On lower bounds for the matching number of subcubic graphs

P.E. Haxell* and A.D. Scott†

Monday 16th May, 2016

Abstract

We give a complete description of the set of triples (α, β, γ) of real numbers with the following property. There exists a constant K such that $\alpha n_3 + \beta n_2 + \gamma n_1 - K$ is a lower bound for the matching number $\nu(G)$ of every connected subcubic graph G, where n_i denotes the number of vertices of degree i for each i.

Keywords: matching, subcubic graph, polyhedron

1 Introduction

A graph is said to be *subcubic* if its maximum degree is at most three. In this paper we consider lower bounds for the maximum size $\nu(G)$ of a matching in subcubic graphs G.

Various lower bounds on $\nu(G)$ for subcubic graphs G appear in the literature. For example, the following theorem is due to Biedl, Demaine, Duncan, Fleischer and Kobourov [1]. Here n_i denotes the number of vertices of degree i in G, and ℓ_2 denotes the number of end-blocks in the block-cutvertex tree of G.

*Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. pehaxell@math.uwaterloo.ca; Partially supported by NSERC.

†Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Rd, Oxford, UK OX2 6GG. scott@maths.ox.ac.uk

Theorem 1. *Let G be a connected graph with n vertices.*

1. *If G is cubic then $\nu(G) \geq 4(n-1)/9$.*
2. *If G is subcubic then $\nu(G) \geq n_3/2 + n_2/3 + n_1/2 - \ell_2/3$, and $\nu(G) \geq (n-1)/3$.*

They also asked whether $\nu(G) \geq (3n + n_2)/9$ for every subcubic graph. It will turn out below that this is not the case.

Generalisations of [1] to regular graphs of higher degree were given by Henning and Yeo in [5] (see also O and West [7]). Lower bounds in terms of other parameters of G have been given, for example, in [7] and [4].

Our aim in this paper is to give a *complete* description of the set L of 3-tuples of real coefficients (α, β, γ) for which there exists a constant K such that $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - K$ for every connected subcubic graph G. (Note that this is equivalent to saying $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - Kc(G)$ for every subcubic graph G, where $c(G)$ denotes the number of components of G.) Our work here is similar in spirit to a result of Chvátal and McDiarmid [2], who addressed a similar question for cover numbers of hypergraphs in terms of their number of vertices and number of edges. We will find, as in [2], that L is a convex set, but in contrast to [2] where the number of extreme points is infinite, in our case L is a certain 3-dimensional polyhedron with a relatively simple description.

We define the polyhedron $P \subset \mathbb{R}^3$ to be the intersection of the six half-spaces

$$\begin{aligned} x_3 &\leq 4/9, \\ x_2 &\leq 1/2, \\ x_3 + x_1 &\leq 2/3, \\ x_3 + 3x_2/2 &\leq 1, \\ x_3 + x_2 + x_1 &\leq 1, \\ x_3 + x_2/6 &\leq 1/2. \end{aligned}$$

We let P_+ be the intersection of P with the nonnegative orthant $[0, \infty)^3$ in $\mathbb{R}^3$. It is easily seen that P is unbounded. However, it follows from the first three inequalities above that P_+ is a bounded subset of the nonnegative orthant.

The main aim of this paper is to prove the following theorem.

Theorem 2. $P = L$.

We will prove that $P \subseteq L$ in Section 2, and $L \subseteq P$ in Section 4.

Our proof that $P \subseteq L$ will need the fact that five specific points belong to L. This is a consequence of the following stronger result, which we prove in Section 3.

Theorem 3. *Let G be a subcubic graph with $c = c(G)$ components. Then*

$$\nu(G) \geq n_2/2 + n_1/2 - c/2, \tag{1}$$
$$\nu(G) \geq n_2/3 + 2n_1/3 - c, \tag{2}$$
$$\nu(G) \geq n_3/4 + n_2/2 + n_1/4 - c/2, \tag{3}$$
$$\nu(G) \geq 7n_3/16 + 3n_2/8 + 3n_1/16 - c/8, \tag{4}$$
$$\nu(G) \geq 4n_3/9 + n_2/3 + 2n_1/9 - c/9. \tag{5}$$

All five of these bounds are sharp: (4) is attained by the triangle, (1) and (3) by any odd cycle, and (1), (2) and (5) by the claw $K_{1,3}$. Furthermore, for a subcubic graph G, each of the bounds is sharp for G if and only if it is sharp for every component of G. We will give further connected, sharp examples for (1), (2), (3), (5) in Section 4. The proof of Theorem 3 is given in Section 3, where we will also note the following corollary concerning the constant K from the definition of L.

Corollary 4. *Let (α, β, γ) be an element of P.*

1. *If $\alpha \geq 0$ then $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - 1$ for every connected subcubic graph G.*

2. *If $\alpha < 0$ then $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - (2|\alpha| + 1)$ for every connected subcubic graph G.*

Note in particular that if G is a connected subcubic graph then $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - 1$ for every $(\alpha, \beta, \gamma) \in P_+$. Note also that if we consider $G = K_{1,3}$ and $(\alpha, \beta, \gamma) = (-\lambda, 0, \lambda + 2/3)$ (which is in P for all $\lambda \geq 0$), then the first bound in Lemma 4 is sharp for $\lambda = 0$, and the second is sharp for all $\lambda > 0$.

In the other direction, the fact that $L \subseteq P$ is a consequence of the following result, which we will prove in Section 4.

