Clustered Colouring of Graph Classes with Bounded Treedepth or Pathwidth

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Abstract. The *clustered chromatic number* of a class of graphs is the minimum integer k such that for some integer c every graph in the class is k-colourable with monochromatic components of size at most c. We determine the clustered chromatic number of any minor-closed class with bounded treedepth, and prove a best possible upper bound on the clustered chromatic number of any minor-closed class with bounded pathwidth. As a consequence, we determine the fractional clustered chromatic number of every minor-closed class.

1 Introduction

This paper studies improper vertex colourings of graphs with bounded monochromatic degree or bounded monochromatic component size. This topic has been extensively studied recently [1, 3, 4, 6, 8, 10–19, 21–23]; see [24] for a survey.

A *k-colouring* of a graph G is a function that assigns one of k colours to each vertex of G. In a coloured graph, a *monochromatic component* is a connected component of the subgraph induced by all the vertices of one colour.

A colouring has *defect* d if each monochromatic component has maximum degree at most d. The *defective chromatic number* of a graph class $\mathcal{G}$, denoted by $\chi_\Delta(\mathcal{G})$, is the minimum integer k such that, for some integer d, every graph in $\mathcal{G}$ is k-colourable with defect d.

A colouring has *clustering* c if each monochromatic component has at most c vertices. The *clustered chromatic number* of a graph class $\mathcal{G}$, denoted by $\chi_\star(\mathcal{G})$, is the minimum integer k such that, for some integer c, every graph in $\mathcal{G}$ has a k-colouring with clustering c. We shall consider such colourings, where the goal is to minimise the number of colours, without optimising the clustering value.

Every colouring of a graph with clustering c has defect $c-1$. Thus $\chi_\Delta(\mathcal{G}) \leqslant \chi_\star(\mathcal{G})$ for every class $\mathcal{G}$.

The following is a well-known and important example in defective and clustered graph colouring. Let T be a rooted tree. The *depth* of T is the maximum number of vertices

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on a root–to–leaf path in T. The *closure* of T is obtained from T by adding an edge between every ancestor and descendent in T. For $h, k \geqslant 1$, let $C\langle h, k\rangle$ be the closure of the complete k-ary tree of depth h, as illustrated in Figure 1.

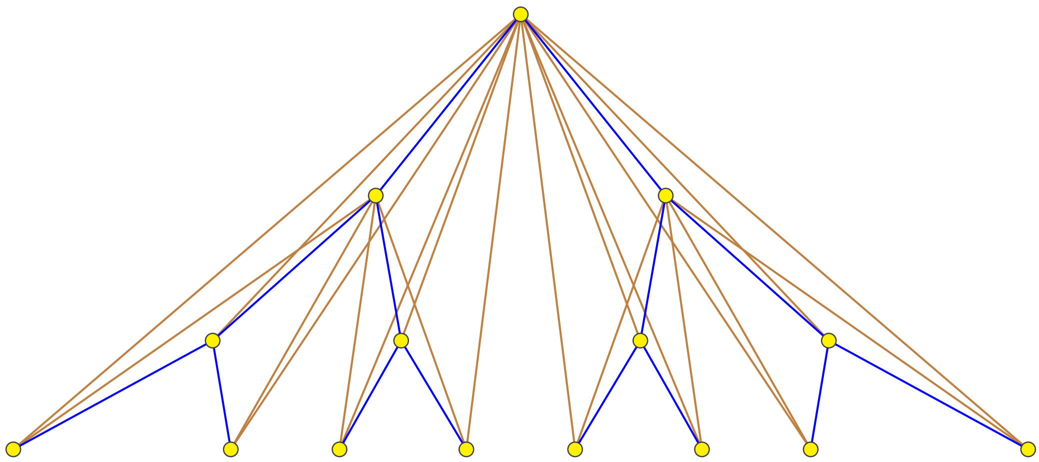

Figure 1: The standard example $C\langle 4, 2\rangle$.

It is well known and easily proved (see [24]) that there is no $(h-1)$-colouring of $C\langle h, k\rangle$ with defect $k-1$, which implies there is no $(h-1)$-colouring of $C\langle h, k\rangle$ with clustering k. This says that if a graph class $\mathcal{G}$ includes $C\langle h, k\rangle$ for all k then the defective chromatic number and the clustered chromatic number are at least h. Put another way, define the *tree-closure-number* of a graph class $\mathcal{G}$ to be

$$\operatorname{tcn}(\mathcal{G}) := \min\{h : \exists k \, C\langle h, k\rangle \notin \mathcal{G}\} = \max\{h : \forall k \, C\langle h, k\rangle \in \mathcal{G}\} + 1;$$

then

$$\chi_\star(\mathcal{G}) \geqslant \chi_\Delta(\mathcal{G}) \geqslant \operatorname{tcn}(\mathcal{G}) - 1.$$

Our main result, Theorem 1 below, establishes a converse result for minor-closed classes with bounded treedepth. First we explain these terms. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from some subgraph of G by contracting edges. A class of graphs $\mathcal{M}$ is *minor-closed* if for every graph $G \in \mathcal{M}$ every minor of G is in $\mathcal{M}$, and $\mathcal{M}$ us *proper* minor-closed if, in addition, some graph is not in $\mathcal{M}$. The *connected treedepth* of a graph H, denoted by $\overline{\operatorname{td}}(H)$, is the minimum depth of a rooted tree T such that H is a subgraph of the closure of T. This definition is a variant of the more commonly used definition of the *treedepth* of H, denoted by $\operatorname{td}(H)$, which equals the maximum connected treedepth of the connected components of H. (See [20] for background on treedepth.) If H is connected, then $\operatorname{td}(H) = \overline{\operatorname{td}}(H)$. In fact, $\operatorname{td}(H) = \overline{\operatorname{td}}(H)$ unless H has two connected components H_1 and H_2 with $\operatorname{td}(H_1) = \operatorname{td}(H_2) = \operatorname{td}(H)$, in which case $\overline{\operatorname{td}}(H) = \operatorname{td}(H) + 1$. It is convenient to work with connected treedepth to avoid this distinction. A class of graphs has *bounded treedepth* if there exists a constant c such that every graph in the class has treedepth at most c.

