assigned colour $c$. Each event is mutually independent of all but at most $k\Delta^2$ other events.

We now prove that $P(A_{v,c}) \leq (4k\Delta^2)^{-1}$. Since $P(A_{v,c})$ is increasing with $\deg(v)$, we may assume that $\deg(v) = \Delta$. Say $w_1, \ldots, w_\Delta$ are the neighbours of $v$. For $i \in [\Delta]$, let $X_i := 1$ if $w_i$ is coloured $c$, otherwise let $X_i := 0$. Then $P(X_i) = p := \frac{1}{k}$. Let
\[ X := \sum_{i=1}^{\Delta} X_i. \] Then \( A_{v,c} \) holds if and only if \( X \geq d \). Let \( \delta := \frac{d k}{\Delta} - 1 \), so \( d = (1 + \delta)p\Delta \).

Then \( \mathbb{P}(A_{v,c}) = \mathbb{P}(X \geq d) = \mathbb{P}(X \geq (1 + \delta)p\Delta) \). Now

\[
\frac{\delta^2 p\Delta}{3} = \frac{1}{3} \left( \frac{dk}{\Delta} - 1 \right)^2 p\Delta = \log(4k\Delta^2).
\]

By Lemma 4 with \( n = \Delta \),

\[ \mathbb{P}(A_{v,c}) \leq e^{-\delta^2 p\Delta/3} = (4k\Delta^2)^{-1}, \]

as claimed. By Lemma 3, with positive probability no event occurs, implying the desired partition exists. \( \square \)

3. Proof of Theorem 1

Our proof works by considering sets of orderings with stronger properties than separation. We start with a lemma about complete graphs.

\textbf{Lemma 6.} Let \( G \) be the complete graph on \( n \) vertices including loops. Then for some integer \( p \leq 10 \log n \), there are linear orderings \( \prec_1, \ldots, \prec_p \) of \( V(G) \), such that:

(1) every pair of disjoint edges \( e, f \in E(G) \) are separated in some \( \prec_i \), and

(2) for every vertex \( v \in V(G) \) and distinct vertices \( u, w \in V(G) \setminus \{v\} \), for some \( i \in [p] \) we have \( u \prec_i v \prec_i w \) or \( w \prec_i v \prec_i u \).

\textbf{Proof.} Let \( p := \lfloor 10 \log n \rfloor \). For \( i \in [p] \), let \( \prec_i \) be a random linear ordering of \( V(G) \).

Let \( e \) and \( f \) be edges in \( G \) with no common endpoint. If neither \( e \) nor \( f \) are loops, then the probability that \( e \) and \( f \) are separated in \( \prec_i \) is \( \frac{1}{3} \). If \( e \) is a loop and \( f \) is a non-loop, then the probability that \( e \) and \( f \) are separated in \( \prec_i \) is \( \frac{2}{3} \). If both \( e \) and \( f \) are loops, then they are always separated in \( \prec_i \). Thus the probability that \( e \) and \( f \) are not separated in \( \prec_i \) is at most \( \frac{2}{3} \). Hence the probability that (1) fails for \( e \) and \( f \) is at most \( \left( \frac{2}{3} \right)^p \).

Now consider a vertex \( v \in V(G) \) and distinct vertices \( u, w \in V(G) \setminus \{v\} \). For each \( i \in [p] \) the probability that \( u \prec_i v \prec_i w \) or \( w \prec_i v \prec_i u \) is \( \frac{1}{3} \). Hence the probability (2) fails for every \( i \in [p] \) is at most \( \left( \frac{2}{3} \right)^p \).

By the union bound, the probability that both (1) and (2) fail is at most \( \left( \binom{|E(G)|}{2} \right) \left( \frac{2}{3} \right)^p + n \left( \begin{array}{c} n-2 \\ 2 \end{array} \right) \left( \frac{2}{3} \right)^p = \left( \begin{array}{c} n(n+1)/2 \\ 2 \end{array} \right) + n \left( \begin{array}{c} n-2 \\ 2 \end{array} \right) \left( \frac{2}{3} \right)^p < n^4 \left( \frac{2}{3} \right)^p < 1 \). Thus there exists linear orderings \( \prec_1, \ldots, \prec_p \) such that (1) and (2) hold. \( \square \)

Note that we need \( \Omega(\log n) \) orderings in Lemma 6 because of (2): if \( p < \log_2(n-1)-1 \) then for any vertex \( v \) and any set of \( p \) orderings, there are distinct vertices \( x, y \) are on the same side of \( v \) in each of the orderings.

