Powers of paths in tournaments

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Abstract

In this short note we show that every tournament contains the k-th power of a directed path of linear length. This improves upon recent results of Yuster and of Girão.

1 Introduction

One of the main themes in extremal graph theory is the study of embedding long paths and cycles in graphs. Some of the classical examples include the Erdős–Gallai theorem that every n-vertex graph with average degree d contains a path of length d, and Dirac's theorem that every graph with minimum degree $n/2$ contains a Hamilton cycle. A famous generalization of this, conjectured by Pósa and Seymour, and proved for large n by Komlós, Sárközy and Szemerédi [3], asserts that if the minimum degree is at least $kn/(k+1)$, then the graph contains the k-th power of a Hamilton cycle.

In this note, we are interested in embedding directed graphs in a tournament. A tournament is an oriented complete graph. The k-th power of the directed path $\vec{P}_\ell = v_0 \dots v_\ell$ of length ℓ is the graph $\vec{P}_\ell^k$ on the same vertex set containing a directed edge $v_i v_j$ if and only if $i < j \leq i + k$. The k-th power of a directed cycle is defined analogously. An old result of Bollobás and Häggkvist [1] says that, for large n, every n-vertex tournament with all indegrees and outdegrees at least $(1/4 + \varepsilon)n$ contains the k-th power of a Hamilton cycle (the constant $1/4$ is optimal). However, we cannot expect to find powers of directed cycles in general, as the transitive tournament contains no cycles at all.

What about powers of directed paths? It is well-known that every tournament contains a directed Hamilton path, but as Yuster [4] recently observed, some tournaments are quite far from containing the square of a Hamilton path: there is an n-vertex tournament that does not even contain the square of $\vec{P}_{2n/3}$, and more generally, for every $k \geq 2$, there are tournaments with n vertices and no k-th power of a path with more than $nk/2^{k/2}$ vertices. In the other direction, Yuster proved that every tournament with n vertices contains the square of a path of length $n^{0.295}$. This was improved very recently by Girão [2], who showed

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that for fixed k, every tournament on n vertices contains the k-th power of a path of length $n^{1-o(1)}$. Both papers noted that no sublinear upper bound is known. Our main result shows that the maximum length is in fact linear in n.

Theorem 1. *Let $n, k \geq 2$. Every n-vertex tournament contains the k-th power of a directed path of length $n/2^{2^{3k}}$.*

It would be interesting to determine the optimal constant c_k such that every tournament on n vertices contains the k-th power of a directed path of length $(1 + o(1))c_k n$. Theorem 1 and Yuster's construction show that c_k lies between $1/2^{2^{3k}}$ and $k/2^{k/2}$.

2 Proofs

Let us start with the following, fairly intuitive, lemma, which takes care of a technical detail in our proof.

Lemma 2. *Let A, B be disjoint vertex sets in a directed graph G such that every vertex in A has at least $|B|/3$ outneighbours in B. Then A contains a subset of $r = \lfloor |A|/6 \rfloor$ vertices that have at least $|B|/2^{8r}$ common outneighbours in B.*

Proof. We may assume that $r \geq 2$ and $|A| = 6r$. Let B_0 be the set of vertices in B with fewer than r inneighbours in A. Then $B_1 = B \setminus B_0$ touches at least $6r|B|/3 - r|B_0| \geq r|B|$ edges coming from A, and hence $|B_1| \geq r|B|/|A| = |B|/6$.

Let us delete edges so that every vertex in B_1 has exactly r inneighbours in A. As there are $\binom{6r}{r} \leq 2^{6r}$ different r-subsets in A, one of them appears as the inneighbourhood of at least $|B_1|/2^{6r} \geq |B|/2^{8r}$ different vertices in B_0. This subset satisfies the lemma. $\square$

We define the r-blowup $\vec{B}_\ell^r$ of the path $\vec{P}_\ell$ as a graph on $r(\ell+1)$ vertices that consists of $\ell + 1$ independent sets $X_0, \ldots, X_\ell$ of size r each, and all the directed edges of the form vv' with $v \in X_{i-1}$, $v' \in X_i$ for some $i \in [\ell]$.