Theorem 1. *For every minor-closed class $\mathcal{G}$ with bounded treedepth,*

$$\chi_\Delta(\mathcal{G}) = \chi_\star(\mathcal{G}) = \operatorname{tcn}(\mathcal{G}) - 1.$$

Our second result concerns pathwidth. A *path-decomposition* of a graph G consists of a sequence $(B_1, \dots, B_n)$, where each B_i is a subset of $V(G)$ called a *bag*, such that for every vertex $v \in V(G)$, the set $\{i \in [1, n] : v \in B_i\}$ is an interval, and for every edge $vw \in E(G)$ there is a bag B_i containing both v and w. Here $[a, b] := \{a, a+1, \dots, b\}$. The *width* of a path decomposition $(B_1, \dots, B_n)$ is $\max\{|B_i| : i \in [1, n]\} - 1$. The *pathwidth* of a graph G is the minimum width of a path-decomposition of G. Note that paths (and more generally caterpillars) have pathwidth 1. A class of graphs has *bounded pathwidth* if there exists a constant c such that every graph in the class has pathwidth at most c.

Theorem 2. *For every minor-closed class $\mathcal{G}$ with bounded pathwidth,*

$$\chi_\Delta(\mathcal{G}) \leqslant \chi_\star(\mathcal{G}) \leqslant 2\operatorname{tcn}(\mathcal{G}) - 2.$$

Theorems 1 and 2 are respectively proved in Sections 2 and 3. These results are best possible and partially resolve a number of conjectures from the literature, as we now explain.

Ossona de Mendez et al. [22] studied the defective chromatic number of minor-closed classes. For a graph H, let $\mathcal{M}_H$ be the class of H-minor-free graphs (that is, not containing H as a minor). Ossona de Mendez et al. [22] proved the lower bound, $\chi_\Delta(\mathcal{M}_H) \geqslant \overline{\operatorname{td}}(H) - 1$ and conjectured that equality holds.

Conjecture 3 ([22]). *For every graph H,*

$$\chi_\Delta(\mathcal{M}_H) = \overline{\operatorname{td}}(H) - 1.$$

Note that Conjecture 3 is known to hold in some special cases. Edwards et al. [8] proved it if $H = K_t$; that is, $\chi_\Delta(\mathcal{M}_{K_t}) = t - 1$, which can be thought of as a defective version of Hadwiger's Conjecture; see [23] for an improved bound on the defect in this case. Ossona de Mendez et al. [22] proved Conjecture 3 if $\overline{\operatorname{td}}(H) \leqslant 3$ or if H is a complete bipartite graph. In particular, $\chi_\Delta(\mathcal{M}_{K_{s,t}}) = \min\{s, t\}$.

Norin et al. [21] studied the clustered chromatic number of minor-closed classes. They showed that for each $k \geqslant 2$, there is a graph H with treedepth k and connected treedepth k such that $\chi_\star(\mathcal{M}_H) \geqslant 2k - 2$. It is easily seen that the corresponding graphs have bounded pathwidth (at most $2k - 3$ to be precise). Thus the upper bound on $\chi_\star(\mathcal{G})$ in Theorem 2 is best possible.

Norin et al. [21] conjectured the following converse upper bound (analogous to Conjecture 3):

Conjecture 4 ([21]). *For every graph H,*

$$\chi_\star(\mathcal{M}_H) \leqslant 2\,\overline{\operatorname{td}}(H) - 2.$$

While Conjectures 3 and 4 remain open, Norin et al. [21] showed in the following theorem that $\chi_\Delta(\mathcal{M}_H)$ and $\chi_\star(\mathcal{M}_H)$ are controlled by the treedepth of H:

Theorem 5 ([21]). *For every graph H, $\chi_\star(\mathcal{M}_H)$ is tied to the (connected) treedepth of H. In particular,*

$$\overline{\mathrm{td}}(H) - 1 \leqslant \chi_\star(\mathcal{M}_H) \leqslant 2^{\overline{\mathrm{td}}(H)+1} - 4.$$

Theorem 1 gives a much more precise bound than Theorem 5 under the extra assumption of bounded treedepth.

Our third main result concerns fractional colourings. For real $t \geqslant 1$, a graph G is *fractionally t-colourable with clustering c* if there exist $Y_1, Y_2, \dots, Y_s \subseteq V(G)$ and $\alpha_1, \dots, \alpha_s \in [0, 1]$ such that[1]:

- Every component of $G[Y_i]$ has at most c vertices,
- $\sum_{i=1}^{s} \alpha_i \leqslant t$,
- $\sum_{i:v\in Y_i} \alpha_i \geqslant 1$ for every $v \in V(G)$.

The *fractional clustered chromatic number* $\chi^f_\star(\mathcal{G})$ of a graph class $\mathcal{G}$ is the infimum of $t > 0$ such that there exists $c = c(t, \mathcal{G})$ such that every $G \in \mathcal{G}$ is fractionally t-colourable with clustering c. *Fractional defective chromatic number* $\chi^f_\Delta(\mathcal{G})$ is defined in exactly the same way, except the condition on the component size is replaced by "the maximum degree of $G[Y_i]$ is at most d". The following theorem determines the fractional clustered chromatic number and fractional defective chromatic number of any proper minor-closed class.

Theorem 6. *For every proper minor-closed class $\mathcal{G}$,*

$$\chi^f_\Delta(\mathcal{G}) = \chi^f_\star(\mathcal{G}) = \mathrm{tcn}(\mathcal{G}) - 1.$$

This result is proved in Section 4.

2 Treedepth

Say G is a subgraph of the closure of some rooted tree T. For each vertex $v \in V(T)$, let T_v be the subtree of T rooted at v (consisting of v and all its descendents), and let $G[T_v]$ be the subgraph of G induced by $V(T_v)$.