The following definition is a key to the proof of Theorem 1. A representation \( \prec_1, \ldots, \prec_p \) of a graph \( G \) is \textit{strongly separating} if:

(a) for all disjoint edges \( vw, xy \in E(G) \), for some ordering \( \prec_i \), we have \( v, w \prec_i x, y \) or \( x, y \prec_i v, w \), and

(b) for every edge \( vw \in E(G) \) and vertex \( x \in V(G) \setminus \{v, w\} \), we have \( x \prec_i v, w \) and \( v, w \prec_j x \) for some \( i, j \in [p] \).
We define the strong separation dimension of a graph $G$ to be the minimum number of linear orderings in a strongly separating representation of $G$. Clearly the separation dimension of a graph is at most its strong separation dimension, and it will be helpful to work with the latter.

**Lemma 7.** Every graph $G$ with maximum degree $\Delta$ has strong separation dimension at most the separation dimension of $G$ plus $2\Delta + 2$.

*Proof.* Say $G$ has separation dimension $d$. By Vizing’s Theorem, there is a partition $E_1, \ldots, E_{\Delta + 1}$ of $E(G)$ into matchings. Starting from a separating representation of $G$ in $d$ dimensions, we now add two orderings $<_i$ and $<_i'$ for each $i \in [\Delta + 1]$. Say $E_i = \{v_1w_1, \ldots, v_nw_n\}$. Let $<_i v_1, w_1, \ldots, v_nw_n$ followed by $V(G) \setminus \{v_1, w_1, \ldots, v_n, w_n\}$ in any ordering. Let $<_i'$ be the reverse of $<_i$. Every edge $vw$ of $G$ is in some $E_i$. Since $v$ and $w$ are consecutive in $<_i$, for each vertex $x \in V(G) \setminus \{v, w\}$, we have $v, w < x$ and $x < _i v, w$, or $v, w < _i' x$ and $x < _i v, w$. Hence we have a strongly separating representation of $G$ with $d + 2\Delta + 2$ orderings in total. □

**Lemma 8.** Let $G_1, \ldots, G_k$ be the connected components of a graph $G$. For $a \in [k]$, let $p_a$ be the strong separation dimension of $G_a$. Then $G$ has strong separation dimension at most $\max\{p_1, \ldots, p_k, 2\}$. Moreover, there is such a representation such that in each ordering, $V(G_1) < V(G_2) < \cdots < V(G_k)$ or $V(G_k) < V(G_{k-1}) < \cdots < V(G_1)$.

*Proof.* Let $p := \max\{p_1, \ldots, p_k, 2\}$. For $a \in [k]$, let $\{<_1^a, \ldots, <_p^a\}$ be a strongly separating representation of $G_a$. For $j \in [p - 1]$, let $<_j$ be the ordering of $V(G)$ with $V(G_1) <_j \cdots <_j V(G_k)$, where $V(G_a)$ is internally ordered according to $<_j^a$, for $a \in [k]$. Finally, let $<_p$ be the ordering of $V(G)$ with $V(G_k) <_p \cdots <_p V(G_1)$, where $V(G_a)$ is internally ordered according to $<_p^a$, for $a \in [k]$. Thus $\{<_1, \ldots, <_p\}$ is a representation of $G$, which we now show is strongly separating. Consider disjoint edges $vw, xy \in E(G)$. If $vw$ and $xy$ are in the same component, then (a) holds by assumption. Otherwise, $vw$ and $xy$ are in distinct components, implying that $v, w < x, y$ or $x, y < v, w$, and again (a) holds. Now consider an edge $vw \in E(G)$ and vertex $x \in V(G) \setminus \{v, w\}$. If $vw$ and $x$ are in the same component, then (b) holds by assumption. So we may assume that $vw \in E(G_a)$ and $x \in V(G_b)$ for distinct $a, b \in [k]$. If $a < b$ then $v, w < x$ and $x < v, w$. If $b < a$ then $v, w < v, x$ and $x < v, w$. Thus (b) holds, and $\{<_1, \ldots, <_p\}$ is strongly separating. □

Note that every connected graph with at least three vertices has strong separation dimension at least 2, so Lemma 8 implies that for every graph $G$ with at least three vertices in some component, the strong separation dimension of $G$ equals the maximum strong separation dimension of the components of $G$.

