Pure pairs. IV. Trees in bipartite graphs

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Abstract

In this paper we investigate the bipartite analogue of the strong Erdős-Hajnal property. We prove that for every forest H and every τ with $0 < \tau \leq 1$, there exists $\varepsilon > 0$, such that if G has a bipartition (A, B) and does not contain H as an induced subgraph, and has at most $(1-\tau)|A| \cdot |B|$ edges, then there is a stable set X of G with $|X \cap A| \geq \varepsilon|A|$ and $|X \cap B| \geq \varepsilon|B|$. No graphs H except forests have this property.

1 Introduction

All graphs in this paper are finite and have no loops or parallel edges. If G, H are graphs, we say G *contains* H if some induced subgraph of G is isomorphic to H, and G is H*-free* otherwise. We denote by $\alpha(G), \omega(G)$ the cardinalities of the largest stable sets and largest cliques in G respectively. Two disjoint sets A, B are *complete* if every vertex in A is adjacent to every vertex in B, and *anticomplete* if no vertex in A has a neighbour in B; and we say A *covers* B if every vertex in B has a neighbour in A. A pair (A, B) of subsets of $V(G)$ is *pure* if A, B are complete or anticomplete to each other. We denote the complement graph of H by $\overline{H}$. We denote the number of vertices of G by $|G|$.

The Erdős-Hajnal conjecture [6, 7] asserts that:

1.1 Conjecture: *For every graph H, there exists $c > 0$ such that every H-free graph G satisfies*

$$\alpha(G)\omega(G) \geq |G|^c.$$

An *ideal* or *hereditary class* of graphs is a class $\mathcal{G}$ of graphs such that if $G \in \mathcal{G}$ and H is isomorphic to an induced subgraph of G then $H \in \mathcal{G}$. We say that an ideal $\mathcal{G}$ of graphs has the *Erdős-Hajnal property* if there exists $c > 0$ such that every $G \in \mathcal{G}$ satisfies $\alpha(G)\omega(G) \geq |G|^c$. Thus the Erdős-Hajnal conjecture states that the class of H-free graphs has the Erdős-Hajnal property.

One way to prove that a class of graphs has the Erdős-Hajnal property is to prove something stronger. We say that a class $\mathcal{G}$ has the *strong Erdős-Hajnal property* if there is some $\varepsilon > 0$ such that every graph $G \in \mathcal{G}$ with at least two vertices contains disjoint sets A, B that have size at least $\varepsilon|G|$ and are either complete or anticomplete (that is, (A, B) is a pure pair). It is easy to prove that, for an ideal $\mathcal{G}$, the strong Erdős-Hajnal property implies the Erdős-Hajnal property (see [2, 9]). Unfortunately, it is also easy to prove (by considering a sparse random graph with girth larger than $|H|$) that if the class of H-free graphs has the strong Erdős-Hajnal property then H must be a forest; and (by considering the complement of a sparse random graph) that $\overline{H}$ must also be a forest. Thus H has at most four vertices, and the Erdős-Hajnal conjecture was already known for these graphs.

But what if we exclude *both* a forest and the complement of a forest? Then there is some good news. In an earlier paper [4], with Maria Chudnovsky, we proved that this implies the strong Erdős-Hajnal property:

1.2 *For every forest H, there exists $\varepsilon > 0$ such that for every graph G that is both H-free and $\overline{H}$-free with $|G| \geq 2$, there is a pure pair (Z_1, Z_2) of subsets of $V(G)$ with $|Z_1|, |Z_2| \geq \varepsilon|G|$.*

We also proved the stronger result that, if G is not too dense, then it is enough just to exclude a forest.

1.3 *For every forest H there exists $\varepsilon > 0$ such that for every H-free graph G with $|G| \geq 2$, either*

- *some vertex has degree at least $\varepsilon|G|$; or*
- *there exist disjoint $Z_1, Z_2 \subseteq V(G)$ with $|Z_1|, |Z_2| \geq \varepsilon|G|$, anticomplete.*

Neither of these theorems hold for any graph H such that neither of $H, \overline{H}$ is a forest (this follows easily from the random construction by Erdős of graphs with large girth and large chromatic number [5]).

In this paper we look at the analogous question for bipartite graphs. It will help to set up some terminology. A *bigraph* G is a graph with a designated bipartition $(V_1(G), V_2(G))$; thus, $V_1(G), V_2(G)$

are disjoint stable sets of G with union $V(G)$. The bigraph obtained from the same graph by exchanging $V_1(G)$ and $V_2(G)$ is called the *transpose* of G. The *bicomplement* of a bigraph G is the bigraph G' with the same vertex set, and the same bipartition, in which for all $v \in V_1(G)$ and $w \in V_2(G)$, v and w are adjacent in G if and only if they are not adjacent in G'. If G is a bigraph, and $X \subseteq V(G)$, $G[X]$ denotes the bigraph induced on X in the natural sense (that is, the bigraph H with $V(H) = X$, where for all $u, v \in X$, u, v are adjacent in H if and only if they are adjacent in G, and $V_i(H) = V_i(G) \cap X$ for $i = 1, 2$). We call $G[X]$ an *induced sub-bigraph* of G. An *isomorphism* between bigraphs G, H is an isomorphism between the corresponding graphs that maps $V_i(G)$ to $V_i(H)$ for $i = 1, 2$. If G, H are bigraphs, we say that G *contains* H if there is an isomorphism from H to an induced sub-bigraph of G. We say G is *H-free* if G does not contain H. We stress that a bigraph G may contain a bigraph H and not contain the transpose of H.