The *weak closure* of a rooted tree T is the graph G with vertex set $V(T)$, where two vertices $v, w \in V(T)$ are adjacent in G whenever v is a leaf of T and w is an ancestor of v in T. As illustrated in Figure 2, let $W\langle h, k\rangle$ be the weak closure of the complete k-ary tree of height h.

[1] If $c = 1$, then this corresponds to a (proper) fractional t-colouring, and if the α_i are integral, then this yields a t-colouring with clustering c.

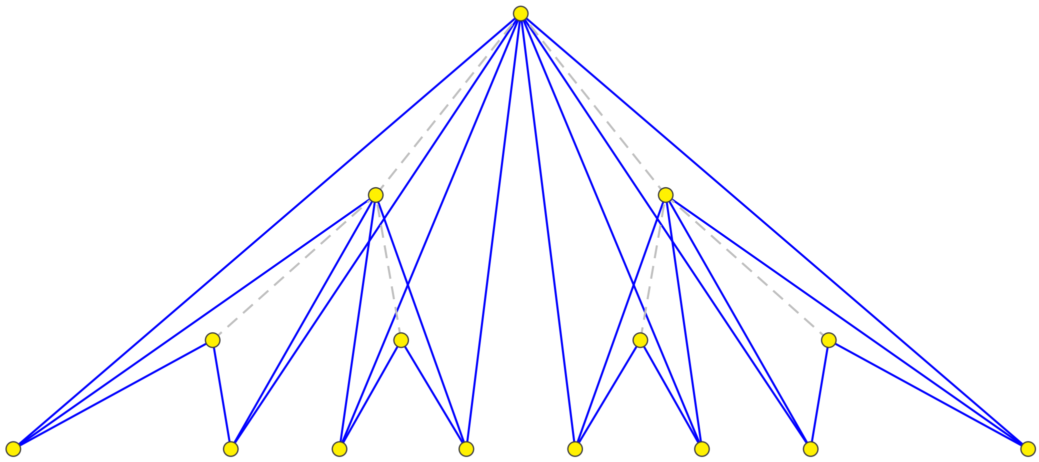

Figure 2: The weak closure $W\langle 4,2\rangle$.

Note that $W\langle h,k\rangle$ is a proper subgraph of $C\langle h,k\rangle$ for $h \geqslant 3$. On the other hand, Norin et al. [21] showed that $W\langle h,k\rangle$ contains $C\langle h,k-1\rangle$ as a minor for all $h,k \geqslant 2$. Therefore Theorem 1 is an immediate consequence of the following lemma.

Lemma 7. *For all $d,k \in \mathbb{N}$ there exists $c = c(d,k) \in \mathbb{N}$ such that for every $h \in \mathbb{N}$ and for every graph G with treedepth at most d, either G contains a $W\langle h,k\rangle$-minor or G is $(h-1)$-colourable with clustering c.*

Proof. Throughout this proof, d is fixed, and we make no attempt to optimise c.

We may assume that G is connected. So G is a subgraph of the closure of some rooted tree of depth at most d. Choose a tree T of depth at most d rooted at some vertex r, such that G is a subgraph of the closure of T, and subject to this, $\sum_{v\in V(T)} \operatorname{dist}_T(v,r)$ is minimal. Suppose that $G[T_v]$ is disconnected for some vertex v in T. Choose such a vertex v at maximum distance from r. Since G is connected, $v \neq r$. By the choice of v, for each child w of v, the subgraph $G[T_w]$ is connected. Thus, for some child w of v, there is no edge in G joining v and $G[T_w]$. Let u be the parent of v. Let T' be obtained from T by deleting the edge vw and adding the edge uw, so that w is a child of u in T'. Note that G is a subgraph of the closure of T' (since v has no neighbour in $G[T_w]$). Moreover, $\operatorname{dist}_{T'}(x,r) = \operatorname{dist}_T(x,r) - 1$ for every vertex $x \in V(T_w)$, and $\operatorname{dist}_{T'}(y,r) = \operatorname{dist}_T(y,r)$ for every vertex $y \in V(T) \setminus V(T_w)$. Hence $\sum_{v\in V(T')} \operatorname{dist}_{T'}(v,r) < \sum_{v\in V(T)} \operatorname{dist}_T(v,r)$, which contradicts our choice of T. Therefore $G[T_v]$ is connected for every vertex v of T.

Consider each vertex $v \in V(T)$. Define the *level* $\ell(v) := \operatorname{dist}_T(r,v) \in [0,d-1]$. Let T_v^+ be the subtree of T consisting of T_v plus the vr-path in T, and let $G[T_v^+]$ be the subgraph of G induced by $V(T_v^+)$. For a subtree X of T rooted at vertex v, define the *level* $\ell(X) := \ell(v)$.

A *ranked graph* (for fixed d) is a triple $(H, L, \prec)$ where:

- H is a graph,
- $L : V(H) \to [0, d-1]$ is a function,
- $\prec$ is a partial order on $V(H)$ such that $L(v) < L(w)$ whenever $v \prec w$.

Up to isomorphism, the number of ranked graphs on n vertices is at most $2^{\binom{n}{2}} d^n 3^{\binom{n}{2}}$. For a vertex v of T, a ranked graph $(H, L, \prec)$ is said to be *contained in* $G[T_v^+]$ if there is an isomorphism ϕ from H to some subgraph of $G[T_v^+]$ such that:

(A) for each vertex $v \in V(H)$ we have $L(v) = \ell(\phi(v))$, and

(B) for all distinct vertices $v, w \in V(H)$ we have that $v \prec w$ if and only if $\phi(v)$ is an ancestor of $\phi(w)$ in T.