For a graph $G$ and disjoint sets $A, B \subseteq V(G)$, let $G[A, B]$ be the bipartite subgraph of $G$ with vertex set $A \cup B$ and edge set $\{vw \in E(G) : v \in A, w \in B\}$.

**Lemma 9.** Fix integers $s, t, k \geq 2$, where $k$ is even. Let $G$ be a graph, and let $V_1, \ldots, V_k$ be a partition of $V(G)$, such that $G[V_i]$ has strong separation dimension at most $s$ for each $i \in [k]$, and $G[V_i, V_j]$ has strong separation dimension at most $t$ for all distinct $i, j \in [k]$. Then $G$ has strong separation dimension at most $2s + (k - 1)t + 20 \log k$. 
Proof. Let $G_0 := \bigcup_{i=1}^k G[V_i]$. Let $H$ be the complete graph with vertex set $[k]$. Let $E_1, \ldots, E_{k-1}$ be a partition of $E(H)$ into perfect matchings, which exists since $k$ is even. For $i \in [k-1]$, let $G_i := \bigcup_{a \in E_i} G[V_a, V_b]$. Note that $V(G_i) = V(G)$ for $i \in [0, k-1]$, and that $G = G_0 \cup G_1 \cup \cdots \cup G_{k-1}$.

Since $s, t \geq 2$, by Lemma 8, $G_0$ has strong separation dimension at most $s$, and $G_1$ has strong separation dimension at most $t$ for each $i \in [k-1]$. This gives $s + (k-1)t$ orderings of $V(G)$. Moreover, by Lemma 8, for each of the $s$ orderings of $G_0$, we have $V_1 \subset \cdots \subset V_k$ or $V_k \subset \cdots \subset V_1$. For each such ordering of $G_0$ of the form $V_1 \subset \cdots \subset V_k$, add the extra ordering $V_k \subset \cdots \subset V_1$ to the representation of $G$. And for each such ordering of $G_0$ of the form $V_k \subset \cdots \subset V_1$, add the extra ordering $V_1 \subset \cdots \subset V_k$ to the representation of $G$. In these extra orderings, each set $V_i$ inherits its ordering from the original. (So the extra ordering is not simply the reverse of the original.) This gives $2s + (k-1)t$ orderings of $V(G)$.

For each $i \in [k]$, let $\overrightarrow{V_i}$ be an arbitrary linear ordering of $V_i$. Let $\overleftarrow{V_i}$ be the reverse ordering. Let $H^+$ be the complete graph on vertex set $[k]$ including loops. By Lemma 6, for some $p \leq 10 \log k$, there is a representation $\{<_1, \ldots, <_p\}$ of $H^+$ such that:

1. each pair of disjoint edges $e, f \in E(H^+)$ are separated in some $<_i$, and
2. for every vertex $v \in V(H^+)$ and for all distinct vertices $u, w \in V(H^+) \setminus \{v\}$, for some $i \in [p]$ we have $u <_i v <_i w$ or $w <_i v <_i u$.

For each $i \in [p]$, introduce two orderings $<_i^+$ and $<_i^-$ of $V(G_i)$ constructed from $<_i$: in the first replace each vertex $i \in V(H^+) \setminus \overrightarrow{V_i}$, and in the second replace each vertex $i \in V(H^+)$ by $\overrightarrow{V_i}$. Together with the previous orderings, this gives a total of at most $2s + (k-1)t + 20 \log k$ orderings of $V(G)$.

We now check that each pair of disjoint edges $vw$ and $xy$ in $G$ are separated in some ordering. Say $v \in V_i$, $w \in V_j$, $x \in V_a$ and $y \in V_b$.

If $i = j$ and $a = b$, then $vw$ and $xy$ are both in $G_0$, and are thus separated in some ordering arising from $G_0$. So we may assume that $i \neq j$ or $a \neq b$. Without loss of generality, $i \neq j$.

If $\{i, j\} = \{a, b\}$ then $ij \in E_\ell$ for some $\ell \in [k-1]$, implying $vw$ and $xy$ are both in $G_\ell$, and are thus separated in some ordering arising from $G_\ell$. So we may assume that $\{i, j\} \neq \{a, b\}$. Thus $ij$ and $ab$ are distinct edges of $H^+$, where $ab$ is possibly a loop.