Theorem 5. *If $(\alpha, \beta, \gamma) \notin P$ then for every constant K there exists a connected subcubic graph G such that $\nu(G) < \alpha n_3 + \beta n_2 + \gamma n_1 - K$.*

Our results generalize previous work. For example, the first bound in Theorem 1 is a special case of (5); the bound $\nu \geq (n-1)/3$ follows from a convex combination of (2) and (5). On the other hand, the answer to the question of Biedl, Demaine, Duncan, Fleischer and Kobourov [1] as to whether $\nu(G) \geq (3n+n_2)/9$ for every subcubic graph is negative by Theorem 2: the vector $(1/3, 4/9, 1/3)$ is not in P as it violates the inequality $x_1+x_2+x_3 \leq 1$, and Example 3 in Section 4 is a counterexample.

2 $P \subseteq L$

In this section we prove one direction of Theorem 2, namely that $P \subseteq L$ (leaving aside the proof of Theorem 3, which we defer to the next section). We will prove that $P \subseteq L$ in two steps. We first show that it is enough to consider just P_+, and then prove that $P_+ \subseteq L$.

We begin with the following simple but useful observation.

Lemma 6. *In any connected subcubic graph G we have $n_3 \geq n_1 - 2$.*

Proof. Let T be a spanning tree of G, and let t_i denote the number of vertices of degree i in T. Then $t_1 \geq n_1$, $t_3 \leq n_3$, and $t_1 = t_3+2$. Thus $n_3 \geq n_1-2$. □

Next we note some closure properties of L.

Lemma 7.

1. *L is convex.*
2. *L is downward closed: if $(a_3, a_2, a_1) \in L$ and $b_i \leq a_i$ for all i then $(b_3, b_2, b_1) \in L$.*
3. *If $(x_3, x_2, x_1) \in L$ then $(x_3 - \lambda, x_2, x_1 + \lambda) \in L$ for all $\lambda \geq 0$.*

Proof. Suppose that $\mathbf{a} = (a_3, a_2, a_1)$, $\mathbf{b} = (b_3, b_2, b_1)$ lie in L, with associated constants K_a, K_b. Thus for every subcubic graph G, say with parameters $\mathbf{n} = (n_3, n_2, n_1)$ and matching number ν, we have $\mathbf{a} \cdot \mathbf{n} \leq \nu + K_a$ and $\mathbf{b} \cdot \mathbf{n} \leq \nu + K_b$. Suppose that $\lambda \in [0, 1]$ and $\mathbf{c} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$. Then

$$\begin{aligned}\mathbf{c} \cdot \mathbf{n} &= \lambda\mathbf{a} \cdot \mathbf{n} + (1-\lambda)\mathbf{b} \cdot \mathbf{n} \\ &\leq \lambda(\nu + K_a) + (1-\lambda)(\nu + K_b) \\ &= \nu + \lambda K_a + (1-\lambda)K_b.\end{aligned}$$

It follows that $\mathbf{c} \in L$, with associated constant $\lambda K_a + (1-\lambda)K_b$. Thus L is convex.

For the second claim, simply note that if $\mathbf{a} \in P$ with associated constant K, then for every subcubic graph G, say with parameters $\mathbf{n} = (n_3, n_2, n_1)$ and matching number ν, we have $\mathbf{b} \cdot \mathbf{n} \le \mathbf{a} \cdot \mathbf{n} \le \nu + K$, so $\mathbf{b} \in L$ with associated constant K.

Now for the final part. Let K be such that $\nu(G) \ge x_3 n_3 + x_2 n_2 + x_1 n_1 - K$ for every connected subcubic graph G. By Lemma 6 we have $n_3 \ge n_1 - 2$, and so $(x_3 - \lambda)n_3 + x_2 n_2 + (x_1 + \lambda)n_1 - (K + 2\lambda) \le x_3 n_3 + x_2 n_2 + x_1 n_1 - K \le \nu(G)$, which shows that $(x_3 - \lambda, x_2, x_1 + \lambda) \in L$. □

The next lemma will allow us to restrict our attention to P_+.

Lemma 8. *If $P_+ \subseteq L$ then $P \subseteq L$.*

Proof. Consider $x = (x_3, x_2, x_1) \in P \setminus L$. Our aim is to find a point in $P_+ \setminus L$. If each x_i is non-negative then x is such a point, so we assume the contrary.

First suppose $x_2 < 0$. We claim that $x' = (x_3, 0, x_1) \in P$. Since $x \in P$, the first and third inequalities defining P are immediate for x', and the second is trivial. The fourth and sixth inequalities follow from the first, and the fifth follows from the third. Therefore $x' \in P$. Now if $x' \in L$ then $x \in L$ because L is downward closed, contradicting our choice of x. Thus $x' \in P \setminus L$.

Therefore we may assume that $x_2 \ge 0$. Next we consider the case in which $x_3 < 0$. Set $\lambda = -x_3$ and let $x' = (x_3 + \lambda, x_2, x_1 - \lambda) = (0, x_2, x_1 + x_3)$. We claim that $x' \in P$. The first inequality for P is trivial, and the second, third and fifth are true because $x \in P$. The fourth and sixth inequalities are implied by the second. Thus $x' \in P$. If $x' \in L$ then by Lemma 7 the point $(x_3 + \lambda - \lambda, x_2, x_1 - \lambda + \lambda) = x \in L$, contradicting our choice of x. Therefore $x' \in P \setminus L$ and we may assume $x_3 \ge 0$.