If $(H, L, \prec)$ is a ranked graph and $i \in [0, d-1]$, then define the *i-splice* of $(H, L, \prec)$ to be the ranked graph $(H', L', \prec')$ obtained from $(H, L, \prec)$ by taking k copies of the subgraph at levels greater than i. More formally, let

$$\begin{aligned} V(H') := &\{(v,0) : v \in V(H), L(v) \in [0,i]\} \cup \\ &\{(v,j) : v \in V(H), L(v) \in [i+1,d], j \in [1,k]\}. \\ E(H') := &\{(v,0)(w,0) : vw \in E(H), L(v) \in [0,i], L(w) \in [0,i]\} \cup \\ &\{(v,0)(w,j) : vw \in E(H), L(v) \in [0,i], L(w) \in [i+1,d], j \in [1,k]\} \cup \\ &\{(v,j)(w,j) : vw \in E(H), L(v) \in [i+1,d], L(w) \in [i+1,d], j \in [1,k]\}. \end{aligned}$$

Define $L'((v,j)) := L(v)$ for every vertex $(v,j) \in V(H')$. Now define the following partial order $\prec'$ on $V(H')$:

- If $v \prec w$ and $L(v), L(w) \in [0,i]$, then $(v,0) \prec' (w,0)$.
- If $v \prec w$ and $L(v) \in [0,i]$ and $L(w) \in [i+1,d]$, then $(v,0) \prec' (w,j)$ for all $j \in [1,k]$.
- If $v \prec w$ and $L(v), L(w) \in [i+1,d]$, then $(v,j) \prec' (w,j)$ for all $j \in [1,k]$.

Note that if $(v,a) \prec' (w,b)$, then $a \leqslant b$ and $v \prec w$ (implying $(L(v) < L(w))$. It follows that $\prec'$ is a partial order on $V(H')$ such that $L'((v,a)) < L'((w,b))$ whenever $(v,a) \prec' (w,b)$. Thus $(H', L', \prec')$ is a ranked graph.

For $\ell \in [0, d-1]$, let
$$N_\ell := (d+1)(h-1)(k+1)^{d-1-\ell}.$$

For each vertex v of T, define the *profile* of v to be the set of all ranked graphs $(H, L, \prec)$ contained in $G[T_v^+]$ such that $|V(H)| \leqslant N_{\ell(v)}$. Note that if v is a desecendant of u, then the profile of v is a subset of the profile of u. For $\ell \in [0, d-1]$, if $N = N_\ell$ then let
$$M_\ell := 2^{2^{\binom{N}{2}} d^N 3^{\binom{N}{2}}}.$$

Then there are at most M_ℓ possible profiles of a vertex at level ℓ.

We now partition $V(T)$ into subtrees. Each subtree is called a *group*. (At the end of the proof, vertices in a single group will be assigned the same colour.) We assign vertices to groups in non-increasing order of their distance from the root. Initialise this process by placing each leaf v of T into a singleton group. We now show how to determine the group of a non-leaf vertex. Let v be a vertex not assigned to a group at maximum distance from r. So each child of v is assigned to a group. Let Y_v be the set of children y of v, such that the number of children of v that have the same profile as y is in the range $[1, k-1]$. If $Y_v = \emptyset$ start a new singleton group $\{v\}$. If $Y_v \neq \emptyset$ then merge all the groups rooted at vertices in Y_v into one group including v. This defines our partition of $V(T)$ into groups. Each group X is *rooted* at the vertex in X closest to r in T. A group Y is *above* a distinct group X if the root of Y is on the path in T from the root of X to r.

The next claim is the key to the remainder of the proof.

Claim 1. *Let $uv \in E(T)$ where u is the parent of v, and u is in a different group to v. Then for every ranked graph $(H, L, \prec)$ in the profile of v, the $\ell(u)$-splice of $(H, L, \prec)$ is in the profile of u.*

Proof. Since $(H, L, \prec)$ is in the profile of v, there is an isomorphism ϕ from H to some subgraph of $G[T_v^+]$ such that for each vertex $x \in V(H)$ we have $L(x) = \ell(\phi(x))$, and for all distinct vertices $x, y \in V(H)$ we have that $x \prec y$ if and only if $\phi(x)$ is an ancestor of $\phi(y)$ in T.

Since u and v are in different groups, there are k children $y_1, \ldots, y_k$ of u (one of which is v) such that the profiles of $y_1, \ldots, y_k$ are equal. Thus $(H, L, \prec)$ is in the profile of each of $y_1, \ldots, y_k$. That is, for each $j \in [1, k]$, there is an isomorphism ϕ_j from H to some subgraph of $G[T_{y_j}^+]$ such that for each vertex $x \in V(H)$ we have $L(x) = \ell(\phi_j(x))$, and for all distinct vertices $x, y \in V(H)$ we have that $x \prec y$ if and only if $\phi_j(x)$ is an ancestor of $\phi_j(y)$ in T.

Let $(H', L', \prec')$ be the $\ell(u)$-splice of $(H, L, \prec)$. We now define a function ϕ' from $V(H')$ to $V(G[T_u^+])$. For each vertex $(x, 0)$ of H' (thus with $x \in V(H)$ and $L(x) \in [0, \ell(u)]$), define $\phi'((x, 0)) := \phi(x)$. For every other vertex (x, j) of H' (thus with $x \in V(H)$ and $L(x) \in [\ell(u) + 1, d-1]$ and $j \in [1, k]$), define $\phi'((x, j)) := \phi_j(x)$.

