Powers of paths and cycles in tournaments

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Abstract

We show that for every positive integer k, any tournament can be partitioned into at most 2^{ck} k-th powers of paths. This result is tight up to the exponential constant. Moreover, we prove that for every $\varepsilon > 0$ and every integer k, any tournament on $n \geq \varepsilon^{-Ck}$ vertices which is ε-far from being transitive contains the k-th power of a cycle of length $\Omega(\varepsilon n)$; both bounds are tight up to the implied constants.

1 Introduction

Tournaments are complete graphs where every edge has an orientation. A simple exercise ([13]) shows that any tournament contains a Hamilton path, i.e., a directed path which passes through every vertex. A natural generalisation of a directed path is a k-th power of a path which consists of a sequence of vertices $x_1, x_2, \ldots, x_n$ with the property that $x_i \to x_j$ for every $1 \leq i < j \leq i + k \leq n$. When $k \geq n - 1$, this k-th power of a path is an n-vertex transitive tournament.

In [14], Yuster investigated the problem of estimating the minimum over all n-vertex tournaments of the maximum length of a power of a path. He showed that any tournament on n vertices must contain a power of a directed path on at least $n^{0.295}$ vertices. Confirming a conjecture of Yuster, Draganić et al. [5] showed that for every k there always exists a k-th power of a path of linear order.

Theorem 1.1 (Draganić et al. [5])**.** *For every positive integer k, any tournament on n vertices contains the k-th power of a path of length $n/2^{(2+o(1))k}$.*

More precisely, for every positive integer k, every tournament on n vertices contains a k-th power of a path on at least $kn/2^{4k+6}$ vertices. Moreover, it was shown in [5] that this is tight up

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to the constant in the exponent. One might be tempted to ask whether the same phenomenon holds for every acyclic digraph with bounded maximum degree. In other words, is it the case that, for any positive integer d, there is a constant $C(d) > 0$ such that for any acyclic digraph D on n vertices with maximum degree d, every tournament on at least $C(d)n$ vertices contains a copy of D? Very recently Fox, He and Widgerson [8] gave a negative answer to this question. Indeed, they showed that for all $\Delta \geq 2$ and every sufficiently large n, there is an acyclic digraph D on n vertices with maximum degree Δ for which there are tournaments on at least $n^{\Omega(\Delta^{2/3-o(1)})}$ vertices that do not contain any copy of D.

In this paper, we build some tools for finding powers of paths and cycles in tournaments and use them to prove two further results. We first extend Theorem 1.1, by showing that every tournament can be partitioned into $2^{O(k)}$ tournaments each of which contains the k-th power of a Hamilton path. We then consider powers of cycles, showing that if a tournament is far from transitive then it must contain the kth power of a long cycle; we give bounds that are essentially tight up to the implicit constants. These results are discussed in the next two subsections.

1.1 Partitions into powers of paths

A common theme in combinatorics is the problem of partitioning the vertex set of a graph into a bounded number of pieces, each satisfying certain properties. A famous problem in this area is Lehel's Conjecture, which states that any graph has a vertex partition into two parts, where one forms a cycle and the other forms an anticycle (a cycle in the complement); equivalently, any 2-edge-coloured graph has a vertex bipartition into two monochromatic cycles of distinct colours This was confirmed for large enough graphs by Łuczak, Rödl, and Szemerédi [12] and for all graphs by Bessy and Thomassé [1].

Similar questions arise for colourings with more colours. An influential result of Erdős, Gyárfás and Pyber [7] shows that any r-edge-coloured graph can be partitioned into at most $O(r^2 \log r)$ monochromatic copies of a cycle. Recently Bustamante, Corsten, Frankl, Pokrovskiy and Skokan [3] extended this to powers of cycles, proving that that for all natural numbers k and r, the vertices of every r-edge-coloured complete graph can be partitioned into a bounded number of k-th powers of cycles. (We refer the reader to the survey of Gyárfás [10], for many other problems dealing with partitions and covers of finitely edge-coloured graphs.)

In light of these results it is very natural to ask whether every tournament has a finite partition of the vertex set into k-th powers of paths. In our first result, we answer this question.

Theorem 1.2. *Every n-vertex tournament T can be partitioned into at most $2^{10^5 k}$ vertex-disjoint k-th powers of directed paths.*

We remark that the bound above is essentially tight, up to the implied constant. To see this, let T_k be a tournament on $2^{k/2}$ vertices which does not contain a transitive tournament on k vertices,

and let T be a tournament consisting of the disjoint union of $n/2^{k/2}$ copies of T_k, where all edges between the copies are oriented from left to right. It is easy to see that T does not contain a k-th power of a path of length greater than $kn/2^{k/2}$.

1.2   Powers of cycles in ε-intransitive tournaments

Generalizing the previous results about path powers to cycle powers is not possible. For example, the transitive tournament does not contain any directed cycle at all. One can only say something about cycles in tournament by making further structural assumptions.

An old result of Bollobás and Häggkvist [2] says that for every k and $\varepsilon > 0$, any tournament on sufficiently many vertices with minimum semi-degree at least $(1/4 + \varepsilon)n$ contains a k-th power of a Hamilton cycle and this is tight up to the $o(1)$ error term. Recently, Draganic, Munhá Correia, and Sudakov [6] were able to find an almost tight bound for the error term.

We complement these results by showing that much milder assumptions are sufficient for the existence of long (linear-length) k-th powers of cycles in tournaments.

We say that an n-vertex tournament is ε-*intransitive* if no matter how we order its vertices, there are always at least εn^2 backward edges. This is a way to measure how far the tournament is from being transitive. Note that a tournament cannot be γ-intransitive for $\gamma > 1/4$, because for any vertex ordering τ, either τ or its reverse induces fewer than $n^2/4$ backward edges. This definition turns out to be quite important and is a sort equivalent of edge density for tournaments. Indeed, as a by-product of a result of Chung and Graham [4], it follows that for any tournament H, there is $\varepsilon(H) \geq 0$ such that every sufficiently large tournament T which is $(\varepsilon(H) + o(1))$-intransitive must contain a copy of H. Fox and Sudakov [9] showed that $\varepsilon(F) = 0$ works for any transitive blow-up F of a directed triangle.

We prove the following result.

Theorem 1.3. *Let $0 < \varepsilon < 1/4$, then every ε-intransitive tournament on $n \geq \varepsilon^{-10^5 k}$ vertices contains the k-th power of a cycle of length at least $\varepsilon n/1500$.*

Furthermore, in Section 4, we show both bounds are essentially tight.

The rest of the paper is organized as follows. In the next section, we develop some machinery that we shall need for our proofs. Theorem 1.2, on partitioning into powers of cycles, is proved in Section 3; and Theorem 1.3, on finding powers of long cycles, is proved in Section 4.

We note that we have not tried to optimize absolute constants in our results.

2   Tools

A tournament is transitive if its vertices can be ordered so that every edge is oriented from its smaller endpoint to its larger endpoint. Note that every subtournament of a transitive tournament

is also transitive.

We use $[n]$ to denote the set of integers $\{1, \ldots, n\}$. T is a tournament. For two vertex sets A, B in T, $e(A, B)$ denotes the number of edges from A to B, and $d(A, B) = \frac{e(A,B)}{|A||B|}$ denotes the density of such edges. Of course, $d(A, B) = 1$ if and only if A and B are disjoint and there is an edge from every vertex in A to every vertex in B. We will denote this by $A \Rightarrow B$.

For convenience, our notion of cycles will include singleton vertices and edges as degenerate cases on one or two vertices. We will also drop floor and ceiling signs when they are not essential.

2.1 Extremal lemmas

Let us start with two well-known facts that we will need for the proof. The first is a basic property of tournaments.

Proposition 2.1. *Every tournament on at least 2^k vertices contains a k-vertex transitive subtournament.*

We will also need the following simple observation about edge densities.

Proposition 2.2. *Let $G = (A \cup B, E)$ be a bipartite graph with $|E| = \beta|A||B|$ edges for some $0 < \beta \leq 1$. Then for every $\varepsilon \geq 0$, A contains at least $(\beta - \varepsilon)|A|$ vertices of degree at least $\varepsilon|B|$.*

Next, we formulate a special case of the Kővári-Sós-Turán theorem, which will be a key tool in our arguments. We give a short proof for completeness.