Finally suppose $x_1 < 0$. Then we claim $x' = (x_3, x_2, 0) \in P \setminus L$. To check $x' \in P$ observe that the first, second, fourth and sixth inequalities are true because $x \in P$. The third follows from the first and the fifth follows from the first and second. Again we may conclude $x' \notin L$ because L is downward closed. Hence $x' \in P \setminus L$ as required, completing the proof that $P_+ \subseteq L$ implies $P \subseteq L$. □

It is therefore enough to prove that $P_+ \subseteq L$. Since L is a convex set, it is enough to show that the extreme points of P_+ all belong to L. The extreme points of P_+ (written as (x_3, x_2, x_1)) are

$$\{(0,1/2,1/2),(0,1/3,2/3),(1/4,1/2,1/4),(7/16,3/8,3/16),\\ (4/9,1/3,2/9),(1/4,1/2,0),(7/16,3/8,0),(0,1/2,0),(4/9,0,0),\\ (0,0,0),(4/9,1/3,0),(0,0,2/3),(4/9,0,2/9)\}.$$

This can be verified by hand, or (as we did) by using a computational package such as *polymake* [3].

Our aim is then to show that all thirteen extreme points of P_+ belong to L. Since L is downward closed, it is enough to consider the points that do not lie below any others: for instance, $(7/16, 3/8, 0)$ lies below $(7/16, 3/8, 3/16)$, so $(7/16, 3/8, 3/16) \in L$ implies that $(7/16, 3/8, 0) \in L$. This leaves us with the following five points:

$$\{(0,1/2,1/2),(0,1/3,2/3),(1/4,1/2,1/4),(7/16,3/8,3/16),(4/9,1/3,2/9)\}.$$

The fact that these points all belong to L follows from Theorem 3, which we prove in the next section. We conclude that $P \subseteq L$.

3 Proofs of Theorem 3 and Corollary 4

First we show how Corollary 4 follows from Theorem 3.

Proof. Let G be a connected subcubic graph. Observe that by Theorem 3 and monotonicity, we have $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - 1$ for each extreme point (α, β, γ) of P_+. By convexity, the same inequality holds for every point $(\alpha, \beta, \gamma) \in P_+$.

Now suppose $(\alpha, \beta, \gamma) \in P$ and $\alpha \geq 0$. Then (arguing as in the proof of Lemma 8) we know that $(\alpha, \beta', \gamma') \in P_+$ where $\beta' = \max\{\beta, 0\}$ and $\gamma' = \max\{\gamma, 0\}$. Hence

$$\nu(G) \geq \alpha n_3 + \beta' n_2 + \gamma' n_1 - 1 \geq \alpha n_3 + \beta n_2 + \gamma n_1 - 1.$$

If $\alpha < 0$ then set $\lambda = |\alpha|$. Then as in the proof of Lemma 8 we find that $(\alpha + \lambda, \beta, \gamma - \lambda) = (0, \beta, \gamma - \lambda) \in P$. Hence by the previous paragraph $\nu(G) \geq \beta n_2 + (\gamma - \lambda) n_1 - 1$. By Lemma 6 we have $2\lambda \geq \lambda n_1 - \lambda n_3$. Summing these two inequalities and rearranging gives $\nu(G) \geq \alpha n_3 + \beta n_2 + \gamma n_1 - (2\lambda + 1)$ as required. □

The remainder of this section is devoted to the proof of Theorem 3.

Lemma 9. *Let G be a connected subcubic graph with n vertices. Suppose $\nu(G) \geq (n-1)/2$. Then G satisfies Theorem 3.*

Proof. Bounds (1) and (3) are immediate. Bound (4) holds unless $7n/16 - 1/8 > n/2 - 1/2$, which implies $n \leq 5$. If (5) fails to hold then $4n/9 - 1/9 > n/2 - 1/2$, which means $n \leq 6$. These cases are easily checked. For (2), using Lemma 6 we find $n_1 \leq n_3 + 2 \leq n - n_1 + 2$, and hence $n_1 \leq 1 + n/2$. Thus $n_2/3 + 2n_1/3 - 1 \leq n/3 + n_1/3 - 1 \leq n/2 + 1/3 - 1$. □

In particular, if G has a perfect matching or if G is hypomatchable (meaning $G - v$ has a perfect matching for every $v \in V(G)$) then Theorem 3 holds.

In our proof we will make use of the Gallai-Edmonds structure theorem (see, for instance, [6]). In the statement below, the sets A, B and C are defined as follows (here $\Gamma(A)$ denotes the neighbourhood of A).

- $A = \{v \in V(G) : \nu(G - v) = \nu(G)\}$,
- $B = \Gamma(A) \setminus A$,
- $C = V(G) \setminus (A \cup B)$.

Theorem 10. *(Gallai-Edmonds) Let G be a graph. Then*

1. *every component of $G[A]$ is hypomatchable,*
2. *every component of $G[C]$ has a perfect matching,*
3. *every $X \subseteq B$ has neighbours in at least $|X| + 1$ components of $G[A]$.*

One consequence of Theorem 10 is that we may assume $B \neq \emptyset$, otherwise each component of G has a perfect matching or is hypomatchable, in which case we are done by Lemma 9. Note also that Part (3) implies that each vertex of B has degree at least two.

It is easy to check that all the bounds in Theorem 3 hold for graphs with at most three vertices, so we assume G has $n \geq 4$ vertices and that the theorem is true for graphs with fewer than n vertices. Since we may consider each component separately, we may assume G is connected. Choose a vertex $v \in B$, and consider $G - v$. Since $v \notin A$ we know $\nu(G - v) = \nu(G) - 1$. Let t_i denote the number of neighbours of v of degree i for $i = 1, 2, 3$. Let U denote the set of neighbours of v of degree 1, so $|U| = t_1$. Then $G' = G - v - U$ satisfies $\nu(G') = \nu(G) - 1$.