We now show that ϕ' is an isomorphism from H' to a subgraph of $G[T_u^+]$. Consider an edge $(x, a)(y, b)$ of H'. Thus $xy \in E(H)$. It suffices to show that $\phi'((x, a))\phi'((y, b)) \in E(G[T_u^+])$. First suppose that $a = b = 0$. So $L(x) \in [0, \ell(u)]$ and $L(y) \in [0, \ell(u)]$. Thus $\phi'((x, a)) = \phi(x)$ and $\phi'((y, b)) = \phi(y)$. Since ϕ is an isomorphism to a subgraph of $G[T_v^+]$, we have $\phi(x)\phi(y) \in E(G[T_v^+])$, which is a subgraph of $G[T_u^+]$. Hence $\phi'((x, a))\phi'((y, b)) \in E(G[T_u^+])$, as desired. Now suppose that $a = 0$ and $b \in [1, k]$. Thus $\phi'((x, a)) = \phi(x)$ and $\phi'((y, b)) = \phi_b(y)$. Moreover, both $\ell(\phi(x))$ and $\ell(\phi_b(x))$ equal $L(x) \in [0, \ell(u)]$. There is only vertex z in T_v^+ with $\ell(z)$ equal to a specific number in $[0, \ell(u)]$. Thus $\phi'((x, a)) = \phi(x) = \phi_b(x)$ $(= z)$. Since ϕ_b is an isomorphism to a subgraph of $G[T_{y_b}^+]$, we have $\phi_b(x)\phi_b(y) \in E(G[T_{y_b}^+])$, which is a subgraph of $G[T_u^+]$. Hence $\phi'((x, a))\phi'((y, b)) \in E(G[T_u^+])$, as desired. Finally, suppose that $a = b \in [1, k]$. Thus $\phi'((x, a)) = \phi_a(x)$ and $\phi'((y, b)) = \phi_b(y) = \phi_a(y)$. Since ϕ_a is an isomorphism to a subgraph of $G[T_{y_a}^+]$, we have $\phi_a(x)\phi_a(y) \in E(G[T_{y_a}^+])$, which is a subgraph of $G[T_u^+]$. Hence $\phi'((x,a))\phi'((y,b)) \in E(G[T_u^+])$, as desired. This shows that ϕ' is an isomorphism from H' to a subgraph of $G[T_u^+]$.

We now verify property (A) for $(H', L', \prec')$. For each vertex $(x, 0)$ of H' (thus with $x \in V(H)$ and $L(x) \in [0, \ell(u)]$) we have $L'((x,0)) = L(x) = \ell(\phi(x)) = \ell(\phi'((x,0)))$, as desired. For every other vertex (x, j) of H' (thus with $x \in V(H)$ and $L(x) \in [\ell(u)+1, d-1]$ and $j \in [1,k]$) we have $L'((x,j)) = L(x) = \ell(\phi_j(x)) = \ell(\phi'((x,j)))$, as desired. Hence property (A) is satisfied for $(H', L', \prec')$.

We now verify property (B) for $(H', L', \prec')$. Consider distinct vertices $(x,a),(y,b) \in V(H')$. First suppose that $a = 0$ and $b = 0$. Then $(x,a) \prec' (y,b)$ if and only if $x \prec y$ if and only if $\phi(x)$ is an ancestor of $\phi(y)$ in T if and only if $\phi'((x,a))$ is an ancestor of $\phi'((y,b))$ in T, as desired. Now suppose that $a = 0$ and $b \in [1,k]$. Then $(x,a) \prec' (y,b)$ if and only if $x \prec y$ if and only if $\phi(x)$ is an ancestor of $\phi_b(y)$ in T if and only if $\phi'((x,a))$ is an ancestor of $\phi'((y,b))$ in T, as desired. Now suppose that $a = b \in [1,k]$. Then $(x,a) \prec' (y,b)$ if and only if $x \prec y$ if and only if $\phi_a(x)$ is an ancestor of $\phi_b(y)$ in T if and only if $\phi'((x,a))$ is an ancestor of $\phi'((y,b))$ in T, as desired. Finally, suppose that $a, b \in [1,k]$ and $a \neq b$. Then (x,a) and (y,b) are incomparable under $\prec'$, and $\phi'((x,a))$ and $\phi'((y,b))$ in T are unrelated in T, as desired. Hence property (B) is satisfied for $(H', L', \prec')$.

So ϕ' is an isomorphism from H' to a subgraph of $G[T_u^+]$ satisfying properties (A) and (B). Thus $(H', L', \prec')$ is contained in $G[T_u^+]$, as desired. Since $(H, L, \prec)$ is in the profile of v, we have $|V(H)| \leqslant (d+1)(h-1)(k+1)^{h-\ell(v)}$. Since $|V(H')| \leqslant (k+1)|V(H)|$ and $\ell(u) = \ell(v) - 1$, we have $|V(H')| \leqslant (d+1)(h-1)(k+1)^{h+1-\ell(v)} = (d+1)(h-1)(k+1)^{h-\ell(u)}$. Thus $(H', L', \prec')$ is in the profile of u. □

The proof now divides into two cases. If some group X_0 is adjacent in G to at least $h-1$ other groups above X_0, then we show that G contains $W\langle h,k\rangle$ as a minor. Otherwise, every group X is adjacent in G to at most $h-2$ other groups above X, in which case we show that G is $(h-1)$-colourable with bounded clustering.

Finding the Minor

Suppose that some group X_0 is adjacent in G to at least $h-1$ other groups $X_1, \dots, X_{h-1}$ above X_0. We now show that G contains $W\langle h,k\rangle$ as a minor; refer to Figure 3.

For $i \in [0, h-1]$, let v_i be the root of X_i. For $i \in [1, h-1]$, let w_i be a vertex in X_i adjacent to some vertex z_i in X_0; since G is a subgraph of the closure of T, w_i and thus v_i are on the v_0r-path in T. For $i \in [0, h-2]$, let u_i be the parent of v_i in T (which exists since $v_{h-2} \neq r$). So u_i is not in X_i (but may be in X_{i+1}). Note that $v_0, u_0, w_1, v_1, u_1, \dots, w_{h-2}, v_{h-2}, u_{h-2}, w_{h-1}, v_{h-1}$ appear in this order on the v_0r-path in T, where $v_0, v_1, \dots, v_{h-1}$ are distinct (since they are in distinct groups).