Lemma 2.3. *Let $G = (A \cup B, E)$ be a bipartite graph such that for some $0 < \beta \leq 1/2$, every vertex in A has at least $\beta|B|$ neighbours in B. If $|A| \geq k/\beta$, then A contains a subset X of size k with at least $\beta^{4k}|B|$ common neighbours in B.*

Proof. We may assume that $|A| = \lceil k/\beta \rceil$. A vertex $v \in B$ sees $\binom{d(v)}{k}$ different k-subsets of A in its neighbourhood. This gives a total of

$$\sum_{v \in B} \binom{d(v)}{k} \geq |B| \binom{\sum d(v)/|B|}{k} \geq |B| \binom{\beta|A|}{k} \geq |B|$$

k-sets over all vertices of B, where we used Jensen's inequality, and that $\sum d(v) \geq \beta|A||B| \geq k|B|$. But there are only $\binom{|A|}{k} \leq \binom{2k/\beta}{k} \leq (\frac{2k/\beta}{k/e})^k \leq \beta^{-4k}$ different k-sets in A, so one of them must have at least $\beta^{4k}|B|$ common neighbours in $|B|$. $\square$

We will also need the following strengthening for tournaments, which comes as a simple application of the dependent random choice method.

Lemma 2.4. *Let A, B be disjoint sets in a tournament. If $d(A, B) \geq \beta$ for some $0 < \beta \leq 1/2$, and $|A|, |B| \geq \beta^{-5k}$, then there are subsets $X \subseteq A$ and $Y \subseteq B$ of size $|X| \geq \beta^{4k}|A|$ and $|Y| = k$ such that Y induces a transitive tournament, and $X \Rightarrow Y$.*

Proof. Let S be a random set of $s = \left\lfloor \frac{1}{2} \log_{1/\beta} |B| \right\rfloor$ independently and uniformly sampled vertices in A, and let $T \subseteq B$ be the set of common outneighbours of S. Then $\mathbb{E}[|T|] = \sum_{v \in B} d(A, v)^s \geq \beta^s |B| \geq |B|^{1/2}$ by Jensen's inequality. Let Z be the number of k-subsets in T with fewer than $\beta^{4k}|A|$ common inneighbours in A. The probability that a given k-subset $Q \subseteq B$ with $\gamma|A|$ common inneighbours in A satisfies $Q \subseteq T$ is γ^s, so we have $\mathbb{E}[Z] = \binom{|B|}{k} \beta^{4k \cdot s} \leq (\beta^{4s}|B|)^k \leq 1$, as $4s \geq \log_{1/\beta}(|B|)$. Hence $\mathbb{E}[|T| - Z] \geq |B|^{1/2} - 1$. Let us fix a random sample where $|T| - Z \geq |B|^{1/2} - 1$.

By deleting a vertex of each k-subset counted by Z from the set T, we obtain a subset $W \subseteq T$ of size at least $|B|^{1/2} - 1$ such that all k-subsets in W have at least $\beta^{4k}|A|$ common inneighbours. As $|W| \geq 2^{2k}$, we can use Proposition 2.1 to find a k-set $Y \subseteq W$ that induces a transitive subtournament. We can choose X to be the common inneighbourhood of Y. $\qquad \square$

Corollary 2.5. *Let A, B be disjoint sets in a tournament. If $d(A, B) \geq \beta$ for some $0 < \beta \leq 1/2$, and $|A|, |B| \geq \beta^{-5k}$, then there are sets $X \subseteq A$ and $Y \subseteq B$ that induce transitive tournaments of size k and satisfy $X \rightrightarrows Y$.*

Proof. This is immediate from Lemma 2.4 noting $\beta^{4k}|A| \geq 2^k$ and applying Proposition 2.1. $\qquad \square$

Our final tool in this section describes a sufficient condition when we can repeatedly apply the previous lemmas to construct a sequence of transitive tournaments.

Lemma 2.6. *Let $A_1, \ldots, A_t$ be disjoint vertex sets of size at least 2^{10k} in a tournament, and suppose there is no $i \in [t-1]$ and sets $B \subseteq A_i$ and $B' \subseteq A_{i+1}$ such that B and B' induce transitive tournaments of size k, and $B' \rightrightarrows B$. Then there are sets $X_i \subseteq A_i$ of size k such that each X_i induces a transitive subtournament, and $X_1 \rightrightarrows \cdots \rightrightarrows X_t$.*

Proof. We proceed by induction on t under the weaker assumption that $|A_t| \geq 2^{6k}$. The $t = 1$ case easily follows from Proposition 2.1, so we assume $t \geq 2$.

If $d(A_t, A_{t-1}) \geq 1/2$, then we can apply Corollary 2.5 to find k-sets $B \subseteq A_{t-1}$ and $B' \subseteq A_t$ that induce transitive tournaments and satisfy $B' \rightrightarrows B$, contradicting our assumption.

So $d(A_t, A_{t-1}) \leq 1/2$, i.e., $d(A_{t-1}, A_t) \geq 1/2$, and we can apply Lemma 2.4 to find a k-subset $X_t \subseteq A_t$ that induces a transitive tournament, and another subset $A'_{t-1} \subseteq A_{t-1}$ of size $|A'_{t-1}| \geq |A_{t-1}|/2^{4k} \geq 2^{6k}$ such that $A'_{t-1} \rightrightarrows X_t$. Applying the induction hypothesis to the sets $A_1, \ldots, A_{t-2}, A'_{t-1}$ yields the result. $\qquad \square$

2.2 Median orderings

A *median ordering* of a tournament T is an ordering $v_1 \prec \cdots \prec v_n$ of the vertices that maximizes the number of forward edges, i.e., edges of the form $v_i v_j$ with $i < j$. Studying such orderings have been very helpful in understanding the structure of tournaments. An interval of vertices with

respect to a median ordering is a sequence $V[i,j] = \{v_i, \ldots, v_j\}$ for some $i \leq j$. For two vertex subsets X, Y we write $X \prec Y$ to denote that $x \prec y$ for any $x \in X$ and $y \in Y$.

Lemma 2.7. *Suppose that in a median ordering of some tournament T, an interval is split into subintervals $A_0 \prec \cdots \prec A_t$ of size m each. Then every vertex $v \in A_0$ has at least $\frac{t-2}{2}m$ outneighbours in $A = A_1 \cup \cdots \cup A_{t-1}$, and every vertex in $v \in A_t$ has at least $\frac{t-2}{2}m$ inneighbours in A.*

Proof. Note that $|A| = (t-1)m$, so if some $v \in A_0$ has fewer than $\frac{t-2}{2}m$ outneighbours in A, then v has at least m fewer outneighbours in A than inneighbours. As v has only $m-1$ neighbours in A_0, moving v to the end of the interval $A_0 \cup A$ (so that $A_{t-1} \prec v \prec A_t$) is guaranteed to increase the number of forward edges. This contradicts our assumption on the ordering. The statement about A_t can be proved analogously. $\square$

Using this lemma, we can obtain an ordered variant of Lemma 2.3. It will be helpful to allow a set F of 'forbidden' vertices.

Lemma 2.8. *Suppose that in a median ordering of some tournament T, an interval A is split into three subintervals $A_0 \cup A_1 \cup A_2$ of size m each, and let $F \subseteq A$ be a set of at most $m/4$ forbidden vertices. Then every set $A_0' \subseteq A_0$ of size $8k$ contains a subset X of size k such that for some $i \in [2]$, X has at least $m/2^{12k}$ common outneighbours in $A_i \setminus F$.*

Proof. By Lemma 2.7, every vertex in A_0' has at least $m/2$ outneighbours in $A_1 \cup A_2$, at least $m/4$ of which lie in $(A_1 \cup A_2) \setminus F$. Applying Lemma 2.3 with $\beta = 1/8$, we get a set $X \subseteq A_0'$ of size k with at least $2m/2^{12k}$ common outneighbours in $(A_1 \cup A_2) \setminus F$. Of course, at least $m/2^{12k}$ of these common outneighbours must lie in the same set $A_i \setminus F$ for some $i \in [2]$. $\square$

We can now easily deduce an ordered variant of Lemma 2.6. In this case there is no need for the assumption on backward edges: the median ordering provides all the structure we need.