Let n'_i denote the number of vertices of degree i in G'. Since each degree-3 neighbour of v becomes a degree-2 vertex, the number of degree-3 vertices drops by t_3, plus one more if v itself has degree 3. Thus $n'_3 = n_3 - t_3 - (d(v) - 2) = n_3 - t_3 - (t_1 + t_2 + t_3 - 2) = n_3 - 2t_3 - t_2 - t_1 + 2$. Each degree-2 neighbour of v becomes a degree-1 vertex, and if v has degree 2 then the number of degree-2 vertices drops by one more. Hence $n'_2 = n_2 + t_3 - t_2 - (3 - d(v)) = n_2 + t_3 - t_2 - (3 - t_1 - t_2 - t_3)) = n_2 + 2t_3 + t_1 - 3$. Finally $n'_1 = n_1 - t_1 + t_2$, and $c' \leq t_3 + t_2$. Then by the induction hypothesis,

1. $$\begin{aligned}\nu(G') &\geq n'_2/2 + n'_1/2 - c'/2 \\ &\geq n_2/2 + (2t_3 + t_1 - 3)/2 + n_1/2 + (t_2 - t_1)/2 - (t_3 + t_2)/2 \\ &= n_2/2 + n_1/2 - 1/2 + (t_3 - 2)/2,\end{aligned}$$

2. $$\begin{aligned}\nu(G') &\geq n'_2/3 + 2n'_1/3 - c' \\ &\geq n_2/3 + (2t_3 + t_1 - 3)/3 + 2n_1/3 + 2(t_2 - t_1)/3 - (t_3 + t_2) \\ &= n_2/3 + 2n_1/3 - 1 - (t_3 + t_2 + t_1)/3,\end{aligned}$$

3. $$\begin{aligned}\nu(G') &\geq n'_3/4 + n'_2/2 + n'_1/4 - c'/2 \\ &\geq n_3/4 + (2 - 2t_3 - t_2 - t_1)/4 + n_2/2 + (2t_3 + t_1 - 3)/2 + n_1/4 \\ &\quad + (t_2 - t_1)/4 - (t_3 + t_2)/2 \\ &= n_3/4 + n_2/2 + n_1/4 - 1/2 - (t_2 + 1)/2,\end{aligned}$$

4. $$\begin{aligned}\nu(G') &\geq 7n'_3/16 + 3n'_2/8 + 3n'_1/16 - c'/8 \\ &\geq 7n_3/16 + 7(2 - 2t_3 - t_2 - t_1)/16 + 3n_2/8 + 3(2t_3 + t_1 - 3)/8 \\ &\quad + 3n_1/16 + 3(t_2 - t_1)/16 - (t_3 + t_2)/8 \\ &= 7n_3/16 + 3n_2/8 + 3n_1/16 - 1/4 - t_3/4 - 3t_2/8 - t_1/4 \\ &= [7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8] - (4t_3 + 6t_2 + 4t_1 + 2)/16,\end{aligned}$$

5. $$\begin{aligned}\nu(G') &\geq 4n'_3/9 + n'_2/3 + 2n'_1/9 - c'/9 \\ &\geq 4n_3/9 + 4(2 - 2t_3 - t_2 - t_1)/9 + n_2/3 + (2t_3 + t_1 - 3)/3 \\ &\quad + 2n_1/9 + 2(t_2 - t_1)/9 - (t_3 + t_2)/9 \\ &= 4n_3/9 + n_2/3 + 2n_1/9 - 1/9 - (t_3 + t_2 + t_1)/3.\end{aligned}$$

Since $\nu(G) = \nu(G') + 1$ and $t_3 + t_2 + t_1 \leq 3$ it follows from the calculations above that bounds (1), (2) and (5) hold for G. (In fact (2) alternatively follows from (5) together with Lemma 7(3)).

We now focus on bounds (3) and (4). Note that in these cases, our inductive statement gives

$$\nu(G') \geq n_3/4 + n_2/2 + n_1/4 - 1/2 - (t_2 + 1)/2,$$

and

$$\nu(G') \geq [7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8] - (4t_3 + 6t_2 + 4t_1 + 2)/16.$$

First we note some consequences of Theorem 10 and the above calculations.

Lemma 11. *1. Every $v \in B$ has at least two neighbours in A.*

2. If $x \in A$ has exactly two neighbours u and w, and if $u \in B$, then $w \in B$ as well.

3. If (4) *fails for G then every $v \in B$ has degree 3.*

4. If one of (3) *and* (4) *fails for G then every $v \in B$ has at least two degree-2 neighbours.*

Proof. We have already noted that the first statement is immediate from Theorem 10(3). To verify the second claim, observe that if $w \in A$ then u and w are both in a component H of $G[A]$, which is hypomatchable by Theorem 10. But x has degree 1 in H, which is not possible in a hypomatchable component. Thus $w \in B$.

If (3) fails then $t_2 \geq 2$; if (4) fails then $4t_3 + 6t_2 + 4t_1 \geq 15$ and so (as $d(v) \leq 3$) we have $t_2 \geq 2$ and $t_1 + t_2 + t_3 = 3$. The last two assertions follow immediately, as the same calculation holds for any vertex of B. □

Next we derive some elementary facts about the neighbours of degree-2 vertices.