Let P_j be the z_jr-path in T for $j \in [1, h-1]$. Let H_0 be the graph with $V(H_0) :=$

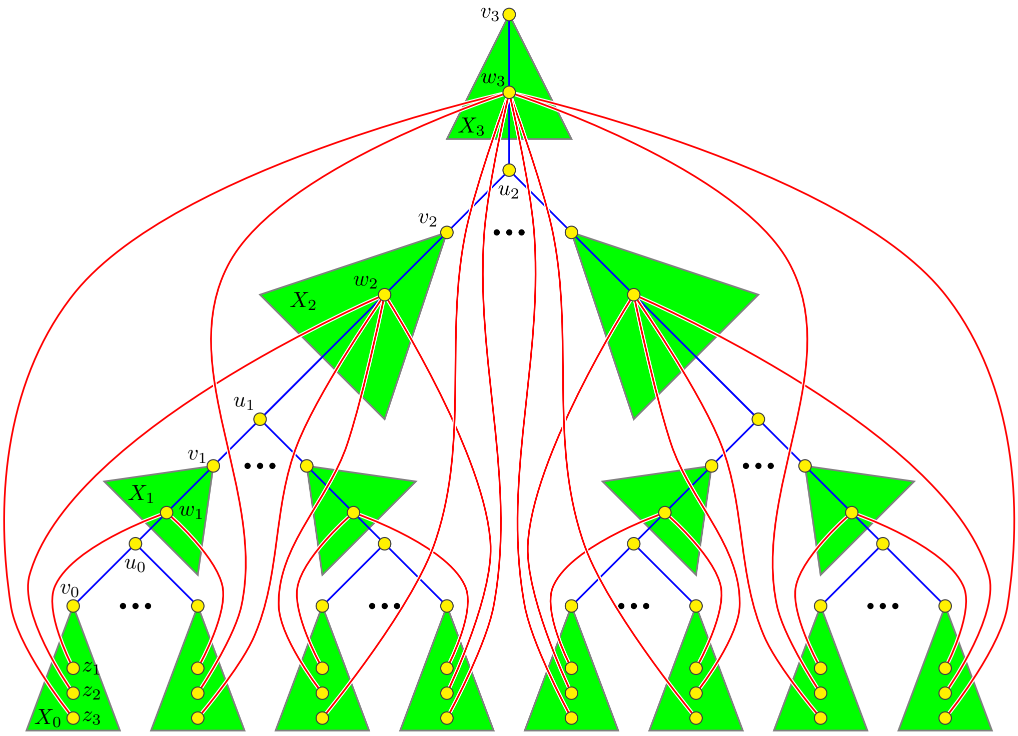

Figure 3: Construction of a $W\langle 4,k\rangle$ minor (where u_i might be in X_{i+1}).

$V(P_1 \cup \cdots \cup P_{h-1})$ and $E(H_0) := \{z_j w_j : j \in [1, h-1]\}$. Define the function $L_0 : V(H_0) \to [0, d-1]$ by $L_0(x) := \ell(x)$ for each $x \in V(H_0)$. Define the partial order $\prec_0$ on $V(H_0)$, where $x \prec_0 y$ if and only if x is ancestor of y in T. Thus $(H_0, L_0, \prec_0)$ is a ranked graph. By construction, $(H_0, L_0, \prec_0)$ is contained in $G[T^+_{v_0}]$. Since H_0 has less than $(d+1)(h-1)$ vertices, H_0 is in the profile of v_0. For $i = 0, 1, \dots, h-2$, let $(H_{i+1}, L_{i+1}, \prec_{i+1})$ be the $\ell(u_i)$-splice of $(H_i, L_i, \prec_i)$.

By induction, using Claim 1 at each step and since $G[T^+_{u_i}] \subseteq G[T^+_{v_{i+1}}]$, we conclude that for each $i \in [0, h-1]$, the ranked graph $(H_i, L_i, \prec_i)$ is in the profile of v_i. In particular, $(H_{h-1}, L_{h-1}, \prec_{h-1})$ is in the profile of v_{h-1}, and H_{h-1} is isomorphic to a subgraph of G. Note that each vertex of H_{h-1} is of the form $(((\dots(x, d_1), d_2), \dots), d_{h-1})$ for some $x \in V(H_0)$ and $d_1, \dots, d_{h-1} \in [0, k]$. For brevity, call such a vertex $x\langle d_1, \dots, d_{h-1}\rangle$. Note that if $x = w_j$ for some $j \in [1, h-1]$, then $d_1 = \cdots = d_j = 0$ (since w_j is above u_i whenever $i < j$, and $(H_{i+1}, L_{i+1}, \prec_{i+1})$ is the $\ell(u_i)$-splice of $(H_i, L_i, \prec_i)$).

For $x \in V(H_0)$, let Λ_x be the set of vertices $x\langle d_1, \dots, d_{h-1}\rangle$ in H_{h-1}. By construction, no two vertices in Λ_x are comparable under $\prec_{h-1}$. Therefore, by property (B), $V(T_a) \cap V(T_b) = \emptyset$ for all distinct $a, b \in \Lambda_x$. In particular, $V(T_a) \cap V(T_b) = \emptyset$ for all distinct $a, b \in \Lambda_{v_0}$. As proved above, $G[T_a]$ is connected for each $a \in V(T)$. Let G' be the graph obtained from G by contracting $G[T_a]$ into a single vertex $\alpha\langle d_1, \dots, d_{h-1}\rangle$, for each $a = v_0\langle d_1, \dots, d_{h-1}\rangle \in \Lambda_{v_0}$. So G' is a minor of G.