Lemma 2.9. *Let $k > 0$, and suppose that in a median ordering of some tournament T, an interval is split into subintervals $A_1 \prec \cdots \prec A_t$ of size $m \geq 2^{20k}$ each. Suppose we have a set F of vertices F such that $|F \cap A_i| \leq m/8$ for every $i \in [t]$, and $X \subseteq A_1 \setminus F$ is a set of size $8k$ that induces a transitive tournament.*

Then there are sets $X_1 \rightrightarrows \cdots \rightrightarrows X_s$ such that $X_1 \subseteq X$, and each X_i induces a transitive tournament of size k in $A_{j_i} \setminus F$, where the indices $1 = j_1 < \cdots < j_s$ satisfy $j_{i+1} \in \{j_i + 1, j_i + 2\}$ for every i, and $j_s \in \{t-1, t\}$.

Proof. Set $j_1 = 1$ and $X_1' = X$. We repeat the following step for every $i = 1, 2, \ldots$ as long as $j_i < t - 1$.

Applying Lemma 2.8 to the interval $A_{j_i} \cup A_{j_i+1} \cup A_{j_i+2}$ with $A'_0 = X'_i$ and forbidden vertices $(F \cap A_{j_i+1}) \cup (F \cap A_{j_i+2})$ gives a k-set $X_i \subseteq X'_i$ with at least $m/2^{12k} \geq 2^{8k}$ common outneighbours in $A_{j_i+1} \setminus F$ for some $j_{i+1} \in \{j_i + 1, j_i + 2\}$. By Proposition 2.1, we can find a subset X'_{i+1} of $8k$ common outneighbours that induce a transitive subtournament.

This process stops with some $j_s \in \{t - 1, t\}$, and we can then choose any k-subset $X_s \subseteq X'_s$ so that $X_1, \ldots, X_s$ satisfy the statement. $\qquad\square$

The next lemma is a key component of our arguments, and perhaps the most technical result in the paper. It says that if two sets are far enough apart in a median ordering, then we can connect them with a blowup of a path.

Lemma 2.10. *Suppose that in a median ordering of some tournament T, an interval is split into subintervals $A_0 \prec \cdots \prec A_t$ of size $m \geq 100 \cdot 2^{40400k}$ each, where $t \geq 50$. Let $A'_0 \subseteq A_0$ and $A'_t \subseteq A_t$ be subsets of size at least 2^{4001k}, and let F be a set of at most $m/2$ vertices in $A = A_1 \cup \cdots \cup A_{t-1}$. Then there is $s \leq 3$ and disjoint sets $X_0 \subseteq A'_0$ and $X_1, \ldots, X_{s-1} \subseteq A \setminus F$ and $X_s \subseteq A'_t$, such that $X_0 \Rightarrow \cdots \Rightarrow X_s$, and each X_i induces a transitive tournament of size k.*

Proof. Let $\varepsilon = 1/100$, and define $A^I \subseteq A \setminus F$ as the set of vertices with at least $\varepsilon|A'_0|$ inneighbours in A'_0, and $A^O \subseteq A \setminus F$ as the set of vertices with at least $\varepsilon|A'_t|$ outneighbours in A'_t. We claim that $|A^I|, |A^O| \geq (\frac{t-3}{2} - (t-1)\varepsilon)m$. Indeed, there are at least $\frac{t-2}{2}m|A'_0| = \frac{t-2}{2t-2}|A||A'_0|$ edges from A'_0 to A by Lemma 2.7, so Proposition 2.2 applied to the bipartite graph induced by these edges with $\beta = \frac{t-2}{2t-2}$ gives $(\frac{t-2}{2} - (t-1)\varepsilon)m$ vertices in A with at least $\varepsilon|A'_0|$ inneighbours in A'_0. Excluding the vertices of F yields the lower bound on $|A^I|$. The bound on $|A^O|$ is analogous.

If A^I and A^O share at least $2^{k/\varepsilon^2}$ vertices, then we are done with $s = 2$ as follows. By Proposition 2.1, there is a set $Y \subseteq A^I \cap A^O$ of size k/ε^2 that induces a transitive tournament. Then every vertex of Y has at least $\varepsilon|A'_t|$ outneighbours in A'_t, so we can apply Lemma 2.3 to the bipartite graph of the edges from Y to A'_t to get sets $Y' \subseteq Y$ and $\tilde{A}_t \subseteq A'_t$ such that $|Y'| \geq k/\varepsilon$ and $|\tilde{A}_t| \geq \varepsilon^{4k/\varepsilon}|A'_t| \geq |A'_t|/2^{4000k} \geq 2^k$ with $Y' \Rightarrow \tilde{A}$. Once again, every vertex of Y' has at least $\varepsilon|A'_0|$ inneighbours in A'_0, so Lemma 2.3 gives $Y'' \subseteq Y'$ and $\tilde{A}_0 \subseteq A'_0$ such that $|Y''| \geq k$ and $|\tilde{A}_0| \geq \varepsilon^{4k}|A'_0| \geq 2^k$ with $\tilde{A}_0 \Rightarrow Y''$. We can then choose X_0 and X_2 to be k-subsets of $\tilde{A}_0$ and $\tilde{A}_t$ that induce transitive tournaments, and X_1 to be Y''.

So we may assume that $|A^I \cap A^O| < 2^{k/\varepsilon^2} < \varepsilon m$. Let $\tilde{A}^I = A^I \setminus A^O$. Then $\tilde{A}^I$ is disjoint from A^O and has size at least $(\frac{t-3}{2} - t\varepsilon)m$. Also, $|\tilde{A}^I \cup A^O| > (t-3)m - (2t-1)\varepsilon m$, so the set $A^X = A \setminus (A^I \cup A^O)$ of leftover vertices (including F) has size at most $2m + (2t - 1)\varepsilon m$.

If $d(\tilde{A}^I, A^O) > \varepsilon$, then we can conclude the argument with $s = 3$ similarly to the previous case: As $|\tilde{A}^I|, |A^O| \geq m \geq \varepsilon^{-5k/\varepsilon}$, we can apply Corollary 2.5 to find sets $Y \subseteq \tilde{A}^I$ and $Z \subseteq A^O$ that induce transitive tournaments of size k/ε and satisfy $Y \Rightarrow Z$. Now every vertex of Y has at least $\varepsilon|A'_0|$ inneighbours in A'_0 and every vertex of Z has at least $\varepsilon|A'_t|$ outneighbours in A'_t, so two independent applications of Lemma 2.3 gives k-sets $Y' \subseteq Y$ and $Z' \subseteq Z$ as well as sets $\tilde{A}_0 \subseteq A'_0$

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and $\tilde{A}_t \subseteq A'_t$ of size $|\tilde{A}_0| \geq \varepsilon^{4k}|A'_0| \geq 2^k$ and $|\tilde{A}_t| \geq \varepsilon^{4k}|A'_t| \geq 2^k$ such that $\tilde{A}_0 \rightrightarrows Y' \rightrightarrows Z' \rightrightarrows \tilde{A}_t$. We then choose X_0 and X_3 to be transitive k-subsets of $\tilde{A}_0$ and $\tilde{A}_t$, respectively, and set $X_1 = Y'$ and $X_2 = Z'$.

So let us also assume that $d(\tilde{A}^I, A^O) \leq \varepsilon$. We will show that we could not have started with a median ordering in this case. Let $B = A_1 \cup \cdots \cup A_{\lfloor (t-1)/2 \rfloor}$ and $C = A_{t-\lfloor (t-1)/2 \rfloor} \cup \cdots \cup A_{t-1}$, and let us first bound the size of $B^I = B \cap A^I$ and $C^O = C \cap A^O$. By Lemma 2.7, there are at least $\frac{t-6}{4}|A'_0|m$ edges from A'_0 to B and $\frac{t-6}{4}|A'_t|m$ edges from C to A'_t, so Proposition 2.2 implies $|B^I|, |C^I| \geq (\frac{t-6}{4} - (t-1)\varepsilon)m$. We then also see that $\tilde{B}^I = B \cap \tilde{A}^I$ has size at least $(\frac{t-6}{4} - t\varepsilon)m$.