Lemma 12. *Suppose G fails to satisfy one of* (3) *and* (4). *Then no two degree-2 vertices of G are adjacent. Furthermore every vertex of B has degree 3.*

Proof. Recall our assumption that G has at least four vertices. If G is a 4-cycle then (3) and (4) are satisfied (by Lemma 9), so let us assume otherwise. Suppose u and w are adjacent degree-2 vertices.

If u and w are not in a triangle or 4-cycle then suppressing u and w (i.e. if u' and v' are the other neighbours of u, v then we replace the path $u'uvv'$ by the edge $u'v'$) gives a connected graph G' with $\nu(G') = \nu(G) - 1$, $n_3' = n_3$, $n_2' = n_2 - 2$, and $n_1' = n_1$. Then by the induction hypothesis for (3), $\nu(G') \geq n_3'/4 + n_2'/2 + n_1'/4 - 1/2 = n_3/4 + n_2/2 + n_1/4 - 1/2 - 1$, showing G satisfies (3). For (4) we have by induction $\nu(G') \geq 7n_3'/16 + 3n_2'/8 + 3n_1'/16 - 1/8 = 7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8 - 6/8$, which also suffices.

If uwx is a triangle then form G' by removing u and w. Then $\nu(G') = \nu(G) - 1$, $n_3' = n_3 - 1$, $n_2' = n_2 - 2$, $n_1' = n_1 + 1$, and $c' = 1$. For (3) we get $\nu(G') \geq n_3'/4 + n_2'/2 + n_1'/4 - 1/2 = n_3/4 - 1/4 + n_2/2 - 1 + n_1/4 + 1/4 - 1/2 = [n_3/4 + n_2/2 + n_1/4 - 1/2] - 1$, showing G satisfies (3). For (4) we have by induction $\nu(G') \geq 7n_3/16 - 7/16 + 3n_2/8 - 6/8 + 3n_1/16 + 3/16 - 1/8 = [7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8] - 1$, as needed.

If u and w are in a 4-cycle $uwxz$ then by assumption (say) x has degree 3. Form G' by removing u and w, so that $\nu(G') = \nu(G) - 1$. If $d(z) = 3$ then G' has $n_3' = n_3 - 2$, $n_2' = n_2$, $n_1' = n_1$, and $c' = 1$. Then using induction for (3) we find $\nu(G') \geq n_3'/4 + n_2'/2 + n_1'/4 - 1/2 = (n_3 - 2)/4 + n_2/2 + n_1/4 - 1/2 = n_3/4 + n_2/2 + n_1/4 - 1/2 - 1/2$, which suffices. For (4) we get $\nu(G') \geq 7n_3/16 - 14/16 + 3n_2/8 + 3n_1/16 - 1/8 = [7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8] - 14/16$ as required.

If $d(z) = 2$ the parameters become $n_3' = n_3 - 1$, $n_2' = n_2 - 2$, and $n_1' = n_1 + 1$, giving for (3) $\nu(G') \geq n_3'/4 + n_2'/2 + n_1'/4 - 1/2 = (n_3 - 1)/4 + (n_2 - 2)/2 + n_1/4 = 1/4 - 1/2 + n_3/4 + n_2/2 + n_1/4 - 1/2 - 1$ as needed. For (4) we get $\nu(G') \geq 7n_3/16 - 7/16 + 3n_2/8 - 6/8 + 3n_1/16 + 3/16 - 1/8 = [7n_3/16 + 3n_2\ /8 + 3n_1/16 - 1/8] - 1$. This completes the proof of the first statement. The second statement now follows using Lemma 11(3),(4). $\square$

Lemma 13. *Suppose G fails to satisfy one of* (3) *and* (4)*. Then each degree-2 vertex w has two degree-3 neighbours.*

Proof. Lemma 12 tells us that w has no degree-2 neighbours. Suppose for a contradiction that w has a degree-1 neighbour x. Then (recalling G has at least four vertices) $G' - \{w, x\}$ has $\nu(G') = \nu(G) - 1$, $n_3' = n_3 - 1$, $n_2' = n_2$, $n_1' = n_1 - 1$, and $c' = 1$. Then using induction for (3) gives $\nu(G') \geq n_3'/4 + n_2'/2 + n_1'/4 - c'/2 \geq n_3/4 - 1/4 + n_2/2 + n_1/4 - 1/4 - 1/2 = [n_3/4 + n_2/2 + n_1/4 - 1/2] - 1/2$, which suffices. For (4) we get $\nu(G') \geq$

$7n_3'/16+3n_2'/8+3n_1'/16-c'/8 \geq 7n_3/16-7/16+3n_2/8+3n_1/16-3/16-1/8 = [7n_3/16+3n_2/8+3n_1/16-1/8]-10/16$. □

Call a degree-3 vertex $v \in G$ *good* if it has two degree-2 neighbours that do not have a common neighbour different from v. Observe that if v has three degree-2 neighbours then either v is good, or $G = K_{2,3}$, in which case (3) and (4) hold.

Lemma 14. *Suppose G fails to satisfy one of* (3) *and* (4). *Then every good vertex v of G has three degree-2 neighbours, all of which are in different components of $G - v$.*

Proof. Let w and x be degree-2 neighbours that are not adjacent and have no common neighbour other than v. As before, we write t_i for the number of degree i neighbours of v, and U for the set of degree 1 neighbours of v. Let G' be the graph obtained by removing $\{v\} \cup U$ and identifying w and x into a new vertex of degree 2. Then $\nu(G') = \nu(G) - 1$, $n_3' = n_3 - t_3 - 1$, $n_2' = n_2 - t_2 + t_3 + 1$, $n_1' = n_1 - t_1 + t_2 - 2$, and $c' \leq 2 - t_1$.