What about the analogue of 1.2 for bigraphs? Let H be a forest bigraph, and suppose that G is a bigraph that contains neither H not its bicomplement: must there then be a pure pair of subsets of $V_1(G)$ and $V_2(G)$ respectively, both of linear size? This seems neither to imply, nor to be implied by, the results of [4] just mentioned, but there has been some previous work on it. It was conjectured in this form by Alecu, Atminas, Lozin and Zamaraev [1], amd Axenovich, Tompkins and Weber [3], and independently Korándi, Pach and Tomon [10] gave a result for bigraphs H in which $|V_1(H)| \le 2$:

1.4 *Let H be a forest bigraph with $|V_1(H)| \le 2$, such that the bicomplement of H is also a forest. Then there exists $\varepsilon > 0$ such that, if G is a bigraph that is H-free then there is a pure pair (Z_1, Z_2) in G with $Z_i \subseteq V_i(G)$ and $|Z_i| \ge \varepsilon |V_i(G)|$ for $i = 1, 2$.*

In this paper we will prove the conjecture of Alecu, Atminas, Lozin and Zamaraev [1], that the bipartite analogue of 1.2 holds in full. Our main result is the following:

1.5 *For every forest bigraph H with bicomplement J, there exists $\varepsilon > 0$ such that, if G is a bigraph that is both H-free and J-free, then there is a pure pair (Z_1, Z_2) in G with $Z_i \subseteq V_i(G)$ and $|Z_i| \ge \varepsilon |V_i(G)|$ for $i = 1, 2$.*

Note that this implies that it is sufficient to exclude any forest H and any forest bicomplement J (as we can always consider the forest obtained from the disjoint union of H and $\overline{J}$). Furthermore, if we exclude any finite set of bigraphs, then the random construction once again shows that one of the graphs must be a forest, and one must be the bicomplement of a forest. Thus 1.5 leads to a characterizion of the finite sets of excluded subgraphs that give the (bipartite) strong Erdős-Hajnal property.

As with 1.3, we will also prove a 'one-sided' version of the result for sparse graphs, where we only exclude a forest. Let us say a bigraph G is *ε-coherent*, where $\varepsilon > 0$, if:

- every vertex in $V_1(G)$ has degree less than $\varepsilon |V_2(G)|$;
- every vertex in $V_2(G)$ has degree less than $\varepsilon |V_1(G)|$;
- there do not exist anticomplete subsets $Z_1 \subseteq V_1(G)$ and $Z_2 \subseteq V_2(G)$, such that $|Z_i| \ge \varepsilon |V_i(G)|$ for $i = 1, 2$.

Thus, if $0 < \varepsilon \le \varepsilon'$ and G is ε-coherent then it is also ε'-coherent. Our main theorem says:

1.6 *For every forest bigraph H, there exists $\varepsilon > 0$ such that every ε-coherent bigraph contains H.*

This can be further strengthened, to prove the statement in the abstract; in the next section we will show that the "sparse" hypothesis can be replaced with a "not very dense" hypothesis, as follows:

1.7 *For every forest bigraph H, and every τ with $0 < \tau \le 1$, there exists $\varepsilon > 0$ such that if G is an H-free bigraph with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges, then there are anticomplete sets $Z_i \subseteq V_i(G)$ with $|Z_i| \ge \varepsilon |V_i(G)|$ for $i = 1, 2$.*

Finally, let us note that there are very interesting questions of this type for *ordered* bigraphs, or equivalently 0-1 matrices. Korándi, Pach and Tomon [10] also made a much stronger conjecture, that the statement of 1.7 still holds if we work with ordered bigraphs (bigraphs G with fixed linear orders on $V_1(G)$ and on $V_2(G)$) and ask for containment that respects the orders. This remains open, and is discussed further in [12].

2 Reducing to the sparse case

In this section we will show that 1.6 implies 1.7 and 1.5. We need the following. For general graphs an analogue was proved by Rödl [11], using the regularity lemma, but for bigraphs its proof is much easier, and does not use the regularity lemma. The result is essentially due to Erdős, Hajnal and Pach [8], but we give a proof since it is short.

2.1 *Let H be a bigraph, and let $\varepsilon > 0$. Then there exists $\delta > 0$ with the following property. Let G be an H-free bigraph with $V_1(G), V_2(G) \ne \emptyset$; then there exists $A_i \subseteq V_i(G)$ with $|A_i| \ge \delta |V_i(G)|$ for $i = 1, 2$, such that either*

- *every vertex in A_1 has fewer than $\varepsilon|A_2|$ neighbours in A_2, and every vertex in A_2 has fewer than $\varepsilon|A_1|$ neighbours in A_1; or*
- *every vertex in A_1 has more than $(1-\varepsilon)|A_2|$ neighbours in A_2, and every vertex in A_2 has more than $(1-\varepsilon)|A_1|$ neighbours in A_1.*

Proof. We may assume that $\varepsilon < 1$ and $V(H) \ne \emptyset$, because otherwise the result is trivial. Let $V_1(H) = \{a_1, \dots, a_k\}$ and $V_2(H) = \{b_1, \dots, b_\ell\}$. If $k = 0$ or $\ell = 0$ the theorem holds with $\delta = \min(1/2, |H|^{-1})$, so we may assume that $k, \ell > 0$. Define $\delta = \min(1/2, |H|^{-1}, (\varepsilon/2)^k/\ell)$. We claim that δ satisfies the theorem. Let G be an H-free bigraph. If $|V_1(G)| < \delta^{-1}$, the result holds with $|A_1| = 1, A_1 = \{v\}$ say, and and A_2 the larger of $N(v)$ and $V_2(G) \setminus N(v)$, since $\delta \le 1/2$; so we may assume that $|V_1(G)| \ge \delta^{-1}$, and similarly $|V_2(G)| \ge \delta^{-1}$.

Let $0 \le i \le k$, let $u_1, \dots, u_i \in V_1(G)$ be distinct, and let $1 \le j \le \ell$. We say that $v \in V_2(G)$ is *j-appropriate* for $(u_1, \dots, u_i)$ if for all h with $1 \le h \le i$:

- if a_h, b_j are adjacent in H then u_h, v are adjacent in G; and
- if a_h, b_j are nonadjacent in H then u_h, v are nonadjacent in G.