Let us now consider the ordering on the vertices of the tournament that moves all vertices in $\tilde{A}^I$ to the end of the interval A, without affecting the relative order of vertices in any other way. So $A'_0 \prec (A^O \cup A^X) \prec \tilde{A}^I \prec A'_t$ in this new ordering. With this reordering, we may lose up to $|\tilde{A}^I||A^X|$ forward edges between $\tilde{A}^I$ and A^X, and up to $\varepsilon|\tilde{A}^I||A^O|$ between $\tilde{A}^I$ and A^O, but we will surely gain at least $|\tilde{B}^I||C^O| - \varepsilon|\tilde{A}^I||A^O|$ forward edges between $\tilde{B}^I$ and C^O. With $t \geq 50$ and $\varepsilon = 1/100$, we can bound the terms as follows:

$$|\tilde{B}^I||C^O| \geq \left(\frac{t-6}{4} - t\varepsilon \right) \left(\frac{t-6}{4} - (t-1)\varepsilon \right) m^2 \geq \frac{(t-2)^2}{25} \cdot m^2$$

$$|\tilde{A}^I||A^O| \leq \left(\frac{t-2}{2} - t\varepsilon \right) \left(\frac{t-2}{2} - (t-1)\varepsilon \right) m^2 \leq \frac{(t-2)^2}{4} \cdot m^2$$

$$|\tilde{A}^I||A^X| \leq \left(\frac{t-2}{2} - t\varepsilon \right) (2 + (2t-1)\varepsilon) m^2 \leq \frac{t-2}{2} \cdot \frac{t-2}{15} \cdot m^2.$$

This means that the new ordering has at least

$$|\tilde{B}^I||C^O| - 2\varepsilon|\tilde{A}^I||A^O| - |\tilde{A}^I||A^X| \geq \left(\frac{1}{25} - \frac{1}{200} - \frac{1}{30} \right)(t-2)^2 m^2 \geq \frac{(t-2)^2 m^2}{600}$$

more forward edges than the median ordering we started with, which is a contradiction. $\qquad \square$

It will be more convenient for us to apply the previous lemma via the following statement.

Corollary 2.11. *Suppose that in a median ordering of some tournament T, an interval is split into subintervals $A_0 \prec \cdots \prec A_t$ of size $m \geq 100 \cdot 2^{40400k}$ each, where $t \geq 60$. Let $X \subseteq A_0$ and $X' \subseteq A_t$ be $4k$-subsets that induce transitive tournaments, and let F be a set of at most $m/2$ forbidden vertices in $A = A_1 \cup \cdots \cup A_{t-1}$. Then there is $s \leq 5$ and disjoint sets $X_0 \subseteq X$ and $X_1, \ldots, X_{s-1} \subseteq A \setminus F$ and $X_s \subseteq X'$, such that $X_0 \rightrightarrows \cdots \rightrightarrows X_s$, and each X_i induces a transitive tournament of size k.*

Proof. By Lemma 2.7, every vertex in X has at least $3m/2$ outneighbours in $A_1 \cup A_2 \cup A_3 \cup A_4$, at least m of which are not in F. So we can apply Lemma 2.3 with $A = X$, $B = (A_1 \cup A_2 \cup A_3 \cup A_4) \setminus F$ and $\beta = 1/4$ to find a k-subset $X_0 \subseteq X$ with at least $3m/2^{8k}$ common outneighbours in B. At least $m/2^{10k} \geq 2^{20001k}$ of these are in the same A_i with $i \leq 4$, let us denote them by A'_i.

The same argument can be applied to X' from the other direction, so we similarly get a k-set $X_t \subseteq X'$ with and another set $A'_{i'} \subseteq A_{i'} \setminus F$ for $i' \geq t - 4$ such that $|A'_{i'}| \geq 2^{2000k}$ and $A'_{i'} \rightrightarrows X_t$.

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But then we can apply Lemma 2.10 to the interval $A_i \cup \cdots \cup A_{i'}$ to find transitive k-sets $Y_0 \rightrightarrows \cdots \rightrightarrows Y_{s'}$ with $s' \leq 3$ such that $Y_0 \subseteq A'_i$, $Y_1, \ldots, Y_{s-1} \subseteq (A_{i+1} \cup \cdots \cup A_{i'-1}) \setminus F$, and $Y_{s'} \subseteq A'_{i'}$. Relabeling $X_0, Y_0, \ldots, Y_{s'}, X_t$ as $X_0, \ldots, X_s$ with $s = s' + 2$ yields the sets we were looking for. $\square$
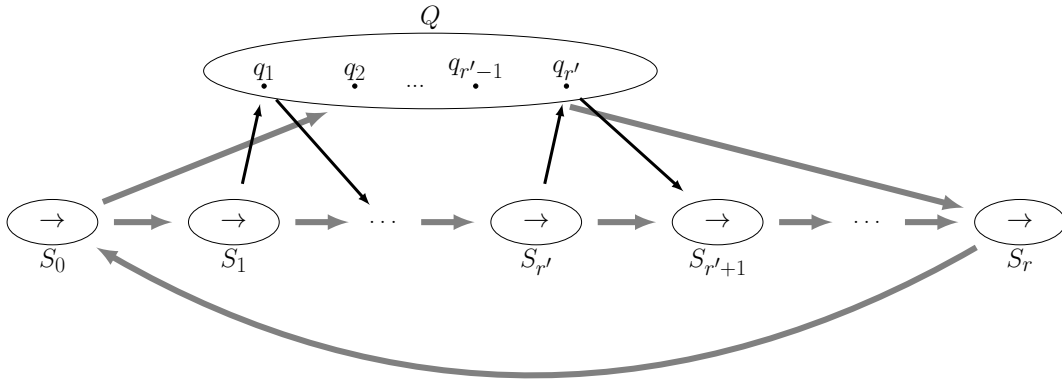
2.3   Absorbers

Definition 2.12. *We say that a tournament H is a k-absorber if there is an r'-set $Q = \{q_1, \ldots, q_{r'}\}$ and a partition $V(H) = S_0 \cup \cdots \cup S_r \cup Q$ such that*

(i) *$|Q| = r' = 2^{10k}$ and $r > r'$.*

(ii) *$S_0 \rightrightarrows S_1 \rightrightarrows \cdots \rightrightarrows S_r \rightrightarrows S_0$, and each S_i induces a transitive tournament of size $2k$.*

(iii) *$S_0 \rightrightarrows Q \rightrightarrows S_{r'+1}$, and $S_i \rightrightarrows q_i \rightrightarrows S_{i+1}$ for $i \in [r']$.*

We will refer to the set $Q = Q(H)$ as the absorbing part of H.

It is easy to see that every k-absorber H contains the k-th power of a Hamilton cycle. In fact, its structure is more robust. The key property of these absorbers is captured by the following statement.

Proposition 2.13. *Let H be a k-absorber, and $X, Y \subseteq Q(H)$ be two vertex sets of size $2k$ in the absorbing part of H that induce transitive subtournaments. Then H contains the k-th power of a Hamilton path whose first k vertices are in Y and last k vertices are in X.*

Proof. Let $Y_0 \subseteq Y$ and $X_0 \subseteq X \setminus Y_0$ be arbitrary disjoint subsets of size k. We will cover the vertices of H by the k-th power of a path whose first k vertices are the ones in Y_0 and last k vertices are the ones in X_0.

Let us split each vertex set S_i of the k-absorber H arbitrarily into two k-sets S_i^1 and S_i^2. The backbone of the path power is given by the relations

$$Y_0 \rightrightarrows S_{r'}^1 \rightrightarrows \cdots \rightrightarrows S_r^1 \rightrightarrows S_0^1 \rightrightarrows S_1 \rightrightarrows \cdots \rightrightarrows S_{r'-1} \rightrightarrows S_{r'}^2 \rightrightarrows \cdots \rightrightarrows S_r^2 \rightrightarrows S_0^2 \rightrightarrows X_0.$$

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As each of these sets induces a transitive subtournament of size at least k, we can combine them into the k-th power of a path touching all vertices in the above order. Finally, we can insert any leftover vertex $q_i \in Q \setminus (X_0 \cup Y_0)$ between S_{i-1} and S_i (or between S_0^1 and S_1 when $i = 1$, and between $S_{r'-1}$ and $S_{r'}^2$ when $i = r'$) by the assumptions on k-absorbers. $\qquad\square$

Lemma 2.14. *Let $H_1, \ldots, H_s$ be vertex-disjoint k-absorbers in a tournament T. Then T contains the k-th power of a directed path with vertex set $V(H_1) \cup \cdots \cup V(H_s)$.*

Proof. Consider the auxiliary tournament on vertex set $[s]$, where ij is an edge if $d(Q(H_i), Q(H_j)) \geq 1/2$ (keeping only one of ij and ji if $d(Q(H_i), Q(H_j)) = 1/2$). Like every tournament, this must contain a Hamilton path, so we may assume that $1, \ldots, s$ is a directed path.