If $\{i, j\} \cap \{a, b\} = \emptyset$ then $ij$ and $ab$ are separated in some ordering $<_h$ arising from $H^+$, implying that $vw$ and $xy$ are also separated (in both $<_h^+$ and $<_h^-$). So we may assume that $\{i, j\} \cap \{a, b\} \neq \emptyset$. Without loss of generality, $i = a$.

First suppose that $a = b$ (= $i$). Then $xy \in E(G_0)$ and $v \in V(G_0)$. Thus for some ordering $<_a$ of $G_0$, we have $v <_a x, y$. By construction, $V_j <_a V_i$ or $V_i <_a V_j$. If $V_j \subset V_i$ then $w <_a v <_a x, y$. Otherwise, $V_i \subset V_j$. Then in the extra ordering associated with $<_a$, we have $w < v < x, y$. In both cases, $vw$ and $xy$ are separated.

So we may assume that $a \neq b$. Thus $j \neq b$, as otherwise $\{i, j\} = \{a, b\}$. By property (2) above, for some $r \in [p]$ we have $j <_r i <_r b$ or $b <_r i <_r j$. Without loss of generality, $j <_r i <_r b$. Since $v < x$ in $\overrightarrow{V_i}$ or in $\overleftarrow{V_i}$, in one of $<_r^+$ and $<_r^-$, we have $w < v < x, y$, implying $vw$ and $xy$ are separated.
It remains to show that for every edge $vw \in E(G)$ and vertex $x \in V(G) \setminus \{v,w\}$, we have $x < v, w$ in some ordering and $v, w < x$ in another ordering. Since $vw \in E(G_i)$ for some $i \in \{0, k - 1\}$, and $x \in V(G_i)$, this property holds by assumption. □

We now prove Theorem 1, which says that every graph with maximum degree $\Delta$ has separation dimension less than $20\Delta$. Recall that Chandran et al. [6] proved the upper bound of $2\Delta(\lceil \log_2 \log_2 \Delta \rceil + 3) + 1$, which is less than $20\Delta$ if $\Delta \leq 2^{17}$. So it suffices to assume that $\Delta \geq 2^{17}$. In this case, to enable an inductive proof, we prove the following strengthening.

Lemma 10. For $\Delta \geq 2^{17}$, every graph with maximum degree at most $\Delta$ has strong separation dimension at most $20\Delta(1 - \Delta^{-1/5})$.

Proof. We proceed by induction on $\Delta$. In the base case, suppose that $2^{17} \leq \Delta \leq 2^{32}$. Let $G$ be a graph with maximum degree $\Delta$. By Lemma 7 and the result of Chandran et al. [6] mentioned above, the strong separation dimension of $G$ is at most

$$2\Delta(\lceil \log_2 \log_2 \Delta \rceil + 4) + 3 = 18\Delta + 3 \leq 20\Delta(1 - \Delta^{-1/5}).$$

So we may assume that $\Delta > 2^{32}$. Let $G$ be a graph with maximum degree $\Delta$. Let $k$ be the largest even integer at most $\Delta^{1/4}$. Let

$$d := (1 + k^{-1}) \frac{\Delta}{k}.$$

By Lemma 5, there is a partition $V_1, \ldots, V_k$ of $V(G)$ such that for every vertex $v \in V(G)$ and integer $i \in [k],$

$$|N_G(v) \cap V_i| < \frac{\Delta}{k} + \sqrt{\frac{3\Delta \log(4k\Delta^2)}{k}} < d,$$

where the final inequality holds since $k \leq \Delta^{1/4}$ and $\Delta > 2^{32}$. Thus $G[V_i]$ and $G[V_i, V_j]$ have maximum degree at most $d$ for all distinct $i, j \in [k]$.

Now $d \geq \frac{\Delta}{k} \geq \Delta^{3/4} \geq 2^{24}$ and $d < \Delta$. By induction, $G[V_i]$ and $G[V_i, V_j]$ both have strong separation dimension at most $20d(1 - d^{-1/5})$ for all distinct $i, j \in [k]$. Since $20d(1 - d^{-1/5}) \geq 2$, by Lemma 9, $G$ has strong separation dimension at most $20(k + 1)d(1 - d^{-1/5}) + 20 \log k$, which is at most $20(k + 2)d(1 - d^{-1/5})$. All that remains is to prove that

$$(k + 2)d(1 - d^{-1/5}) \leq \Delta(1 - \Delta^{-1/5}). \quad (1)$$

Suppose for the sake of contradiction that (1) does not hold. Substituting for $d$ and since $k + 4 \geq (k + 2)(1 + k^{-1}),$

$$(k + 4) \frac{\Delta}{k} (1 - d^{-1/5}) \geq (k + 2)(1 + k^{-1}) \frac{\Delta}{k} (1 - d^{-1/5}) > \Delta(1 - \Delta^{-1/5}).$$