The computation for (3) becomes $\nu(G') \geq n_3'/4 + n_2'/2 + n_1'/4 - c'/2 \geq n_3/4 - t_3/4 - 1/4 + n_2/2 + (t_3+1-t_2)/2 + n_1/4 + (t_2 - t_1 - 2)/4 - (2-t_1)/2 = [n_3/4 + n_2/2 + n_1/4 - 1/2] + t_3/4 - t_2/4 + t_1/4 - 3/4$. Then (3) holds unless $t_2 = 3$ and $c' = 2$.

For (4) we get $\nu(G') \geq 7n_3'/16+3n_2'/8+3n_1'/16-c'/8 \geq 7n_3/16-7t_3/16-7/16 + 3n_2/8 + 3(t_3+1-t_2)/8 + 3n_1/16 + 3(t_2-t_1-2)/16 - (2-t_1)/8 = [7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8] - (t_3 + 3t_2 + t_1 + 9)/16$, so (4) holds unless $t_2 = 3$ and $c' = 2$.

Hence in both cases we may assume that $t_2 = 3$ and so $c' = 2$. Let y be the third neighbour of v. Since $c' = 2$ we know that y is in a different component of G' (and hence of $G - v$) to w and x. In particular, y is not adjacent to w or x and does not share a second common neighbour with either of them. Thus we could apply the above argument with w and y and find that x is in a different component of $G - v$ from both w and y. This completes the proof. □

We may now complete the proof for (3).

Lemma 15. *G satisfies* (3).

Proof. Suppose the contrary. If any degree-3 vertex has another degree-3 vertex in its neighbourhood, then we may verify (3) by considering the graph G' obtained by deleting an edge joining two degree-3 vertices. In this case $n_3' = n_3 - 2$, $n_2' = n_2 + 2$, $n_1' = n_1$ and $c' \leq 2$. Hence using induction we get $\nu(G) \geq \nu(G') \geq n_3'/4 + n_2'/2 + n_1'/4 - 2/2 = n_3/4 + n_2/2 + n_1/4 - 1/2$, proving (3) as required.

Thus we may assume no two degree-3 vertices are adjacent. Next we check that no degree-3 vertex has two degree-1 neighbours. If on the contrary x has degree-1 neighbours v and w, and a third neighbour z (which necessarily has degree 2, or else G is $K_{1,3}$ and satisfies (3)), form G' by removing v, w, and x. Then $n_3' = n_3 - 1$, $n_2' = n_2 - 1$, $n_1' = n_1 - 1$, $c' = 1$ and $\nu(G) = \nu(G') + 1$. Therefore by induction $\nu(G) \geq n_3'/4 + n_2'/2 + n_1'/4 - c'/2 + 1 \geq [n_3/4 + n_2/2 + n_1/4 - 1/2] - 1/4 - 1/2 - 1/4 + 1$, showing (3) holds. Thus every degree-3 vertex has at least two degree-2 neighbours.

Suppose a degree-2 vertex w has neighbours v and z (which both have degree 3 by Lemma 13). If v is good then z is also good, since otherwise every other degree-2 neighbour of z (at least one of which exists) is also a degree-2 neighbour of v, and would therefore be in the same component of $G - v$ as w, contradicting Lemma 14. Therefore there are no good vertices at all, since otherwise (since G has at least one degree-3 vertex, in B) by Lemma 14 we would find that G is a subdivision of a connected 3-regular graph, but removing any degree-3 vertex results in 3 components. This is not possible since, in particular, every connected graph has a vertex whose removal leaves a connected graph.

Since G has no good vertices, in particular no degree-3 vertex can have three degree-2 neighbours. So every degree 3 vertex has exactly two degree 2 neighbours. It follows that G is a cycle (of even length) with a pendant edge attached to every second vertex (these are the graphs $G_3(t)$ in Example 3 in the next section). But (3) holds for this graph, completing the proof. □

We are left to verify (4). We need one more technical lemma.

Lemma 16. *No vertex in B is good.*

Proof. Suppose on the contrary that B contains good vertices. Let $v \in B$ be a good vertex. Let W be the union of the vertex sets of all paths of the form $vw_1w_2\ldots w_r$ where $r \geq 1$, each w_i with i odd is a degree-2 vertex in A, and each w_i with i even is in B. Let H be the subgraph of G induced by W. Then H is connected.

We claim that each vertex of $W \cap B$ is good. To verify this, consider a good vertex $w \in W \cap B$ (for example $w = v$). By Lemma 11(1) we know w

has at least two neighbours u and x in A, and $d(u) = d(x) = 2$ by Lemma 14. Also, Lemma 11(2) implies that the other neighbour z of u is in B and hence is in $W \cap B$. Thus $d(z) = 3$ by Lemma 11(3). If z were not good then every degree-2 neighbour of z different from u (at least one of which exists, by Lemma 11(4)) would be a degree-2 neighbour of w, and would hence be in the same component of $G - w$ as u, contradicting Lemma 14. Hence z is good. Applying this observation repeatedly (moving along the paths used to define H) we find that every vertex of $W \cap B$ is good.

By Lemma 11(2) we know that $A \cap W$ is independent, and each $x \in A \cap W$ has exactly two neighbours in $B \cap W$. Since each $w \in B \cap W$ is good, it has three degree-2 neighbours in G by Lemma 14, at least two of which are in A by Lemma 11(1). So by Lemma 11(3) we know $B \cap W$ is independent. Therefore H is the subdivision of a connected subcubic graph J with vertex set $B \cap W$ and minimum degree at least 2. (Note that J has no multiple edges by Lemma 14 and the fact that each $w \in B \cap W$ is good.)