Now for every $i \in [s-1]$, we can apply Corollary 2.5 to $Q(H_i)$ and $Q(H_{i+1})$ with $\beta = 1/2$ to get $2k$-sets $X_i \subseteq Q(H_i)$ and $Y_{i+1} \subseteq Q(H_{i+1})$ that induce transitive tournaments and satisfy $X_i \rightrightarrows Y_{i+1}$. Let us set $Y_1 = X_1$ and $X_s = Y_s$. Then by Proposition 2.13 each H_i contains the k-th power of some spanning path P_i that starts with k vertices in Y_i and ends with k vertices in X_i. As $X_i \rightrightarrows Y_{i+1}$, the concatenation of these paths satisfies our requirements. $\qquad\square$

The next lemma is our tool for finding k-absorbers in the tournament. Its proof heavily uses the underlying median ordering, but at this point, it is simply a combination of previously established lemmas.

Lemma 2.15. *Suppose that in a median ordering of some tournament T, an interval is split into subintervals $A_0 \prec \cdots \prec A_t$ of size $m \geq 2^{81000k}$ each, where $t \geq 80$. If there are sets $X_0 \subseteq A_0$ and $X_t \subseteq A_t$ of size $|X_0|, |X_t| \geq 8k$ that both induce transitive subtournaments and $X_t \rightrightarrows X_0$, then T contains a k-absorber.*

Proof. By Lemma 2.7, every vertex in X_0 has at least $m/2$ outneighbours in $A_1 \cup A_2$. Then Lemma 2.3 with $\beta = 1/4$ gives subsets $X_0' \subseteq X_0$ and $Y \subseteq A_1 \cup A_2$ of size $|X_0'| \geq 2k$ and $|Y| \geq 2m/2^{16k}$ such that $X_0' \rightrightarrows Y$. At least half of Y must lie in the same A_{i_1} for some $i_1 = 1$ or $i_1 = 2$, so $Y_{i_1} = Y \cap A_{i_1}$ has size $|Y_{i_1}| \geq m/2^{16k}$.

Similarly, Lemma 2.7 implies that every vertex in Y_{i_1} has at least $m/2$ outneighbours in $A_{i_1+1} \cup A_{i_1+2}$, so $d(Y_{i_1}, A_{i_2}) \geq 1/4$ for $i_2 = i_1 + 1$ or $i_2 = i_1 + 2$. We can thus apply Lemma 2.4 with $\beta = 1/4$ to find sets $Y_{i_1}' \subseteq Y_{i_1}$ and $X_{i_2} \subseteq A_{i_2}$ of size $|Y_{i_1}'| \geq |Y_{i_1}|/2^{64k} \geq m/2^{80k}$ and $|X_{i_2}| \geq 8k$ such that X_{i_2} induces a transitive tournament in T, and $Y_{i_2}' \rightrightarrows X_{i_2}$.

We will construct the k-absorber as follows. We set $S_0 = X_0'$, and $S_{r'+1}$ (where $r' = 2^{10k}$) will be a k-subset of X_{i_2} defined later. We find the sets $S_1, \ldots, S_{r'}$ and Q in Y_{i_1}' by applying Theorem 1.1 to obtain a path $v_1 v_2 \ldots v_\ell$ in Y_{i_1}' with $\ell = r'(2k+1) \leq m/2^{90k} \leq |Y_{i_1}'|/2^{10k}$ vertices, whose $4k$-th power is in the tournament. Let us define $S_i = \{v_{i(2k+1)-1}, \ldots, v_{i(2k+1)-2k}\}$ and $q_i = v_{i(2k+1)}$ for every $i = [r']$, and let $Q = \{q_1, \ldots, q_{r'}\}$. Then we know that each S_i induces a transitive tournament

of size $2k$, and we have $S_0 \rightrightarrows Q \rightrightarrows S_{r'+1}$ and $S_0 \rightrightarrows S_1 \rightrightarrows \ldots \rightrightarrows S_{r'+1}$, as well as $S_i \rightrightarrows q_i \rightrightarrows S_{i+1}$ for every $i \in [r']$.

The crucial part of the construction is closing the cycle. To do so, we apply Corollary 2.11 to the interval $A_{i_2} \cup \cdots \cup A_t$ with $X = X_{i_2}$, $X' = X_t$, $F = \varnothing$, and $2k$ in the place of k. As $t - i_2 \geq 60$, $|X_{i_2}|, |X_t| \geq 8k$, and each A_i has size $m \geq 2^{81000k}$, there are sets $S_{r'+1} \subseteq X_{i_2}$, $S_{r'+s+1} \subseteq X_t$ and $S_{r'+2}, \ldots, S_{r'+s} \subseteq A_{i_2+1} \cup \cdots \cup A_{t-1}$ such that each of $S_{r'+1}, \ldots, S_{r'+s+1}$ has size $2k$, induces a transitive subtournament, and $S_{r'+1} \rightrightarrows \ldots \rightrightarrows S_{r'+s+1}$.

By construction, we have $S_{r'+s+1} \rightrightarrows S_0$, so we are done with $r = r' + s + 1$. $\qquad\qquad \square$

3 Proof of the partitioning theorem

Theorem 3.1. *Every n-vertex tournament T can be covered with at most $2^{10^5 k}$ vertex-disjoint k-th powers of directed paths.*

Proof. Let $H_1, \ldots, H_s$ be a maximal collection of vertex-disjoint k-absorbers in T. By Proposition 2.13, the vertices $V(H_1) \cup \cdots \cup V(H_s)$ can be covered by the k-th power of a single directed path P. Let $T' = T - (H_1 \cup \cdots \cup H_s)$ be the subtournament induced by the remaining vertices, and let $n' = |V(T')|$.

Take a median ordering $\prec$ of T', and let us split the vertices into subintervals of size $m = 2^{81000k}$. More precisely, we split $V(T')$ into intervals $A_0 \prec A_1 \prec \cdots \prec A_t$ where $t = \lfloor n'/m \rfloor$, $|A_i| = m$ for $i \in [t]$, and $|A_0| < m$. We can afford to use a single path for each vertex in A_0, so let us focus on covering $A_1 \cup \cdots \cup A_t$.

As T' does not contain any k-absorbers, Lemma 2.15 tells us that there cannot be indices $i, i' \in [t]$ such that $i' \geq i + 80$, and $B' \rightrightarrows B$ for some $8k$-sets $B \subseteq A_i$ and $B' \subseteq A_{i'}$ that both induce transitive subtournaments. We can therefore apply Lemma 2.6 to the sets $A_1, A_{81}, A_{161}, \ldots$ to find $8k$-subsets $X_1 \subseteq A_1, X_{81} \subseteq A_{81}, X_{161} \subseteq A_{161}, \ldots$ that each induce transitive subtournaments, and $X_1 \rightrightarrows X_{81} \rightrightarrows X_{161} \rightrightarrows \ldots$. In fact, we can then apply it again to the sets $A_1 \setminus X_1, A_{81} \setminus X_{81}, A_{161} \setminus X_{161}, \ldots$ to find another sequence of such $8k$-sets $X_1' \subseteq A_1, X_{81}' \subseteq A_{81}, X_{161}' \subseteq A_{161}, \ldots$ disjoint from the X_i, and repeat this as long as each A_i contains at least 2^{80k} unused vertices.

This way we can find subsets $X_i^h \subseteq A_i$ of size $8k$ for every $i \in [t]$ and $h = 1, \ldots, r$ for some r, such that these sets are pairwise disjoint, each of them induces a transitive subtournament, $X_w^h \rightrightarrows X_{w+80}^h \rightrightarrows X_{w+160}^h \rightrightarrows \ldots$ for every $h \in [r]$ and $w \in [80]$, and $2^{81000k} - 8kr < 2^{80k}$, i.e., the set $U_i \subseteq A_i$ of vertices left uncovered by the X_i^h has size $|U_i| < 2^{80k}$ for every $i \in [r]$. Note that each sequence $X_w^h, X_{w+80}^h, \ldots$ contains the k-th power of a *spanning* path, and this holds even if we remove at most $7k$ arbitrary vertices from each X_i^h. It is therefore enough to cover the remaining vertices $U_1 \cup \cdots \cup U_t$ with k-th powers of paths, and we can even use some vertices from the X_i^h for this purpose.