Thus

$$(1 + 4k^{-1})(1 - d^{-1/5}) > 1 - \Delta^{-1/5}.$$  \tag{1}

Hence

$$4k^{-1} + \Delta^{-1/5} > (1 + 4k^{-1})d^{-1/5} > d^{-1/5}.$$
Since \( k \geq \frac{1}{5} \Delta^{1/4} \) and \( d < \frac{3}{2} \Delta^{3/4} \),

\[
5 \Delta^{-1/4} + \Delta^{-1/5} > \left( \frac{3}{2} \Delta^{3/4} \right)^{-1/5},
\]

which is a contradiction since \( \Delta > 2^{32} \). Hence (1) holds, which completes the proof. \( \square \)

4. Proof of Theorem 2

This section shows that graphs with separation dimension 3 have bounded average degree. Much of the proof works in any dimension, so we present it in general. We include proofs of the following two folklore lemmas for completeness.

**Lemma 11.** Every graph with average degree at least \( 2d \) contains a subgraph with minimum degree at least \( d \).

**Proof.** Deleting a vertex of degree less than \( d \) maintains the property that the average degree is at least \( 2d \). Thus, repeatedly deleting vertices of degree less than \( d \) produces a subgraph with average degree at least \( 2d \) and minimum degree at least \( d \). \( \square \)

**Lemma 12.** Every graph with minimum degree at least \( 2d \) contains a bipartite spanning subgraph with minimum degree at least \( d \).

**Proof.** For a partition \( A, B \) of \( V(G) \), let \( e(A, B) \) be the number of edges between \( A \) and \( B \). Let \( A, B \) be a partition of \( V(G) \) maximising \( e(A, B) \). If some vertex \( v \) in \( A \) has fewer than \( d \) neighbours in \( B \), then \( v \) has more than \( d \) neighbours in \( A \), implying that \( e(A \setminus \{v\}, B \cup \{v\}) > e(A, B) \), which contradicts the choice of \( A, B \). Thus each vertex in \( A \) has at least \( d \) neighbours in \( B \), and by symmetry, every vertex in \( B \) has at least \( d \) neighbours in \( A \). The result follows. \( \square \)

Let \( G \) be a bipartite graph with bipartition \((A, B)\). A representation \( \{<1, \ldots, <d\} \) of \( G \) is **consistent** if for every edge \( vw \in E(G) \) with \( v \in A \) and \( w \in B \), we have \( v <_i w \) for all \( i \in [d] \). A representation \( \{<1, \ldots, <d\} \) of \( G \) is \( A \)-homogeneous if there are integers \( a_1, \ldots, a_d \in \{-1, +1\} \), such that for every vertex \( v \in A \), there is a linear ordering \( <_v \) of \( N_G(v) \), with the property that for \( i \in [d] \),

- if \( a_i = 1 \) then \( N_G(v) \) is ordered in \( <_i \) according to \( <_v \), and
- if \( a_i = -1 \) then \( N_G(v) \) is ordered in \( <_i \) according to \( <'_v \),

where \( <'_v \) is the reverse of \( <_v \). The definition of \( B \)-homogeneous is analogous.

**Lemma 13.** Suppose that for some positive integers \( d \) and \( t \), there is a graph \( G \) with average degree at least \( 2^{d+2}(2^{d+1}t)^{2d-1} \) and separation dimension at most \( d \). Then there is a bipartite subgraph \( G' \) of \( G \) with bipartition \((A', B')\), with minimum degree at least \( t \), such that \( G' \) has a \( d \)-dimensional consistent separating representation that is \( A' \)-homogeneous or \( B' \)-homogeneous.