Let U be the tree with vertex set $\{\langle d_1, \dots, d_{h-1}\rangle : \exists j \in [0, h-1]\ d_1 = \cdots = d_j =$

0 and $d_{j+1},\dots,d_{h-1}\in[1,k]\}$, where the parent of $(0,\dots,0,d_{j+1},d_{j+2},\dots,d_{h-1})$ is $(0,\dots,0,d_{j+2},\dots,d_{h-1})$. Then U is isomorphic to the complete k-tree of height h rooted at $\langle 0,\dots,0\rangle$. We now show that the weak closure of U is a subgraph of G', where each vertex $\langle 0,\dots,0,d_{j+1},\dots,d_{h-1}\rangle$ of U with $j\in[1,h-1]$ is mapped to vertex $w_j\langle 0,\dots,0,d_{j+1},\dots,d_{h-1}\rangle$ of G', and each other vertex $\langle d_1,\dots,d_{h-1}\rangle$ of U is mapped to $\alpha\langle d_1,\dots,d_{h-1}\rangle$ of G'. For all $d_1,\dots,d_{h-1}\in[1,k]$ and $j\in[1,h-1]$ the vertex $z_j\langle d_1,\dots,d_{h-1}\rangle$ of G is contracted into the vertex $\alpha\langle d_1,\dots,d_{h-1}\rangle$ of G'. By construction, $z_j\langle d_1,\dots,d_{h-1}\rangle$ is adjacent to $w_j\langle 0,\dots,0,d_{j+1},\dots,d_{h-1}\rangle$ in G. So $\alpha\langle d_1,\dots,d_{h-1}\rangle$ is adjacent to $w_j\langle 0,\dots,0,d_{j+1},\dots,d_{h-1}\rangle$ in G'. This implies that the weak closure of U (that is, $W\langle h,k\rangle$) is isomorphic to a subgraph of G', and is therefore a minor of G.

Finding the Colouring

Now assume that every group X is adjacent in G to at most $h-2$ other groups above X. Then $(h-1)$-colour the groups in order of distance from the root, such that every group X is assigned a colour different from the colours assigned to the neighbouring groups above X. Assign each vertex within a group the same colour as that assigned to the whole group. This defines an $(h-1)$-colouring of G.

Consider the function $s:[0,d-1]\to\mathbb{N}$ recursively defined by

$$s(\ell):=\begin{cases}1 & \text{if } \ell=d-1\\ (k-1)\cdot M_{\ell+1}\cdot s(\ell+1) & \text{if } \ell\in[0,d-2].\end{cases}$$

Then every group at level ℓ has at most $s(\ell)$ vertices. By construction, our $(h-1)$-colouring of G has clustering $s(0)$, which is bounded by a function of d and k, as desired. □

3 Pathwidth

The following lemma of independent interest is the key to proving Theorem 2. Note that Eppstein [9] independently discovered the same result (with a slighly weaker bound on the path length). The decomposition method in the proof has been previously used, for example, by Dujmović, Joret, Kozik, and Wood [5, Lemma 17].

Lemma 8. *Every graph with pathwidth at most w has a vertex 2-colouring such that each monochromatic path has at most $(w+3)^w$ vertices.*

Proof. We proceed by induction on $w\geqslant 1$. Every graph with pathwidth 1 is a caterpillar, and is thus properly 2-colourable. Now assume $w\geqslant 2$ and the result holds for graphs with pathwidth at most $w-1$. Let G be a graph with pathwidth at most w. Let $(B_1,\dots,B_n)$ be a path-decomposition of G with width at most w. Let $t_0,t_1,t_2,\dots,t_m$ be a maximal sequence such that $t_0=0$, $t_1=1$, and for each $i\geqslant 2$, t_i is the minimum integer such that $B_{t_i} \cap B_{t_{i-1}} = \emptyset$. For odd i, colour every vertex in B_{t_i} 'red'. For even i, colour every vertex in B_{t_i} 'blue'. Since $B_{t_i} \cap B_{t_i-1} = \emptyset$, no vertex is coloured twice. Let G' be the subgraph of G induced by the uncoloured vertices. By the choice of B_{t_i}, each bag B_j with $j \in [t_{i-1}+1, t_i - 1]$ intersects $B_{t_{i-1}}$. Thus $(B_1 \cap V(G'), \dots, B_n \cap V(G'))$ is a path-decomposition of G' of width at most $w-1$. By induction, G' has a vertex 2-colouring such that each monochromatic path has at most $(w+3)^{w-1}$ vertices. Since $B_{t_i} \cup B_{t_{i+2}}$ separates $B_{t_i+1} \cup \dots \cup B_{t_{i+2}-1}$ from the rest of G, each monochromatic component of G is contained in $B_{t_i+1} \cup \dots \cup B_{t_{i+2}-1}$ for some $i \in [0, n-2]$. Consider a monochromatic path P in $G[B_{t_i+1} \cup \dots \cup B_{t_{i+2}-1}]$. Then P has at most $w+1$ vertices in $B_{t_{i+1}}$. Note that $P - B_{t_{i+1}}$ is contained in G'. Thus P consists of up to $w+2$ monochromatic subpaths in G' plus $w+1$ vertices in $B_{t_{i+1}}$. Hence P has at most $(w+2)(w+3)^{w-1} + (w+1) < (w+3)^w$ vertices. □

Nešetřil and Ossona de Mendez [20] showed that if a graph G contains no path on k vertices, then $\mathrm{td}(G) < k$ (since G is a subgraph of the closure of a DFS spanning tree with height at most k). Thus Lemma 8 implies:

Corollary 9. *Every graph with pathwidth at most w has a vertex 2-colouring such that each monochromatic component has treedepth at most $(w+3)^w$.*

Proof of Theorem 2. Let $\mathcal{G}$ be a minor-closed class of graphs, each with pathwidth at most w. Let h be the minimum integer such that $C\langle h,k\rangle \notin \mathcal{G}$ for some $k \in \mathbb{N}$. Consider $G \in \mathcal{G}$. By Corollary 9, G has a vertex 2-colouring such that each monochromatic component H of G has treedepth at most $(w+3)^w$. Since $C\langle h,k\rangle$ is not a minor of H, by Lemma 7, H is $(h-1)$-colourable with clustering $c((w+3)^w, k)$. Taking a product colouring, G is $(2h-2)$-colourable with clustering $c((w+3)^w, k)$. Hence $\chi_\Delta(\mathcal{G}) \leqslant \chi_\star(\mathcal{G}) \leqslant 2h-2$. □

Note that Lemma 8 cannot be extended to the setting of bounded tree-width graphs: Esperet and Joret (see [15, Theorem 4.1]) proved that for all positive integers w and d there exists a graph G with tree-width at most w such that for every w-colouring of G there exists a monochromatic component of G with diameter greater than d (and thus with a monochromatic path on more than d vertices, and thus with treedepth at least $\log_2 d$).