Let $m' = 2^{81000k} - 8kr < 2^{80k}$ be the size of each U_i, and denote its vertices as $U_i = \{u_{i,1}, \ldots, u_{i,m'}\}$. By Lemma 2.7, $u_{i,j}$ has at least $79m/2$ outneighbours in the interval $A_{i+1} \cup \cdots \cup A_{i+80}$, at least $m/2$ of which must be in $A_{i+40} \cup \cdots \cup A_{i+80}$. In particular, at least $m/160$ of these outneighbours must fall in some $A_{\alpha(i,j)}$ with $i + 40 \leq \alpha(i,j) \leq i + 80$, and similarly, $u_{i,j}$ has at least $m/160$ inneighbours in some $A_{\beta(i,j)}$ with $i - 80 \leq \beta(i,j) \leq i - 40$. Note that a given index $\gamma \in [t]$ can appear as $\alpha(i,j)$ or $\beta(i,j)$ for no more than $160m'$ different vertices $u_{i,j}$, so we can choose subsets of in- and outneighbourhoods of size $m/(160^2 m') \geq m/2^{100k}$ so that they are all disjoint. Also, each of these subsets must come from at least $(m/2^{100k})/8k \geq 2^{80000k}$ different $8k$-sets X_γ^h. All in all, for every $u_{i,j}$, we can find a set $N_{i,j}^+ \subseteq A_{\alpha(i,j)}$ of outneighbours and another subset $N_{i,j}^- \subseteq A_{\beta(i,j)}$ of inneighbours for $u_{i,j}$ such that these are pairwise disjoint subsets of size at least 2^{80000k} each, and they together contain at most one vertex from each $8k$-set X_γ^h.

Recall that there are no indices i, i' with $i' \geq i + 80$ such that $B' \rightrightarrows B$ for some $8k$-sets $B \subseteq A_i$ and $B' \subseteq A_{i'}$. As $\alpha(i,j) \geq \beta(i,j) + 80$ and $\beta(i+240,j) \geq \alpha(i,j) + 80$ for every i, j, this means that we can apply Lemma 2.6 to the sequence $N_{w,j}^-, N_{w,j}^+, N_{w+240,j}^-, N_{w+240,j}^+, N_{w+480,j}^-, N_{w+480,j}^+, \ldots$ to find $8k$-sets $Y_{i,j}^- \subseteq N_{i,j}^-$ and $Y_{i,j}^+ \subseteq N_{i,j}^+$ that induce transitive tournaments, and satisfy $Y_{w,j}^- \rightrightarrows Y_{w,j}^+ \rightrightarrows Y_{w+240,j}^- \rightrightarrows Y_{w+240,j}^+ \rightrightarrows Y_{w+480,j}^- \rightrightarrows Y_{w+480,j}^+ \rightrightarrows \ldots$, for every $w \in [240]$ and $j \in [m']$. As $Y_{i,j}^- \rightrightarrows u_{i,j} \rightrightarrows Y_{i,j}^+$, we can cover the vertices $Y_{w,j}^- \rightrightarrows u_{w,j} \rightrightarrows Y_{w,j}^+ \rightrightarrows Y_{w+240,j}^- \rightrightarrows u_{w+240,j} \rightrightarrows Y_{w+240,j}^+ \rightrightarrows Y_{w+480,j}^- \rightrightarrows u_{w+480,j} \rightrightarrows Y_{w+480,j}^+ \rightrightarrows \ldots$ with the k-th power of a single path.

Putting everything together, we see that we can cover $U_1 \cup \cdots \cup U_t$ (and the vertices in $Y_{i,j}^\pm$) with the k-th powers of $240m'$ vertex-disjoint paths. As noted above, these paths use at most one vertex from each X_i^h, so we can cover the remaining vertices in T' with the k-th powers of $80r$ vertex-disjoint paths, plus one path for each vertex in A_0. Together with P, we obtain no more than $240m' + 80r + m + 1 \leq 2^{90k} + 2^{81010k} + 2^{81000k} + 1 < 2^{10^5 k}$ vertex disjoint paths whose k-th powers cover all vertices of T. $\qquad\square$

4 Proof of the existence of a long cycle power

First, we show that the bounds in Theorem 1.3 are tight up to the implied constants. We can easily construct a roughly ε-intransitive tournament on n vertices with no cycle of length $5\varepsilon n$, as follows. Split the n vertices into $m = 1/(4\varepsilon)$ parts of equal size, say $A_1, \ldots, A_m$. Let T_i be a random tournament on A_i for every i, and orient all A_i-A_j edges from A_i to A_j when $i < j$. It is easy to see that a random tournament is roughly $1/4$-intransitive, so T must be roughly ε-intransitive. On the other hand, every cycle intersects at most one of the T_i, so its length is at most $4\varepsilon n$.

We now prove that the bound on the order of the tournament in Theorem 1.3 is tight up to an absolute constant.

Lemma 4.1. *For every $k \geq 300$ and $0 < \varepsilon < 1/8$, there is an ε-intransitive tournament T on at least $\varepsilon^{-k/50}$ vertices that does not contain the k-th power of any cycle of length longer than k.*

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Proof. Let T be a transitive tournament on $n = \lceil \varepsilon^{-k/50} \rceil$ vertices, and denote the transitive ordering by τ. Let $R(T)$ be a random tournament obtained by independently reversing each edge with probability 2ε.

First, we show that $R(T)$ is ε-intransitive with probability at least $1/2$. In any fixed ordering π of the vertices, there are either at least $n^2/4$ forward edges or $n^2/5$ backward edges. Either way, a standard application of the Chernoff bounds shows that the probability that $R(T)$ contains fewer than εn^2 backward edges is less than $e^{-\varepsilon n^2/100}$. There are $n!$ orderings, so the probability that $R(T)$ is not ε-intransitive is at most $n! e^{-\varepsilon n^2/100} < e^{n \log n - \varepsilon n^2/100} < 1/2$ using $\varepsilon n > 200 \log n$ (which is easy to check with the given parameters).

Claim. *If $R(T)$ contains the k-th power of a cycle of length at least k, then the backward edges of $R(T)$ with respect to τ contain a copy of $K_{k/10,k/10}$.*

Proof. Let $C \subseteq R(T)$ be the k-th power of a cycle of length at least k. Let $x_1, \ldots, x_m$ be an ordering of the vertices of C such that x_i sends edges to $x_{i+1}, \ldots, x_{i+k}$ for every $i \in [m]$ (and taking indices modulo m).

Split C into consecutive intervals of size $k/2$, say $A_1, \ldots, A_{\lfloor 2m/k \rfloor}$ (ignoring any leftover vertices), so that $A_1 \rightrightarrows A_2 \rightrightarrows \cdots \rightrightarrows A_{\lfloor 2m/k \rfloor} \rightrightarrows A_1$. If A_2 contains at least $k/10$ vertices that precede some $k/10$ vertices in A_1 in τ, then we have the desired $K_{k/10,k/10}$ consisting of backward edges. Hence, we may assume that there are sets $A_1' \subseteq A_1$ and $A_2' \subseteq A_2$ of size at least $2k/3$ such that $A_1' \prec A_2'$. By applying the same argument to A_2' and A_3, either we find a $K_{k/10,k/10}$ consisting of backward edges, or there are sets $A_2'' \subseteq A_2'$ and $A_3' \subseteq A_3$ such that A_2'' has size $k/10$, A_3' has size $2k/3$, and $A_2'' \prec A_3'$. Continuing in the same fashion, we obtain a sequence $A_1' \prec A_2'' \prec \cdots \prec A_{\lfloor 2m/k \rfloor}'$. Now $A_{\lfloor 2m/k \rfloor}' \rightrightarrows A_1'$ gives a $K_{k/10,k/10}$ of backward edges, as we wanted to show. $\qquad\square$

To finish the proof, it is enough to show that with probability at least $1/2$, there is no complete bipartite $K_{k/10,k/10}$ consisting of backward edges in τ, and hence $R(T)$ does not contains the k-th power of any cycle of length at least k. There are at most $\binom{n}{k/10}^2$ possible complete bipartite graphs on n vertices, and each of them appears in $R(T)$ with probability $(2\varepsilon)^{k^2/100}$. Now it easy to check that

$$\binom{n}{k/10}^2 \cdot (2\varepsilon)^{k^2/100} < n^{k/5} \cdot \varepsilon^{k^2/200} = \varepsilon^{-k^2/250 + k^2/200} < 1/2.$$

This shows that there is an instance of $R(T)$ that is ε-intransitive but does not contain the k-th power of any cycle of length at least k, as we wanted to show. $\qquad\square$

Let us now turn to the proof of Theorem 1.3. The general idea is the following. We first find two vertex subsets that are relatively far from each other in a median ordering, but are connected by many backward edges. We then use our tools from Section 2 to assemble the k-th power of a long cycle as follows. The cycle will start with a complete bipartite graph of backward edges

between the two subsets that we can find using Corollary 2.5. We continue the cycle with forward edges. Lemma 2.9 allows us to touch many of the vertices between the two subsets. Finally, we can apply Corollary 2.11 to close the cycle. This method yields a cycle of length $c_k \varepsilon n$. In order to make the constant independent of k, we will repeat the above argument several times, "wrapping around" the two subsets.