For all i with $0 \le i \le k$ let $n_i = (\varepsilon/2)^i |V_2(G)|$. Thus $n_k = (\varepsilon/2)^k |V_2(G)| \ge (\varepsilon/2)^k \delta^{-1} \ge \ell$. For $0 \le i \le k$, we say a sequence $(u_1, \dots, u_i)$ of vertices in $V_1(G)$ is *i-good* if $u_1, \dots, u_i$ are all distinct, and for $1 \le j \le \ell$ there are at least n_i vertices in $V_2(G)$ that are j-appropriate for $(u_1, \dots, u_i)$.

The null sequence is 0-good, since $n_0 = |V_2(G)|$; and so we may choose i with $0 \le i \le k$ maximum such that there is an i-good sequence $(u_1, \dots, u_i)$. Suppose that $i = k$. For $1 \le j \le \ell$ choose $v_j \in V_2(G)$, j-appropriate for $(u_1, \dots, u_k)$, such that $v_1, \dots, v_\ell$ are all different (the last is possible since $n_k \ge \ell$); then the subgraph induced on $\{u_1, \dots, u_k, v_1, \dots, v_\ell\}$ is isomorphic to H, a contradiction. So $i < k$.

From the maximality of i, for each $u \in V_1(G) \setminus \{u_1, \dots, u_i\}$, there exists $j \in \{1, \dots, \ell\}$ such that fewer than n_{i+1} vertices in $V_2(G)$ are j-appropriate for $(u_1, \dots, u_i, u)$. Hence there exist $A_1 \subseteq V_1(G) \setminus \{u_1, \dots, u_i\}$ with $|A_1| \ge (|V_1(G)| - k)/\ell$, and $j \in \{1, \dots, \ell\}$, such that for each $u \in A_1$, fewer than n_{i+1} vertices in $V_2(G)$ are j-appropriate for $(u_1, \dots, u_i, u)$. Let A'_2 be the set of vertices in $V_2(G)$ that are j-appropriate for $(u_1, \dots, u_i)$; thus $|A'_2| \ge n_i$, from the choice of i. Consequently

- if a_{i+1}, b_j are adjacent in H, then each $u \in A_1$ has fewer than n_{i+1} neighbours in A'_2;
- if a_{i+1}, b_j are nonadjacent in H, then each $u \in A_1$ has more than $|A'_2| - n_{i+1}$ neighbours in A'_2.

By taking bicomplements if necessary, we may assume the former. There are fewer than $n_{i+1}|A_1|$ edges between A_1, A'_2, and so at least $|A'_2|/2$ vertices in A'_2 have fewer than $2n_{i+1}|A_1|/|A'_2|$ neighbours in A_1. Let A_2 be the set of vertices in A'_2 with fewer than $\varepsilon|A_1|$ neighbours in A_1. Since $\varepsilon \ge 2n_{i+1}/|A'_2|$ (because $|A'_2| \ge n_i = (2/\varepsilon)n_{i+1}$), it follows that $|A_2| \ge |A'_2|/2 \ge n_i/2$. Thus:

- $|A_1| \ge \delta|V_1(G)|$, since $|A_1| \ge (|V_1(G)| - k)/\ell \ge \varepsilon|V_1(G)|$ (because $|V_1(G)| \ge \varepsilon^{-1} \ge k + \ell$);
- $|A_2| \ge \delta|V_2(G)|$, since $|A_2| \ge n_i/2 \ge n_k = (\varepsilon/2)^k|V_2(G)| \ge \delta|V_2(G)|$;
- every vertex in A_1 has fewer than $n_{i+1} = \varepsilon n_i/2 \le \varepsilon|A_2|$ neighbours in A_2; and
- every vertex in A_2 has fewer than $\varepsilon|A_1|$ neighbours in A_1.

This proves 2.1. ∎

Proof of 1.5, assuming 1.6. Let H be a forest bigraph with bicomplement J. By 1.6 there exists $\varepsilon > 0$ such that every ε-coherent bigraph contains H. By 2.1 there exists $\delta' > 0$ such that taking $\delta = \delta'$ satisfies 2.1. Let $\delta = \varepsilon\delta'$; we claim that δ satisfies 1.5.

Let G be a bigraph that is both H-free and J-free. We must show that there exist $B_1 \subseteq V_1(G)$ and $B_2 \subseteq V_2(G)$ such that $|B_i| \ge \delta|V_i(G)|$ for $i = 1, 2$, and B_1, B_2 are complete or anticomplete. We may assume that $V_i(G) \neq \emptyset$ for $i = 1, 2$. By the choice of δ', there exist $A_i \subseteq V_1(G)$ for $i = 1, 2$, such that $|A_i| \ge \delta'|V_i(G)|$ for $i = 1, 2$, and either

- every vertex in A_1 has fewer than $\varepsilon|A_2|$ neighbours in A_2, and every vertex in A_2 has fewer than $\varepsilon|A_1|$ neighbours in A_1; or
- every vertex in A_1 has more than $(1 - \varepsilon)|A_2|$ neighbours in A_2, and every vertex in A_2 has more than $(1 - \varepsilon)|A_1|$ neighbours in A_1.

Suppose that the first holds. By applying 1.6 to the subgraph G' of G induced on $A_1 \cup A_2$, we deduce that G' is not ε-coherent; and so there exist anticomplete subsets $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$, such that $|B_i| \ge \varepsilon|A_i|$ for $i = 1, 2$. But $\varepsilon|A_i| \ge \varepsilon\delta'|V_i(G)| = \delta|V_i(G)|$, as required.

If the second holds, then we apply 1.6 to the bicomplement G' of the subgraph of G induced on $A_1 \cup A_2$, and deduce that G' is not ε-coherent (since it is H-free, because G is J-free). So there exist subsets $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$, such that $|B_i| \ge \varepsilon|A_i|$ for $i = 1, 2$, and B_1, B_2 are anticomplete in G' and hence complete in G. Since $|B_i| \ge \delta|V_i(G)|$ for $i = 1, 2$, the result follows. This proves 1.5. ∎

We can in fact weaken the hypothesis of 1.6 that G is sparse. In 1.7 we replace this by the hypothesis that G is not very dense, that the bicomplement of G has at least τn^2 edges. The "not very dense" hypothesis is as good as the "sparse" hypothesis because of the next result.