4 Fractional Colouring

This section proves Theorem 6. The starting point is the following key result of Dvořák and Sereni [7].[2]

[2] Dvořák and Sereni [7] expressed their result in the terms of "treedepth fragility". The sentence "proper minor-closed classes are fractionally treedepth-fragile" after Theorem 31 in [7] is equivalent to Theorem 10. Informally speaking, Theorem 10 shows that the fractional "treedepth" chromatic number of every minor-closed class equals 1.

Theorem 10 ([7]). *For every proper minor-closed class $\mathcal{G}$ and every $\delta > 0$ there exists $d \in \mathbb{N}$ satisfying the following. For every $G \in \mathcal{G}$ there exist $s \in \mathbb{N}$ and $X_1, X_2, \dots, X_s \subseteq V(G)$ such that:*

- $\operatorname{td}(G[X_i]) \leqslant d$*, and*
- *every $v \in V(G)$ belongs to at least $(1-\delta)s$ of these sets.*

We now prove a lower bound on the fractional defective chromatic number of the closure of complete trees of given height.

Lemma 11. *Let $\mathcal{C}_h := \{C\langle h,k\rangle\}_{k\in\mathbb{N}}$. Then $\chi^f_\Delta(\mathcal{C}_h) \geqslant h$.*

Proof. We show by induction on h that if $C\langle h,k\rangle$ is fractionally t-colourable with defect d, then $t \geqslant h-(h-1)d/k$. This clearly implies the lemma. The base case $h = 1$ is trivial.

For the induction step, suppose that $G := C\langle h,k\rangle$ is fractionally t-colourable with defect d. Thus there exist $Y_1, Y_2, \dots, Y_s \subseteq V(G)$ and $\alpha_1, \dots, \alpha_s \in [0,1]$ such that:

- every component of $G[Y_i]$ has maximum degree at most d,
- $\sum_{i=1}^{s} \alpha_i \leqslant t$, and
- $\sum_{i: v\in Y_i} \alpha_i \geqslant 1$ for every $v \in V(G)$.

Let r be the vertex of G corresponding to the root of the complete k-ary tree and let $H_1, \dots, H_k$ be the components of $G - r$. Then each H_i is isomorphic to $C\langle h-1,k\rangle$. Let $J_0 := \{j : r \in Y_j\}$, and let $J_i := \{j : Y_j \cap V(H_i) \neq \emptyset\}$ for $i \in [1,k]$. Denote $\sum_{j\in J_i} \alpha_j$ by $\alpha(J_i)$ for brevity. Thus $\alpha(J_0) \geqslant 1$. For $i \in [1,k]$, the subgraph H_i is $\alpha(J_i)$-colourable with defect d, and thus $\alpha(J_i) \geqslant h-1-(h-2)d/k$ by the induction hypothesis. Thus

$$(k-d)\alpha(J_0) + \sum_{i=1}^{k} \alpha(J_i) \geqslant (k-d) + k(h-1) - (h-2)d = kh - (h-1)d.$$

If $j \in J_0$ then Y_j intersects at most d of $H_1, \dots, H_k$ (since $G[Y_j]$ has maximum degree at most d). Thus every α_j appears with coefficient at most k in the left side of the above inequality, implying

$$(k-d)\alpha(J_0) + \sum_{i=1}^{k} \alpha(J_i) \leqslant k \sum_{i=1}^{s} \alpha_i \leqslant kt.$$

Combining the above inequalities yields the claimed bound on t. □

Proof of Theorem 6. By Lemma 11,

$$\chi^f_\star(\mathcal{G}) \geqslant \chi^f_\Delta(\mathcal{G}) \geqslant \operatorname{tcn}(\mathcal{G}) - 1.$$

It remains to show that $\chi^f_\star(\mathcal{G}) \leqslant \operatorname{tcn}(\mathcal{G}) - 1$. Equivalently, we need to show that for all $h,k \in \mathbb{N}$ and $\varepsilon > 0$, if $C\langle h,k\rangle \notin \mathcal{G}$ then there exists c such that every graph in $\mathcal{G}$

is fractionally $(h-1+\varepsilon)$-colourable with clustering c. This is trivial for $h=1$, and so we assume $h \geqslant 2$.

Let $d \in \mathbb{N}$ satisfy the conclusion of Theorem 10 for the class $\mathcal{G}$ and $\delta = 1 - \frac{1}{1+\varepsilon/(h-1)}$. Choose $c = c(d, k+1)$ to satisfy the conclusion of Lemma 7. We show that c is as desired.

Consider $G \in \mathcal{G}$. By the choice of d there exists $s \in \mathbb{N}$ and $X_1, X_2, \dots, X_s \subseteq V(G)$ such that:

- $\mathrm{td}(G[X_i]) \leqslant d$, and
- every $v \in V(G)$ belongs to at least $(1-\delta)s$ of these sets.

Since $C\langle h, k\rangle \notin \mathcal{G}$, we have $W\langle h, k+1\rangle \notin \mathcal{G}$, and by the choice of c, for each $i \in [1, s]$ there exists a partition $(Y_i^1, Y_i^2, \dots, Y_i^{h-1})$ of X_i such that every component of $G[Y_i^j]$ has at most c vertices. Every vertex of G belongs to at least $(1-\delta)s$ sets Y_i^j where $i \in [1, s]$ and $j \in [1, h-1]$. Considering these sets with equal coefficients $\alpha_i^j := \frac{1}{(1-\delta)s}$, we conclude that G is fractionally $\frac{h-1}{1-\delta}$-colourable with clustering c, as desired (since $\frac{h-1}{1-\delta} = h-1+\varepsilon$). □

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