Our main tool for finding two sets with many backward edges is the following density-increment lemma, which is inspired by a similar tool of Long [11, Lemma 5].

We define the *length* $\text{len}_\tau(e)$ of an edge $e = v_i v_j$ with respect to a given ordering $\tau = v_1 \prec \cdots \prec v_n$ of the vertices as the distance $|i - j|$ of its endpoints.

Lemma 4.2. *Let $0 < \varepsilon \leq 1/4$ and let T be a tournament on n vertices that is ε-intransitive. Then for every $0 < c < 1/3$, T satisfies at least one of the following properties.*

(P1) In every median ordering τ of T, there is a set E_τ consisting of backward edges such that $|E_\tau| \geq c\varepsilon n^2/4$ and $\text{len}_\tau(e) \geq cn/4$, for every $e \in E_\tau$.

(P2) There is a sub-tournament $T' \subseteq T$ of order at least $n/2$ such that T' is $2(1-c)\varepsilon$-intransitive.

Proof. Let $\tau = v_1 \prec \cdots \prec v_n$ be a median ordering of T. By assumption, there are at least εn^2 backward edges in this ordering. Let E_τ be the set of backward edges whose endpoints are at distance at least $cn/4$. If $|E_\tau| \geq \frac{c\varepsilon}{4} n^2$, then (P1) is satisfied, so we may assume that $|E_\tau| < \frac{c\varepsilon}{4} n^2$. In particular, we may assume that $cn/4 > 1$ (i.e., $n > 4/c$), as otherwise E_τ contains at least εn^2 backward edges.

Now let $I_1 = \{v_1, \ldots, v_{\lceil n/2 \rceil}\}$ and $I_2 = \{v_{\lfloor n/2 \rfloor + 1}, \ldots, v_n\}$, and let E_1 and E_2 be the respective sets of backward edges induced by them. We may assume that $|E_1| \geq |E_2|$. Finally, let F be the set of backward edges not contained in either of E_1, E_2 and E_τ. Then the edges in F must go from I_2 to I_1 and have length less than $cn/4$, so they are all induced by the interval $J = \{v_1, \ldots, v_{\lfloor (1+c/2)n/2 \rfloor}\}$.

We have $|E_1| + |E_2| + |E_\tau| + |F| = \varepsilon n^2$ with $|E_1| \geq |E_2|$ and $|E_\tau| < \frac{c\varepsilon}{4} n^2$. If $|F| < \frac{c\varepsilon}{4} n^2$, then $|E_1| > \frac{\varepsilon(1-c/2)}{2} n^2$. Using $n > 4/c$, it is easy to check that $\frac{cn^2}{2} \geq (1-c)(2n+1)$, so we have $|E_1| \geq \frac{\varepsilon(1-c)}{2}(n+1)^2 \geq 2(1-c)\varepsilon |I_1|^2$. As τ is a median ordering of the subtournament induced by any interval, we can choose $T' = T[I_1]$ to satisfy (P2). Otherwise, $|F| \geq \frac{c\varepsilon}{4} n^2$, so $|E_1| + |F| \geq \varepsilon n^2/2 \geq (1-c)(1+c/2)^2 \varepsilon n^2/2 \geq 2(1-c)\varepsilon |J|^2$. We can then choose $T' = T[J]$. $\square$

Corollary 4.3. *Let $0 < \varepsilon \leq 1/4$, and let T be an n-vertex tournament that is ε-intransitive. Then, for some $\tilde{\varepsilon} \geq \varepsilon$, T contains a subtournament $\tilde{T}$ on $\tilde{n}$ vertices that is $\tilde{\varepsilon}$-intransitive such that $\tilde{\varepsilon}\tilde{n} \geq \varepsilon n/5$, and for every median ordering τ of $\tilde{T}$ there is a set E_τ consisting of $\tilde{\varepsilon}^2 \tilde{n}^2/4$ backward edges such that $\text{len}_\tau(e) \geq \tilde{\varepsilon}\tilde{n}/4$, for every $e \in E_\tau$.*

Proof. Let us repeatedly apply Lemma 4.2 with $c = \varepsilon$ as long as (P2) holds, i.e., let $T = T_0 \supseteq T_1 \supseteq \ldots \supseteq T_p$ be a longest sequence of tournaments such that $n_i = |V(T_i)| \geq |V(T_{i-1})|/2$ and T_i is

ε_i-intransitive for some $\varepsilon_i \geq 2(1 - \varepsilon_{i-1})\varepsilon_{i-1}$, for every $i \in [p]$. We claim that $\tilde{T} = T_p$ satisfies the conditions.

To see this, first note that if T_i is ε_i-intransitive, then $\varepsilon_i \leq 1/4$. Since $\varepsilon_{i-1} \leq \frac{2}{3}\varepsilon_i$ for every $i \in [p]$, we have $\sum_{i=0}^{p} \varepsilon_i \leq \varepsilon_p \sum_{i=0}^{p} (2/3)^i \leq 3/4$. We also know that $n_p \geq n_{p-1}/2 \geq \cdots \geq n_0/2^p$ and that

$$1/4 \geq \varepsilon_p \geq 2\varepsilon_{p-1}(1 - \varepsilon_{p-1}) \geq \cdots \geq 2^p \varepsilon_0 \prod_{i=0}^{p-1}(1 - \varepsilon_i) \geq 2^p \varepsilon e^{-2\sum \varepsilon_i} \geq 2^p \varepsilon / e^{3/2},$$

so $\tilde{\varepsilon}\tilde{n} \geq \varepsilon n / e^{3/2} > \varepsilon n / 5$ for $\tilde{\varepsilon} = \varepsilon_p$ and $\tilde{n} = n_p$.

As we chose a maximal sequence of tournaments, $\tilde{T}$ does not satisfy (P2) with $c = \tilde{\varepsilon}$. But then Lemma 4.2 implies (P1) for $\tilde{T}$ with $c = \tilde{\varepsilon}$, which is exactly what we wanted to show. □

We are now ready to prove our theorem.

Theorem 4.4. *Every ε-intransitive tournament on $n \geq \varepsilon^{-41000k}$ vertices contains the k-th power of a cycle of length at least $\varepsilon n / 1500$.*

Proof. Let us apply Corollary 4.3 to T, and let $\tilde{T}$ be the $\tilde{\varepsilon}$-intransitive $\tilde{n}$-vertex subtournament we obtain, where $\tilde{\varepsilon}\tilde{n} \geq \varepsilon n / 5$ and $1/4 \geq \tilde{\varepsilon} \geq \varepsilon$. Note that this implies $\tilde{\varepsilon}\tilde{n} \geq \varepsilon^{-40950k} \geq \tilde{\varepsilon}^{-40950k}$.

Fix any median ordering τ of $\tilde{T}$. Then there is a set E_τ of $\tilde{\varepsilon}^2 \tilde{n}^2 / 4$ backward edges in $\tilde{T}$ such that every edge $e \in E_\tau$ has length $\text{len}_\tau(e) \geq \tilde{\varepsilon}\tilde{n}/4$.

Let us split the $\tilde{n}$ vertices of $\tilde{T}$ into $t = 12/\tilde{\varepsilon}$ intervals $A_1 \prec \cdots \prec A_t$ of size $m = \tilde{n}/t = \tilde{\varepsilon}\tilde{n}/12 \geq \tilde{\varepsilon}^{-40900k}$ each. Then every edge of E_τ must connect two intervals A_i and $A_{i'}$ with at least 2 other intervals in between (i.e., $i' \geq i + 3$). In particular, there must be at least $|E_\tau|/t^2 \geq \tilde{\varepsilon}^2 m^2 / 200$ backward edges going from A_b to A_a for some a, b satisfying $b \geq a + 3$. Now, split each A_i into $t' = m/\tilde{\varepsilon}^{20400k} \geq \tilde{\varepsilon}^{-20500k}$ consecutive subintervals $A_{i,1} \prec \cdots \prec A_{i,t'}$ of size $m' = \tilde{\varepsilon}^{-20400k}$.