2.2 *For all $c, \varepsilon, \tau > 0$ with $\varepsilon < \tau \leq 8/9$, there exists $\delta > 0$ with the following property. Let G be a bigraph with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges and with $V_1(G), V_2(G) \neq \emptyset$. Then there exist $Z_i \subseteq V_i(G)$ with $|Z_i| \geq \delta|V_i(G)|$ for $i = 1, 2$, such that there are fewer than $(1-\varepsilon)|Y_1| \cdot |Y_2|$ edges between Y_1, Y_2 for all subsets $Y_i \subseteq Z_i$ with $|Y_i| \geq c|Z_i|$ for $i = 1, 2$.*

Proof. By reducing c, we may assume that $c \leq 1/3$. Let

$$\lambda = 1 - \frac{(\tau - \varepsilon)c^2}{(1-c^2)(1-\tau)}.$$

It follows that $0 \leq \lambda < 1$ (since $c^2/(1-c^2) \leq 1/8$ and $(\tau - \varepsilon)/(1-\tau) < 8$). Choose an integer $n \geq 0$ such that $\lambda^n(1-\tau) \leq (1-\varepsilon)c/2$. Let $\delta = \min(c^n, \tau)$. We will show that δ satisfies the theorem.

Let G be a bigraph with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges. Choose an integer $t \geq 0$ with $t \leq n$, maximum such that there are subsets $Z_i \subseteq V_i(G)$ with $|Z_i| \geq c^t|V_i(G)|$ for $i = 1, 2$, where the number of edges between Z_1, Z_2 is at most $\lambda^t(1-\tau)|Z_1| \cdot |Z_2|$. (This is possible since we may take $t = 0$ and $Z_i = V_i(G)$ for $i = 1, 2$.)

(1) *If $t = n$ then the theorem holds.*

Suppose that $t = n$. Thus there are at most $\lambda^n(1-\tau)|Z_1| \cdot |Z_2| \leq (1-\varepsilon)(c/2)|Z_1| \cdot |Z_2|$ edges between Z_1, Z_2. At least half of the vertices in Z_1 have at most $(1-\varepsilon)c|Z_2|$ neighbours in Z_2; choose $Z_1' \subseteq Z_1$ with $|Z_1'| \geq |Z_1|/2 \geq \delta|V_1(G)|$ such that every vertex in Z_1' has at most $(1-\varepsilon)c|Z_2|$ neighbours in Z_2. Now let $Y_1 \subseteq Z_1'$, and let $Y_2 \subseteq Z_2$ with $|Y_2| \geq c|Z_2|$. Each vertex in Y_1 has at most $(1-\varepsilon)c|Z_2| \leq (1-\varepsilon)|Y_2|$ neighbours in Y_2, and so the number of edges between Y_1, Y_2 is at most $(1-\varepsilon)|Y_1| \cdot |Y_2|$. Since $|Z_1'| \geq \delta|V_1(G)|$ and $|Z_2| \geq \delta|V_2(G)|$, the pair Z_1', Z_2 satisfies the theorem. This proves (1).

(2) *If $|V_1(G)| \leq 1/\delta$ or $|V_2(G)| \leq 1/\delta$ then the theorem holds.*

Suppose that $|V_1(G)| \leq 1/\delta$, say. Since G has at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges, and $V_1(G) \neq \emptyset$, some vertex $v_1 \in V_1(G)$ has at most $(1-\tau)|V_2(G)|$ neighbours in $V_2(G)$; and so there is a set $Z_2 \subseteq V_2(G)$ with $|Z_2| \geq \tau|V_2(G)| \geq \delta|V_2(G)|$ that is anticomplete to $Z_1 = \{v_1\}$. Since $|Z_1| = 1 \geq \delta|V_1(G)|$, the theorem holds. This proves (2).

(3) *If $t < n$ then the theorem holds.*

Suppose that $t < n$. Since

$$|Z_i| \geq c^t|V_i(G)| \geq c^{n-1}|V_i(G)| \geq (\delta/c)|V_i(G)| \geq \delta|V_i(G)|$$

for $i = 1, 2$, it suffices to show that there do not exist subsets $Y_i \subseteq Z_i$ with $|Y_i| \geq c|Z_i|$ for $i = 1, 2$, such that the number of edges between Y_1, Y_2 is at least $(1-\varepsilon)|Y_1| \cdot |Y_2|$. Suppose that such subsets exist. By averaging, we may assume that $|Y_i| = \lceil c|Z_i|\rceil$ for $i = 1, 2$; by (2) we may assume that $\delta|V_i(G)| > 1$ for $i = 1, 2$, and therefore $|Z_i| \geq (\delta/c)|V_i(G)| > 1/c \geq 3$; and since $c \leq 1/3$, it follows that

$$\lceil c|Z_i|\rceil \leq c|Z_i| + 1 \leq (1-c)|Z_i|.$$

Let $X_i = Z_i \setminus Y_i$, and let $y_i = |Y_i|$ and $x_i = |X_i|$ for $i = 1, 2$. Thus $x_i, y_i \geq c|Z_i| \geq c^{t+1}|V_i(G)|$, for $i = 1, 2$. From the choice of t, it follows that

- there are more than $\lambda^{t+1}(1-\tau)y_1x_2$ edges between Y_1 and X_2;
- there are more than $\lambda^{t+1}(1-\tau)x_1y_2$ edges between X_1 and Y_2; and
- there are more than $\lambda^{t+1}(1-\tau)x_1x_2$ edges between X_1 and X_2.