The next lemma says that tournaments contain r-blowups of very long paths. This is the main tool in our proof, and will readily imply Theorem 1.

Lemma 3. *Let $n, r \geq 2$. Every n-vertex tournament contains an r-blowup $\vec{B}_\ell^r$ of a path of length $\ell \geq n/2^{12r}$.*

Proof. Let us take an ordering $v_0, \ldots, v_{n-1}$ of the vertices that maximizes the number of forward edges, i.e., the number of edges $v_i v_j$ such that $i < j$. We will use the notation $V[i, j) = \{v_i, \ldots, v_{j-1}\}$ for an "interval" of vertices with respect to this ordering, where $0 \leq i < j \leq n$.

We will embed the blowup path $\vec{B}_\ell^r$ inductively using the following claim.

Claim. Let $t = 6r \cdot 2^{8r}$, and $0 \leq i \leq n - 4t$. Then for every subset $A \subseteq V[i, i + t)$ of size $6r$, there is an index $i + t \leq j \leq i + 3t$ such that some subset $A' \subseteq A$ of size r has at least $6r$ common outneighbours in the interval $V[j, j + t)$.

Proof. Let $B = V[i+t, i+4t)$, and note that every vertex $v \in A$ has at least $t = |B|/3$ outneighbours in B. Indeed, otherwise v has at least $2t$ inneighbours in the interval $V[i, i+4t)$, so moving v to the end of this interval would increase the number of forward edges in the ordering, contradicting our assumption.

We can thus apply Lemma 2 to find a subset $A' \subseteq A$ of size r that has at least $|B|/2^{8r} = 18r$ common outneighbours in B. But then one of the intervals $V[i+t, i+2t), V[i+2t, i+3t), V[i+3t, i+4t)$ contains at least $6r$ of these outneighbours, and we can choose j accordingly. $\square$

Let $i_0 = 0$ and $A_0 = V[0, 6r)$. Applying the Claim with $i = i_0$ and $A = A_0$, we get a set A' of size r with at least $6r$ common outneighbours in some interval $V[j, j+t)$ with $j \leq i_0 + 3t$. Let us define $X_0 = A'$, $i_1 = j$, and choose A_1 to be any $6r$ of the common outneighbours. We can then apply the Claim again with $i = i_1$ and $A = A_1$, and repeat this process until some i_ℓ is above $n - 4t$. Choose X_ℓ to be an arbitary r-subset of A_ℓ. Then the sequence $X_0, \dots X_\ell$ gives the vertex classes of a copy of $\vec{B}_\ell^r$ with $\ell \geq n/(4t) \geq n/2^{12r}$. $\square$

The final ingredient we need for the proof of Theorem 1 is the folklore fact that every tournament on 2^k vertices contains a transitive subtournament of size $k+1$ (isomorphic to $\vec{P}_k^k$). This is easily seen by taking a vertex of outdegree at least 2^{k-1} as the first vertex of the subtournament, and then recursing on the outneighbourhood.

Proof of Theorem 1. Let us apply Lemma 3 to the tournament with $r = 2^k$. We then obtain disjoint sets $X_0, \dots, X_\ell$ of size 2^k for $\ell \geq n/2^{12 \cdot 2^k} \geq n/2^{2^{3k}}$ such that for all $i \in [\ell]$, X_i is contained in the outneighbourhood of every vertex in X_{i-1}.

Let us choose a $(k+1)$-subset $Y_i \subseteq X_i$ for every i so that Y_i induces a transitive subtournament, i.e., the k-th power of some path $v_0^i \dots v_k^i$. Then $v_0^0 \dots v_k^0 v_0^1 \dots v_k^1 \dots v_0^\ell \dots v_k^\ell$ is a directed path of length at least $n/2^{2^{3k}}$ whose k-th power is contained in the tournament. $\square$

References

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