Claim. *There are at least $\frac{\tilde{\varepsilon}^2 t'}{800}$ disjoint pairs $\{A_{a,j}, A_{b,j'}\}$ such that $d(A_{b,j'}, A_{a,j}) \geq \frac{\tilde{\varepsilon}^2}{400}$.*

Proof. Note that $\sum_{j,j'} d(A_{b,j'}, A_{a,j}) = t'^2 \cdot d(A_b, A_a) \geq \tilde{\varepsilon}^2 t'^2 / 200$. This means that at least $\tilde{\varepsilon}^2 t'^2 / 400$ of the pairs satisfy $d(A_{b,j'}, A_{a,j}) \geq \tilde{\varepsilon}^2 / 400$. We can then greedily choose $\tilde{\varepsilon}^2 t' / 800$ of these pairs so that they are disjoint from each other. □

Let $r = \frac{m'}{50k} \leq \tilde{\varepsilon}^{-20400k} \leq \frac{\tilde{\varepsilon}^2 t'}{800}$ and let $\{A_{a,j_1}, A_{b,j_1'}\}, \ldots, \{A_{a,j_r}, A_{b,j_r'}\}$ be disjoint pairs provided by the claim. We may assume that $j_1 > \cdots > j_r$.

As $m' > (\frac{\tilde{\varepsilon}}{400})^{-160k}$, we can apply Corollary 2.5 to find sets $Z_i \subseteq A_{a,j_i}$ and $Z_i' \subseteq A_{b,j_i'}$, for every $i \in [r]$, that induce transitive tournaments of size $32k$ and satisfy $Z_i' \rightrightarrows Z_i$.

Next, we apply Lemma 2.9 one by one for every $i \in [r]$ to the interval $A_{a,j_i} \cup A_{a,j_i+1} \cup \cdots \cup A_{a+1,t'}$ with $X = Z_i$ to find $4k$-sets $X_{i,1}, \ldots, X_{i,s_i}$ such that each of them induces a transitive tournament, and they satisfy $X_{i,1} \subseteq Z_i$ and $X_{i,1} \rightrightarrows \cdots \rightrightarrows X_{i,s_i}$. The lemma also implies that for every $j \in [t'-1]$,

one of these sets is contained in $A_{a+1,j} \cup A_{a+1,j+1}$ (for example, $X_{i,s_i} \subseteq A_{a+1,t'-1} \cup A_{a+1,t'}$). In particular, $X_{i,1} \cup \cdots \cup X_{i,s_i}$ contains at least $2kt'$ vertices from A_{a+1}.

Moreover, by defining the set of forbidden vertices in the lemma as $F = \bigcup_{i'<i}(X_{i',1} \cup \cdots \cup X_{i',s_{i'}})$ (which is allowed because $|F \cap A_{\alpha,j}| \le 4kr \le \frac{m'}{8}$ for every $\alpha \in \{a, a+1\}$ and $j \in [t']$), we can ensure that the path blowups $X_{i,1} \rightrightarrows \cdots \rightrightarrows X_{i,s_i}$ are vertex-disjoint over $i \in [r]$. Combining this with the fact that each such path blowup contains at least $2kt'$ vertices, we see that the r path blowups together cover at least $2kt'r = \frac{t'm'}{25} = \frac{m}{25} = \frac{\tilde{\varepsilon}\tilde{n}}{300} \ge \frac{\varepsilon n}{1500}$ vertices.

All we are left to do is connect the path blowups into one big cycle power. We use Corollary 2.11 with $X = X_{i,s_i}$ and $X' = Z'_{i+1}$ for every $i \in [r]$. More precisely, let $t_i \in \{t-1, t\}$ be the index such that $X = X_{i,s_i} \in A_{a+1,t_i}$, and let X' be any $4k$-set in Z'_{i+1} (or Z'_1 if $i = r$). We will apply Corollary 2.11 to the interval $A_{a+1,t_i} \cup \cdots \cup A_{b,j'_{i+1}}$ to get k-sets $Y_{i,0} \rightrightarrows \cdots \rightrightarrows Y_{i,s'_i}$ such that $Y_{i,0} \subseteq X_{i,s_i}$ and $Y_{i,s'_i} \subseteq Z'_{i+1}$ and each of these sets induces a transitive subtournament. This is possible because the interval (which contains the entire A_{a+2}) is split into at least $t' > 60$ subintervals of size $m' \ge 2^{40500k}$.

To make the path blowups $Y_{i,0} \rightrightarrows \cdots \rightrightarrows Y_{i,s'_i}$ vertex-disjoint over $i \in [r]$, we just need to forbid all vertices $F = \left(\bigcup_{i'\in[r]} X_{i',s_{i'}}\right) \cup \left(\bigcup_{i'\in[r]} Z'_{i'}\right) \cup \left(\bigcup_{i'<i}(Y_{i',0} \cup \cdots \cup Y_{i',s'_{i'}})\right)$ that are already used by other path blowups. As $s'_i \le 5$ for every $i \in [r]$, this set contains $|F| \le 4kr + 8kr + 6kr = 18kr \le m'/2$ vertices, so can indeed be used in our applications of Corollary 2.11.

All in all, we found disjoint vertex sets $X_{i,j}$ and $Y_{i,j}$ that each induce transitive subtournaments of size at least k, together contain at least $\frac{\varepsilon n}{1500}$ vertices, and satisfy

$$X_{1,1} \rightrightarrows \cdots \rightrightarrows X_{1,s_1-1} \rightrightarrows Y_{1,0} \rightrightarrows \cdots \rightrightarrows Y_{1,s'_1} \rightrightarrows X_{2,1} \rightrightarrows \cdots \rightrightarrows Y_{r,s'_r} \rightrightarrows X_{1,1}$$

This sequence contains the blowup of a cycle of length $\frac{\varepsilon n}{1500}$, as needed. $\square$

References

[1] S. Bessy and S. Thomassé, *Partitioning a graph into a cycle and an anticycle, a proof of Lehel's conjecture*, J. Combin. Theory Ser. B **100** (2010), 176–180. 2

[2] B. Bollobás and R. Häggkvist, *Powers of Hamilton cycles in tournaments*, J. Combin. Theory Ser. B **50** (1990), 309–318. 3

[3] S. Bustamante, J. Corsten, N. Frankl, A. Pokrovskiy, and J. Skokan, *Partitioning edge-coloured hypergraphs into few monochromatic tight cycles*, SIAM J. Discrete Math. **34** (2020), 1460–1471. 2

[4] F.R.K Chung and, R. L. Graham, *Quasi-random tournaments*, J. Graph Theory **15** (1991), 173–198. 3

[5] N. Draganić, F. Dross, J. Fox, A. Girão, F. Havet, D. Korándi, W. Lochett, D. Munhá Correia, A. Scott, and B. Sudakov, *Powers of paths in tournaments,* Combin. Probab. Comput., to appear. 1

[6] N. Draganić, D. Munhá Correia, and B. Sudakov, *Tight bounds for powers of Hamilton cycles in tournaments,* (2021) arXiv:2103.10414 preprint. 3

[7] P. Erdős, A. Gyárfás, and L. Pyber, *Vertex coverings by monochromatic cycles and trees,* J. Combin. Theory Ser. B **51** (1991), 90–95. 2

[8] J. Fox, X. He, and Y. Widgerson, *Ramsey numbers of sparse digraphs,* (2021) arXiv:2105.02383 preprint. 2

[9] J. Fox, B. Sudakov, *Unavoidable patterns,* J. Combin. Theory, Series A **115** (2008), 1561–1569. 3

[10] A. Gyárfás, *Vertex covers by monochromatic pieces—a survey of results and problems,* Discrete Math. **339** (2016), 1970–1977. 2

[11] E. Long, *Large unavoidable subtournaments,* Combin. Probab. Comput. **26** (2017), 68–77. 14

[12] T. Łuczak, V. Rödl, E. Szemerédi, *Partitioning two-colored complete graphs into two monochromatic cycles,* Combin. Probab. Comput. **7** (1998), 423–436. 2

[13] L. Rédei, *Ein kombinatorischer Satz* Acta Litt. Sci. Szeged **7** (1934), 39–43. 1

[14] R. Yuster, *Paths with many shortcuts in tournaments,* Discrete Math. **334** (2021), 112168. 1