Adding, we deduce that there are more than

$$\lambda^{t+1}(1-\tau)(y_1x_2 + x_1y_2 + x_1x_2) + (1-\varepsilon)y_1y_2$$

edges between Z_1, Z_2, and so

$$\lambda^{t+1}(1-\tau)(y_1x_2 + x_1y_2 + x_1x_2) + (1-\varepsilon)y_1y_2 < \lambda^t(1-\tau)|Z_1| \cdot |Z_2|.$$

Since $|Z_i| = x_i + y_i$ for $i = 1, 2$, it follows that

$$(1-\varepsilon-\lambda^{t+1}(1-\tau))y_1y_2 < \lambda^t(1-\lambda)(1-\tau)(x_1+y_1)(x_2+y_2),$$

and since $y_1y_2 \geq c^2(x_1+y_1)(x_2+y_2)$, we deduce that

$$(1-\varepsilon-\lambda^{t+1}(1-\tau))c^2 < \lambda^t(1-\lambda)(1-\tau).$$

Since $\lambda^{t+1} \leq \lambda$, and $\lambda^t \leq 1$, it follows that $(1-\varepsilon-\lambda(1-\tau))c^2 < (1-\lambda)(1-\tau)$, and so

$$\lambda(1-\tau)(1-c^2) < (1-\tau) - (1-\varepsilon)c^2,$$

contradicting the definition of λ. This proves (3).

From (1) and (3), this proves 2.2. ∎

Let us deduce 1.7, which we restate:

2.3 *For every forest bigraph H, and every $\tau > 0$, there exists $\varepsilon > 0$ such that if G is an H-free bigraph with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges, then there are anticomplete sets $Z_i \subseteq V_i(G)$ with $|Z_i| \geq \varepsilon|V_i(G)|$ for $i = 1, 2$.*

Proof, assuming 1.6. Let H be a forest bigraph, and let $\tau > 0$. By reducing τ, we may assume that $\tau \leq 8/9$. By 1.6 there exists $\eta > 0$ such that every η-coherent bigraph contains H. By reducing η, we may assume that $\eta < \tau$. Choose $c > 0$ such that setting $\delta = c$ satisfies 2.1 with ε replaced by η. Let δ satisfy 2.2 with ε replaced by η. Let $\varepsilon = c\delta\eta$.

Now let G be an H-free bigraph with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges. We may assume that $V_1(G), V_2(G) \neq \emptyset$. By 2.2, there exist $Z_i \subseteq V_i(G)$ with $|Z_i| \geq \delta|V_i(G)|$ for $i = 1, 2$, such that there are fewer than $(1-\eta)|Y_1| \cdot |Y_2|$ edges between Y_1, Y_2 for all subsets $Y_i \subseteq Z_i$ with $|Y_i| \geq c|Z_i|$ for $i = 1, 2$. By 2.1, applied to the sub-bigraph induced on $Z_1 \cup Z_2$, there exists $Y_i \subseteq Z_i$ with $|Y_i| \geq c|Z_i|$ for $i = 1, 2$, such that either

- every vertex in Y_1 has fewer than $\eta|Y_2|$ neighbours in Y_2, and every vertex in Y_2 has fewer than $\eta|Y_1|$ neighbours in Y_1; or
- every vertex in Y_1 has more than $(1-\eta)|Y_2|$ neighbours in Y_2, and every vertex in Y_2 has more than $(1-\eta)|Y_1|$ neighbours in Y_1.

The second is impossible, since for all such Y_1, Y_2 there are fewer than $(1-\eta)|Y_1| \cdot |Y_2|$ edges between Y_1, Y_2. Thus the first bullet holds. Since the sub-bigraph induced on $Y_1 \cup Y_2$ is H-free, it is not η-coherent; and so there exist anticomplete sets $X_i \subseteq Y_i$ with

$$|X_i| \geq \eta|Y_i| \geq \eta(c\delta|V_i(G)|) \geq \varepsilon|V_i(G)|$$

for $i = 1, 2$. This proves 2.3. ∎

3 Parades and concavity

In this section we carry out the main step in the proof of 1.6. The general approach is similar to the main proof in [4], and we apologize for repeating some material and ideas from there. But adapting the proof of [4] to work for bipartite graphs was nontrivial, and there seems no way to present the tricky new parts of the argument without also including the straightforward parts.

We will need a number of definitions. A *parade* in a bigraph G means a sequence

$$(A_1, \ldots, A_K; B_1, \ldots, B_L)$$

of pairwise disjoint nonempty subsets of $V(G)$, such that

- $A_1, \ldots, A_K \subseteq V_1(G)$, and $B_1, \ldots, B_L \subseteq V_2(G)$;
- $A_1, \ldots, A_K$ all have the same cardinality, and $B_1, \ldots, B_L$ all have the same cardinality (possibly different).

Its *length* is the pair (K, L), and its *width* is the pair $(|A_1|, |B_1|)$. (For convenience in handling widths, let us $(W_1', W_2') \leq (W_1, W_2)$ if $W_i' \leq W_i$ for $i = 1, 2$, and define $\lambda(W_1, W_2) = (\lambda W_1, \lambda W_2)$ for $\lambda \geq 0$.) We call the sets A_i, B_i *blocks* of the parade. We are interested in parades of some fixed length, and width at least linear in $(|V_1(G)|, |V_2(G)|)$. (We remark that in later papers of this series we use "parade" to mean the same thing with the second bullet above removed; but here it is convenient to include the second bullet in the definition.)

Here are two useful ways to make smaller parades from larger. Let $\mathcal{P} = (A_1, \ldots, A_K; B_1, \ldots, B_L)$ be a parade. First, let $1 \leq r_1 < r_2 \cdots < r_k \leq K$, and $1 \leq s_1 < \cdots < s_\ell \leq L$; then $\mathcal{P}' = (A_{r_1}, \ldots, A_{r_k}; B_{s_1}, \ldots, B_{s_\ell})$ is a parade, of smaller length but of the same width, and we call it a *sub-parade* of $\mathcal{P}$. Let $I = \{r_1, \ldots, r_k\}$ and $J = \{s_1, \ldots, s_\ell\}$; then $\mathcal{P}[I; J] = (A_i\ (i \in I); B_j\ (j \in J))$ denotes the same subparade $\mathcal{P}'$. Second, for $1 \leq i \leq K$ let $A_i' \subseteq A_i$, all of the same cardinality, and for $1 \leq j \leq L$ let $B_j' \subseteq B_j$, all of the same cardinality; then the sequence $(A_1', \ldots, A_K'; B_1', \ldots, B_L')$ is a parade, of the same length but of smaller width, and we call it a *contraction* of $\mathcal{P}$. A contraction of a sub-parade (or equivalently, a sub-parade of a contraction) we call a *minor* of $\mathcal{P}$. (Thus a minor of a minor is a minor.)