Since each $w \in B \cap W$ is good, the graph J has the property that $J - y$ has $d(y) \geq 2$ components for every vertex y of J. Such a graph cannot exist, so the proof is complete. □

We may therefore assume that no vertex in B has three degree-2 neighbours. Choose $v \in B$. By Lemma 12 we have $d(v) = 3$, and by Lemma 11(4) we know that v has at least two degree-2 neighbours, say w and x. By Lemma 11(1) at least one of them, say w, is in A. Since v is not good, the other neighbour z of v is not a degree-2 vertex, and w and x have another common neighbour y. By Lemma 11(2) we know y is in B. Then by Lemma 11(3) we have that y has another neighbour u, and $d(u) \neq 2$ since y is not good. Since (4) holds for K_4 with one edge deleted, we may assume $u \neq v$. If G consists of a 4-cycle plus two pendant edges attached to non-adjacent vertices then (4) holds, so we may assume without loss of generality that z has degree 3.

If $z = u$ remove v, w, x, y. Then $n_3' = n_3 - 3$, $n_2' = n_2 - 2$, $n_1' = n_1 + 1$, $c' = 1$ and $\nu(G') = \nu(G) - 2$. Then by induction $\nu(G) \geq \nu(G') + 2 \geq 7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8 - 30/16 + 2$, which implies our result.

If u has degree 1 we remove u, v, w, x, y. Then $n_3' = n_3 - 3$, $n_2' = n_2 - 1$, $n_1' = n_1 - 1$, $c' = 1$ and $\nu(G') = \nu(G) - 2$. Then by induction $\nu(G) \geq \nu(G') + 2 \geq 7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8 - 30/16 + 2$, as needed.

Otherwise $z \neq u$, and $d(z) = d(u) = 3$. In this case we remove v, w, x, y. Then $n_3' = n_3 - 4$, $n_2' = n_2$, $n_1' = n_1$, $c' \leq 2$ and $\nu(G') = \nu(G) - 2$. Then by induction $\nu(G) \geq \nu(G') + 2 \geq 7n_3/16 + 3n_2/8 + 3n_1/16 - 1/8 - 30/16 + 2$, which completes the proof of Theorem 3.

4 $L \subseteq P$

The fact that $L \subseteq P$ is an immediate consequence of Theorem 5, which we prove in this section.

Suppose that $(x_3, x_2, x_1) \in L$, so there is some real number K such that

$$\nu(G) \geq x_3 n_3(G) + x_2 n_2(G) + x_1 n_1(G) - K \tag{6}$$

for every connected subcubic graph G (where $n_i(G)$ denotes the number of vertices of G of degree i). We fix a choice of (x_3, x_2, x_1) and K for the rest of this section.

We will consider six special families of graphs: each family will show that (x_3, x_2, x_1) must satisfy one of the inequalities in the definition of P. An example from each family is shown in the figures.

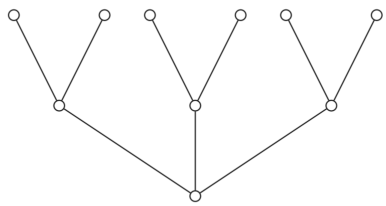
Example 1. Let t be an odd positive integer. The graph $G_1(t)$ is the tree with a root plus $t+1$ levels, indexed by $i = 0, \ldots, t$, in which level i contains $3 \cdot 2^i$ vertices, and all vertices except the leaves have degree 3. Thus $G_1(t)$ is (internally) a cubic tree and has depth $t+1$. Then $n_1 = 3 \cdot 2^t$, $n_2 = 0$ and $n_3 = 1 + 3(2^t - 1) = 3 \cdot 2^t - 2$. Since $G_1(t)$ is bipartite with one partition class S formed by the vertices at levels $0, 2, \ldots, t-1$ we see $\nu(G_1(t)) \leq |S| = 3(4^{(t+1)/2} - 1)/3 = 2^{t+1} - 1$. By (6) we must have

$$(3 \cdot 2^t - 2)x_3 + 3 \cdot 2^t x_1 - K \leq 2 \cdot 2^t - 1,$$

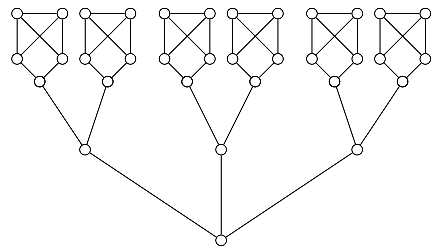
and so, dividing by $3 \cdot 2^t$ and letting $t \to \infty$, we see that

$$x_3 + x_1 \leq 2/3.$$

Example 2. Let J denote the graph obtained by subdividing one edge of K_4, and let x denote the single vertex of degree 2 in J. We define the graph $G_2(t)$, again for odd t, by identifying each leaf in $G_1(t)$ with the vertex x in a copy of the graph J, such that all copies are disjoint from each other and the rest of the graph. Then for this graph $n_1 = n_2 = 0$, and $n_3 = 3 \cdot 2^t - 2 + 15 \cdot 2^t = 9 \cdot 2^{t+1} - 2$. The same set S as before now has the property that removing it leaves $1 + 3(2 + 2^3 + \ldots + 2^t) = 1 + 6(4^{(t+1)/2} -$