**Proof.** Let \( \{<1, \ldots, <d\} \) be a separating representation of \( G \). By Lemma 12, \( G \) contains a bipartite spanning subgraph \( G_1 \) with average degree at least \( 2^{d+1}(2^{d+1}t)^{2d-1} \). Then \( \{<1, \ldots, <d\} \) is a separating representation of \( G_1 \). Let \((A_1, B_1)\) be the bipartition of \( G_1 \).
For each edge \( vw \in E(G_1) \) with \( v \in A_1 \) and \( w \in B_1 \), let \( f(vw) = (f_1(vw), \ldots, f_d(vw)) \), where \( f_i(vw) := 1 \) if \( v <_i w \), and \( f_i(vw) := -1 \) if \( w <_i v \) (for \( i \in [d] \)). Since \( f \) takes at most \( 2^d \) values, there is a set \( E_2 \subseteq E(G_1) \) with \( f(vw) = f(xy) \) for all \( vw, xy \in E_2 \), and \( |E_2| \geq |E(G_1)|/2^d \). Let \( G_2 \) be the spanning subgraph of \( G_1 \) with edge set \( E_2 \). Thus \( G_2 \) has average degree at least \( 2(2^{d+1}t)^{2d-1} \). For \( i \in [d] \), if \( f_i(vw) = -1 \) for \( vw \in E_2 \), then replace \( <_i \) by \( <'_i \). Thus \( \langle 1, \ldots, <_d \rangle \) is a consistent separating representation of \( G_2 \). This property is maintained for all subgraphs of \( G_2 \).

By Lemma 11, \( G_2 \) contains a subgraph \( G_3 \) with minimum degree at least \( (2^{d+1}t)^{2d-1} \). Let \( A_3 := A_2 \cap V(G_3) \) and \( B_3 := B_2 \cap V(G_3) \). Thus \( (A_3, B_3) \) is a bipartition of \( G_3 \). Without loss of generality, \( |A_3| \geq |B_3| \).

For each vertex \( v \in A_3 \), by the Erdős-Szekeres Theorem [7] applied \( d - 1 \) times, there is a subset \( M_v \) of \( N_{G_3}(v) \) that is monotone with respect to \( <_1 \) in each ordering \( <_2, \ldots, <_d \), and
\[
|M_v| \geq (\deg_{G_3}(v))^{1/2d-1} \geq 2^{d+1}t.
\]

Let \( g(v) = (g_2(v), \ldots, g_d(v)) \), where \( g_i(v) := 1 \) if \( M_v \) is forward in \( <_i \), and \( g_i(v) := -1 \) if \( M_v \) is backward in \( <_i \), for \( i \in [2, d] \). Since \( g \) takes at most \( 2^{d-1} \) values, there is a subset \( A_4 \) of \( A_3 \) such that \( g(v) = g(x) \) for all \( v, x \in A_4 \), and \( |A_4| \geq |A_3|/2^{d-1} \). Let \( A_4 := 1 \) and for \( i \in [2, d] \), let \( g_i(v) := g_i(v) \) for \( v \in A_4 \). For \( v \in A_4 \), let \( <_v \) be the ordering of \( M_v \) in \( <_1 \). Let \( B_4 := \bigcup_{v \in A_4} M_v \). Let \( G_4 \) be the bipartite subgraph with bipartition \( (A_4, B_4) \), where \( E(G_4) := \{vw : v \in A_4, w \in M_v\} \). By construction, \( \langle 1, \ldots, <_d \rangle \) is an \( A_4 \)-homogeneous consistent separating representation of \( G_4 \). This property is maintained for all subgraphs of \( G_4 \).

Note that every vertex in \( A_4 \) has degree at least \( 2^{d+1}t \) in \( G_4 \), and that
\[
|V(G_4)| = |A_4| + |B_4| \leq |A_4| + |B_3| \leq |A_4| + |A_3| \leq (1 + 2^{d-1})|A_4| \leq 2^d|A_4|.
\]
Hence \( G_4 \) has average degree
\[
\frac{2|E(G_4)|}{|V(G_4)|} \geq \frac{2^{d+1}t|A_4|}{2^d|A_4|} = 2t.
\]
By Lemma 11, \( G_4 \) contains a subgraph \( G_5 \) with minimum degree at least \( t \). Let \( A_5 := A_4 \cap V(G_5) \). Then \( \langle 1, \ldots, <_d \rangle \) is an \( A_5 \)-homogeneous consistent separating representation of \( G_5 \).

We now prove Theorem 2.