Let $\mathcal{P} = (A_1, \ldots, A_K; B_1, \ldots, B_L)$ be a parade in a bigraph G. We say an induced sub-bigraph H of G is *$\mathcal{P}$-rainbow* or *rainbow relative to* $\mathcal{P}$ if each vertex of H belongs to some block of $\mathcal{P}$, and no two vertices belong to the same block. A *copy* of a bigraph T in a bigraph G is a bigraph isomorphic to T that is contained in G.

An *ordered bigraph* is a bigraph T with linear orders imposed on $V_1(T)$ and on $V_2(T)$. Let $\mathcal{P} = (A_1, \ldots, A_K; B_1, \ldots, B_L)$ be a parade in G. If H is an $\mathcal{P}$-rainbow induced sub-bigraph of G, then there is an associated linear order $<$ on $V_1(H)$ defined by $u < v$ if $u \in A_i$ and $v \in A_j$ for some i, j with $i < j$; and similarly for $V_2(H)$. This gives an ordered bigraph that we call the *$\mathcal{P}$-ordering* of H; and if the $\mathcal{P}$-ordering of H is isomorphic to some ordered bigraph T we say that H is a *copy* of T.

A *rooted bigraph* H is a pair $(H^-, r(H))$, where H^- is a bigraph and $r(H) \in V(H^-)$; we call $r(H)$ the *root*. Thus, the root might belong to $V_1(H)$ or to $V_2(H)$. If H_1, H_2 are rooted bigraphs, by an *isomorphism* between them we mean an isomorphism between H_1^- and H_2^- that takes root to root.

An induced rooted sub-bigraph H of G is *$\mathcal{P}$-left-rainbow* if

- it is $\mathcal{P}$-rainbow; and
- if the root of H belongs to A_h, then $h \le i$ for all $i \in \{1, \ldots, K\}$ with $V(H) \cap A_i \neq \emptyset$; and if the root of H belongs to B_h, then $h \le j$ for all $j \in \{1, \ldots, L\}$ with $V(H) \cap B_j \neq \emptyset$.

We define *$\mathcal{P}$-right-rainbow* similarly, requiring $h \ge i$ and $h \ge j$ instead.

Let $\mathcal{P} = (A_1, \ldots, A_K; B_1, \ldots, B_L)$ be a parade in a bigraph G. If T is an induced subgraph that is $\mathcal{P}$-rainbow, its *support* is the pair (I, J) of subsets of $\{1, \ldots, K\}$ where

- I is the set of all $i \in \{1, \ldots, K\}$ such that $V(T) \cap A_i \neq \emptyset$, and
- J is the set of all $j \in \{1, \ldots, L\}$ such that $V(T) \cap B_j \neq \emptyset$.

If S is an ordered bigraph, and $\mathcal{P}$ is as above, we define the *trace* of S (relative to $\mathcal{P}$) to be the set of supports of all $\mathcal{P}$-rainbow copies of S in G.

We say $\mathcal{P}$ is *τ-support-uniform* if for every ordered tree bigraph T with at most τ vertices, either the trace of T (relative to $\mathcal{P}$) is empty, or it consists of all pairs (I, J) with $I \subseteq \{1, \ldots, K\}$ and $J \subseteq \{1, \ldots, L\}$ of cardinalities $|V_1(T)|, |V_2(T)|$ respectively.

Let $0 < \kappa \le 1$, and let $\mathcal{P} = (A_1, \ldots, A_K; B_1, \ldots, B_L)$ be a parade in a bigraph G. We say that $\mathcal{P}$ is *(κ, τ)-support-invariant* if it has the following property: for every contraction $\mathcal{P}' = (A'_1, \ldots, A'_K; B'_1, \ldots, B'_L)$ of $\mathcal{P}$ such that $|A'_i| \ge \kappa |A_i|$ for $1 \le i \le K$ and $|B'_j| \ge \kappa |B_j|$ for $1 \le j \le L$, and for every ordered tree bigraph T with at most τ vertices, the trace of T relative to $\mathcal{P}$ equals the trace of T relative to $\mathcal{P}'$.

If G is a bigraph and $X \subseteq V_1(G)$ and $Y \subseteq V_2(G)$, or vice versa, and $0 \le \lambda \le 1$, we say that X *λ-covers* Y if there are at least $\lambda|Y|$ vertices in Y with a neighbour in X, and X *λ-misses* Y if there are at least $\lambda|Y|$ vertices in Y with no neighbour in X.

A parade $\mathcal{P} = (A_1, \ldots, A_K; B_1, \ldots, B_L)$ is *λ-top-concave* if it has the following very strong property: for every $Y \subseteq B_1 \cup \cdots \cup B_L$, there do not exist h_1, h_2, h_3 with $1 \le h_1 < h_2 < h_3 \le K$, such that Y λ-covers A_{h_2} and Y λ-misses A_{h_1} and A_{h_3}. We define *λ-bottom-concave* similarly. We say $\mathcal{P}$ is *λ-concave* if it is both λ-top-concave and λ-bottom-concave.

A bigraph G is *balanced* if $|V_1(G)| = |V_2(G)|$; and we say a parade $(A_1, \ldots, A_K; B_1, \ldots, B_L)$ is *balanced* if $K = L$ and all its blocks have the same cardinality (that is, $|A_1| = |B_1|$).

The following theorem, which is the main step in the proof, says that we can find a rooted tree bigraph T in any parade that is sufficiently well-behaved. In later sections, we will show that it is possible to find such a parade.