$G_1(1)$



$G_2(1)$

$1)/3 = 2^{t+2} - 1$ odd components. Therefore any maximum matching in G must leave exposed at least $2^{t+2} - 1 - |S| = 2^{t+1}$ vertices. This tells us $\nu(G_2(t)) \leq (9 \cdot 2^{t+1} - 2 - 2^{t+1})/2 = (2^{t+4} - 2)/2 = 2^{t+3} - 1$. So (6) implies that

$$(9 \cdot 2^{t+1} - 2)x_3 - K \leq 2^{t+3} - 1.$$

Dividing by $9 \cdot 2^{t+1}$ and taking a limit gives

$$x_3 \leq 4/9.$$

Example 3. Let $t \geq 2$ be a positive integer. The graph $G_3(t)$ is obtained from the cycle with $2t$ vertices by attaching a pendant edge to every second vertex. Then $n_1 = n_2 = n_3 = t$. The graph is bipartite with one vertex class consisting of the vertices of degree 3, so $\nu(G_3(t)) \leq n_3 = t$. Thus

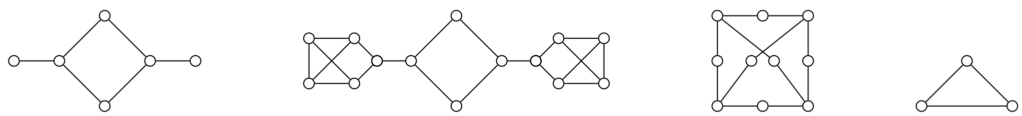
$$x_3 t + x_2 t + x_1 t - K \leq t.$$

Dividing by t and taking a limit gives

$$x_3 + x_2 + x_1 \leq 1.$$

Example 4. The graph $G_4(t)$ is obtained from $G_3(t)$ by adding t disjoint copies of J, identifying the vertex x in each copy with the leaf of a pendant edge. Then $n_1 = 0$, $n_2 = t$ and $n_3 = 6t$. The set of degree-3 vertices on the cycle has size t and leaves $2t$ odd components when deleted, showing $\nu(G_4(t)) \leq (7t - t)/2 = 3t$. Thus

$$6x_3 t + x_2 t - K \leq 3t.$$



$G_3(2)$ $G_4(2)$ $G_5(4)$ $G_6(3)$

Dividing by $6t$ and taking a limit gives

$$x_3 + x_2/6 \leq 1/2.$$

Example 5. For each even integer $t \geq 4$, let $G_5(t)$ be obtained from a cubic graph H on t vertices by subdividing every edge of H exactly once (for sake of definiteness, we may may take H to be a cycle of length t with opposite vertices joined). Then $n_1 = 0$, $n_2 = e(H) = 3t/2$ and $n_3 = t$. Then $G_5(t)$ is bipartite with one vertex class $V(H)$ of size t, so $\nu(G) \leq t$. Thus

$$x_3 t + 3x_2 t/2 - K \leq t.$$

Dividing by t and taking a limit gives

$$x_3 + 3x_2/2 \leq 1.$$

Example 6. Finally, for odd integers $t \geq 3$, we let $G_6(t)$ be the odd cycle of length t. Then $n_1 = n_3 = 0$ and $n_2 = t$, while $\nu = (t-1)/2$. Thus

$$x_2 t/2 - K \leq t/2 - 1/2.$$

Dividing by $t/2$ and taking a limit gives

$$x_2 \leq 1/2.$$

The proof of Theorem 5 is now immediate.

Proof of Theorem 5. If $(x_3, x_2, x_1) \notin P$ then it fails to satisfy one of the inequalities used to define P. Therefore, taking the example above that corresponds to this inequality (and noting that all the examples are connected) we see that by taking t large we can force K to be arbitrarily large. $\square$

In fact it is easy to see that equality holds in each expression bounding $\nu(G_i(t))$, but we do not need this fact. Finally, we note that Example 1 is sharp for (2) and (5); Example 2 is sharp for (5); and Example 6 is sharp for (1) and (3).

Acknowledgements. The authors would like to thank Günter Ziegler for recommending *polymake*, and also an anonymous referee for a very careful reading.

References

[1] T. Biedl, E. Demaine, C. Duncan, R. Fleischer, S. Kobourov, Tight bounds on maximal and maximum matchings, *Discrete Math.* 285 (2004), 7–15.

[2] V. Chvátal and C. McDiarmid, Small transversals in hypergraphs, *Combinatorica* **12** (1992) 19–26.

[3] Ewgenij Gawrilow and Michael Joswig, polymake: a framework for analyzing convex polytopes, Polytopes − combinatorics and computation (Oberwolfach, 1997), 4373, DMV Sem., 29, Birkhäuser, Basel, 2000. MR1785292 (2001f:52033).

[4] M. Henning, C. Löwenstein, D. Rautenbach, Independent sets and matchings in subcubic graphs, *Discrete Math.* 312 (2012), 1900–1910.

[5] M. Henning, A. Yeo, Tight lower bounds on the size of a matching in a regular graph, *Graphs Comb.* 23 (2007), 647–657.

[6] L. Lovász, M. D. Plummer, Matching theory. North-Holland Mathematics Studies **121**, *Annals of Discrete Mathematics* **29**, North-Holland Publishing Co., Amsterdam; Akadémiai Kiadó (Publishing House of the Hungarian Academy of Sciences), Budapest, 1986, xxvii+544 pp.

[7] S. O, D. West, Balloons, cut-edges, matchings, and total domination in regular graphs of odd degree, *J. Graph Th.* 64 (2010), 116–131.