**Lemma 14.** Every graph with separation dimension 3 has average degree less than \( 2^{29} \).

**Proof.** Suppose for the sake of contradiction that there is a graph with separation dimension 3 and average degree at least \( 2^{29} = 2^{3+2(2^3+1)}2^{3-1} \). By Lemma 13, without loss of generality (possibly exchanging the roles of \( A \) and \( B \)), there is a bipartite graph \( G \) with bipartition \( (A, B) \), with minimum degree at least 4, such that \( G \) has a 3-dimensional \( A \)-homogeneous consistent separating representation \( \langle 1, <_2, <_3 \rangle \). Thus there are integers \( a_1, a_2, a_3 \in \{-1, +1\} \), such that for every vertex \( v \in A \), there is a linear ordering \( <_v \) of \( N_G(v) \), with the property that for \( i \in [3] \),
- if \( a_i = 1 \) then \( N_G(v) \) is ordered in \( <_i \) according to \( <_v \), and
- if \( a_i = -1 \) then \( N_G(v) \) is ordered in \( <_i \) according to \( <'_v \).
By symmetry (since we may reverse all orders $<_v$), we may assume that at least two of $a_1, a_2, a_3$ are $+1$. Reordering leaves two cases: $a_1 = a_2 = a_3 = 1$, or $a_1 = a_2 = 1$ and $a_3 = -1$.

Case 1. $a_1 = a_2 = a_3 = 1$: Let $v$ be a vertex in $A$. Let $b, c$ be neighbours of $v$ with $b <_v c$. Since $a_1 = a_2 = a_3 = 1$, we have $v <_i b <_i c$ for each $i \in [3]$. Let $x$ be a neighbour of $b$ other than $v$ (which exists since $G$ has minimum degree at least 3).

Then $vc$ and $bx$ are separated in no ordering, which is a contradiction.

Case 2. $a_1 = a_2 = 1$ and $a_3 = -1$: For each vertex $v \in A$, mark the rightmost edge incident with $v$ according to the ordering $<_v$ of $N_G(v)$. Since $G$ has at least $2|V(G)|$ edges and at most $|V(G)|$ edges are marked, $G$ contains a cycle $C$ of unmarked edges.

As shown above, $C$ is not a 4-cycle. So $|C| \geq 6$.

Let $v$ be the leftmost vertex in $C$ in $<_1$. Let $b$ and $c$ be the neighbours of $v$ in $C$. Without loss of generality, $b <_v c$. Since $a_1 = a_2 = 1$ and $a_3 = -1$, we have that $v <_1 b <_1 c$ and $v <_2 b <_2 c$ and $v <_3 c <_3 b$. Let $w$ be the neighbour of $b$ in $C$, such that $w \neq v$. Note that $v, w \in A$ and $b, c \in B$. Since $b$ is between $v$ and $c$ in $<_1$ and $<_2$, the edges $vc$ and $wb$ are not separated in $<_1$ and $<_2$. Thus $vc$ and $wb$ are separated in $<_3$, implying $v <_3 c <_3 w <_3 b$ by consistency. By the choice of $v$ and by consistency, $v <_1 w <_1 b <_1 c$. And by consistency, $v <_2 w <_2 b$ or $w <_2 v <_2 b$.

Let $b'$ be the rightmost neighbour of $w$ in $<_w$. Thus $wb'$ is marked. Since $w$ is between $v$ and $b$ in $<_1$ and $<_3$, the edges $vb$ and $wb'$ are not separated in $<_1$ and $<_3$. Thus $vb$ and $wb'$ are separated in $<_2$. Since $a_2 = +1$ and $b'$ is the rightmost neighbour of $w$ in $<_w$, we have $b <_2 b'$. Thus $v <_2 w <_2 b <_2 b'$ or $w <_2 v <_2 b <_2 b'$. In both cases, $vb$ and $wb'$ are not separated in $<_2$, which is a contradiction. □

Alon et al. [2] state that it is open whether graphs with bounded separation dimension have bounded chromatic number. Since separation dimension is non-decreasing under taking subgraphs, Lemma 14 implies:

**Corollary 15.** Every graph with separation dimension 3 is $2^{20}$-colourable.

Recall that Alon et al. [2] proved that every $n$-vertex graph with separation dimension $s \geq 2$ has average degree $O(\log^{s-2} n)$. Their proof is by induction on $s$. Applying Theorem 2 in the base case leads to the following result:

**Corollary 16.** For $s \geq 3$, every $n$-vertex graph with separation dimension $s$ has average degree $O(\log^{s-3} n)$.

For each $s \geq 4$, it remains open whether graphs of separation dimension at most $s$ satisfy analogues of Lemma 14 and Corollary 15.

**References**