3.1 *Let $\delta \geq 2$ and $\eta \geq 0$ be integers. Let T be a rooted tree bigraph, such that every vertex has degree at most $\delta + 1$, the root has degree at most δ, and every path from root to leaf has length less than η. Let $\tau = \delta^{\eta+1}$, and let $0 < \lambda \leq 2^{-30\delta}\delta^{-1-\eta}$. Let G be a balanced bigraph with a balanced parade $\mathcal{P}$ of length (K, K) where $K = (32\delta + 4)\tau + 2$, such that $\mathcal{P}$ is λ-concave, $(2^{-30\delta}, \tau)$-support-invariant and τ-support-uniform. Let $\mathcal{P}$ have width (W, W). If G is ε-coherent where $\varepsilon \leq 2^{-30\delta}$, then there is a $\mathcal{P}$-rainbow copy of T.*

Proof. We will be looking at rooted tree bigraphs in which every vertex has degree exactly $\delta + 1$ except the root and the leaves; but the root might have degree different from δ, and not all paths from root to leaf will necessarily have the same length. Let us first set up some notation for such trees.

If $a \geq 2$ is an integer, let $T(a, 0)$ be the rooted tree bigraph H with $|V_1(H)| = 1$ and $V_2(H) = \emptyset$ (thus, a is irrelevant, but this will be convenient). If $a \geq 2$ and $b \geq 1$ are integers, let $T(a, b)$ be the rooted tree bigraph H with root in $V_1(H)$, such that the root has degree a, every vertex different from the root has degree $a+1$ or 1, and every path from root to leaf has length exactly b. We denote the transpose of $T(a, b)$ by $\tilde{T}(a, b)$.

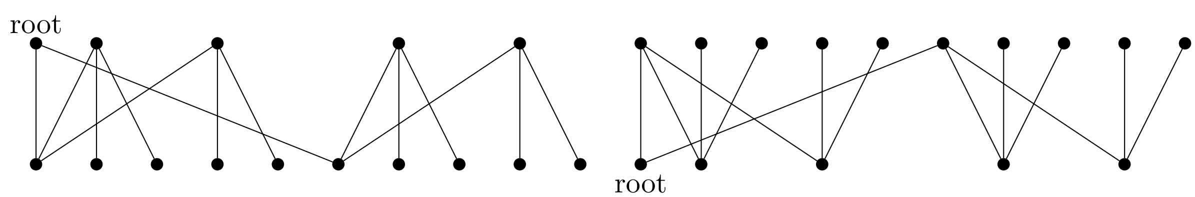

Figure 1: $T(2, 3)$ and $\tilde{T}(2, 3)$.

Now let $a_1, a_2, b_1, b_2 \geq 0$ be integers. Let $T(a_1, b_1, a_2, b_2)$ be the rooted tree bigraph obtained from a_1 copies of $\tilde{T}(\delta, b_1)$ and a_2 copies of $\tilde{T}(\delta, b_2)$, all pairwise disjoint, by adding a new root adjacent to all the old roots. We denote its transpose by $\tilde{T}(a_1, b_1, a_2, b_2)$.

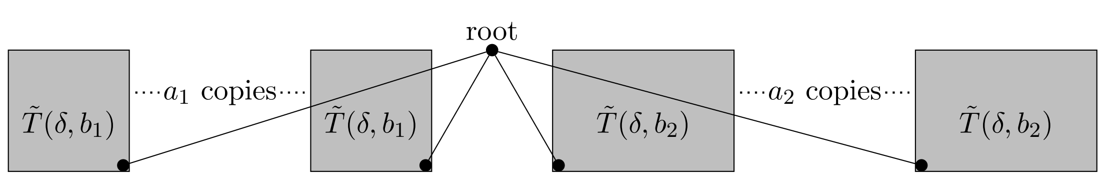

Figure 2: $T(a_1, b_1, a_2, b_2)$.

Let $\mathcal{P} = (A_1, \ldots, A_K; B_1, \ldots, B_K)$ as in the theorem, where $K = (32\delta + 4)\tau + 2$, and let its width be (W, W). We observe that, since G is ε-coherent, fewer than εW vertices in A_1 have no neighbour in B_i for $1 \leq i \leq K$; and since $\delta \leq K$, and $\varepsilon\delta < 1$, there is a vertex in A_1 with a neighbour in each of $B_1, \dots, B_\delta$. Hence there is a copy of $T(\delta, 1)$ that is $\mathcal{P}$-left-rainbow and $\mathcal{P}$-right-rainbow; and similarly there is a copy of $\tilde{T}(\delta, 1)$ that is $\mathcal{P}$-left-rainbow and $\mathcal{P}$-right-rainbow. Consequently we may choose $\rho, \tilde{\rho}, \sigma, \tilde{\sigma}$ as below.

- Choose $\rho \geq 1$ maximum such that there is a copy of $T(\delta, \rho)$ that is $\mathcal{P}$-left-rainbow (and therefore there is a $\mathcal{P}$-left-rainbow copy of $T(0, \rho, \delta, \rho - 1)$); and choose $\phi \geq 0$ maximum such that there is a copy of $T(\phi, \rho, \delta - \phi, \rho - 1)$ that is $\mathcal{P}$-left-rainbow.
- Choose $\tilde{\rho} \geq 1$ maximum such that there is a copy of $\tilde{T}(\delta, \tilde{\rho})$ that is $\mathcal{P}$-left-rainbow; and choose $\tilde{\phi}$ maximum such that there is a copy of $\tilde{T}(\tilde{\phi}, \tilde{\rho}, \delta - \tilde{\phi}, \tilde{\rho} - 1)$ that is $\mathcal{P}$-left-rainbow.
- Choose $\sigma \geq 1$ maximum such that there is a copy of $T(\delta, \sigma)$ that is $\mathcal{P}$-right-rainbow; and choose $\psi \geq 0$ maximum such that there is a copy of $T(\psi, \sigma, \delta - \psi, \sigma - 1)$ that is $\mathcal{P}$-right-rainbow.
- Choose $\tilde{\sigma} \geq 1$ maximum such that there is a copy of $\tilde{T}(\delta, \tilde{\sigma})$ that is $\mathcal{P}$-right-rainbow; and choose $\tilde{\psi}$ maximum such that there is a copy of $\tilde{T}(\tilde{\psi}, \tilde{\sigma}, \delta - \tilde{\psi}, \tilde{\sigma} - 1)$ that is $\mathcal{P}$-right-rainbow.

We suppose for a contradiction that there is no $\mathcal{P}$-rainbow copy of $T(\delta, \eta)$ or its transpose, and so $\rho, \tilde{\rho}, \sigma, \tilde{\sigma} < \eta$. Also $\phi < \delta$, since otherwise $T(\phi, \rho, \delta - \phi, \rho - 1)$ contains $T(\delta, \rho + 1)$, contrary to the maximality of ρ; and similarly $\tilde{\phi}, \psi, \tilde{\psi} < \delta$.

Let us partition $\{2, \dots, K-1\}$ into $16\delta + 2$ intervals, each of length 2τ, and numbered in order. Thus, $I_0, \dots, I_{16\delta+1}$ are pairwise disjoint subsets of $\{2, \dots, K-1\}$, with union $\{2, \dots, K-1\}$, each of cardinality 2τ, and such that $x < y$ for all $i, j \in \{0, \dots, 16\delta + 1\}$ with $i < j$ and all $x \in I_i$ and $y \in I_j$.

Let $a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \geq 0$ be integers, all at most δ, and let e be the sum of these eight integers. Let I be the union of $\{1, K\}$ and the sets I_j for all $j \in \{0, \dots, 8\delta - e\} \cup \{8\delta + e + 1, \dots, 16\delta + 1\}$. Let $\mathcal{P}' = (A'_i \ (i \in I); B'_i \ (i \in I))$ be a balanced minor of $\mathcal{P}$, where $A'_i \subseteq A_i$ and $B'_i \subseteq B_i$ for $i \in I$. We say that $\mathcal{P}'$ is $(a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$-*anchored* if the following hold:

- $\mathcal{P}'$ has width at least $2^{-3e}(W, W)$.
- $Y \subseteq \bigcup_{i \in \{1, \dots, K\} \setminus I} A_i \cup B_i$, and Y is anticomplete to $A'_i \cup B'_i$ for all $i \in I \setminus \{1, K\}$.
- For every $v \in A'_1$ there is a $\mathcal{P}$-left-rainbow copy of $T(\tilde{a}, \tilde{\rho}, \tilde{b}, \tilde{\sigma})$ in $G[Y \cup \{v\}]$ with root v.
- For every $v \in B'_1$ there is a $\mathcal{P}$-left-rainbow copy of $\tilde{T}(a, \rho, b, \sigma)$ in $G[Y \cup \{v\}]$ with root v.
- For every $v \in A'_K$ there is a $\mathcal{P}$-right-rainbow copy of $T(\tilde{c}, \tilde{\rho}, \tilde{d}, \tilde{\sigma})$ in $G[Y \cup \{v\}]$ with root v.
- For every $v \in B'_K$ there is a $\mathcal{P}$-right-rainbow copy of $\tilde{T}(c, \rho, d, \sigma)$ in $G[Y \cup \{v\}]$ with root v.

Let $|V_1(G)| = |V_2(G)| = n$. Choose $a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ with maximum sum such that some balanced minor is $(a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$-anchored (this is possible since $\mathcal{P}$ is $(0, \dots, 0)$-anchored), and let $I, \mathcal{P}', a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, e$ and Y be as above. Let $\mathcal{P}'$ have width (W', W'). Since $e \leq 8\delta$, it follows that $W' \geq 2^{-24\delta} W$.

For a bigraph H, we call the pair $(|V_1(H)|, |V_2(H)|)$ its *part size*. Let

$$T(\phi, \rho, \delta - \phi, \rho - 1), \tilde{T}(\tilde{\phi}, \tilde{\rho}, \delta - \tilde{\phi}, \tilde{\rho} - 1), T(\psi, \sigma, \delta - \psi, \sigma - 1), \tilde{T}(\tilde{\psi}, \tilde{\sigma}, \delta - \tilde{\psi}, \tilde{\sigma} - 1)$$

have part size $(p_1, q_1), (p_2, q_2), (p_3, q_3), (p_4, q_4)$ respectively. Each of these eight numbers is at most τ. Choose

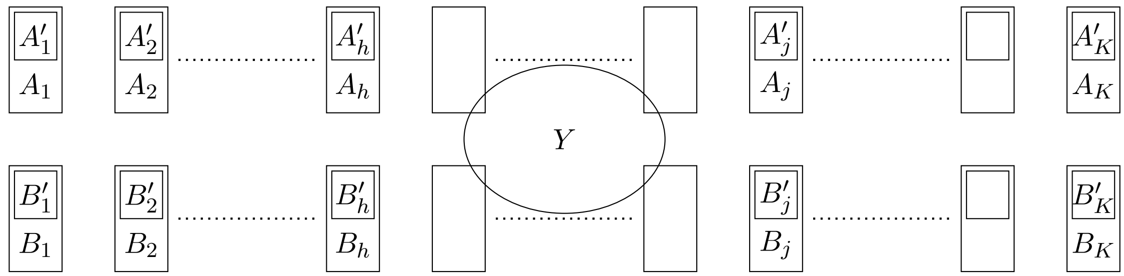

Figure 3: Figure for "anchored". Here, $h = 2\tau(8\delta - e) + 1$, and $j = 2\tau(8\delta + e) + 2$.

- disjoint subsets P_1, P_2 of $I_{8\delta-e}$ of cardinality p_1, p_2 respectively, with the smallest member of $I_{8\delta-e}$ in P_1;
- disjoint subsets P_3, P_4 of $I_{8\delta+e+1}$ of cardinality p_3, p_4 respectively, with the largest member of $I_{8\delta+e+1}$ in P_3;
- disjoint subsets Q_1, Q_2 of $I_{8\delta-e}$ of cardinality q_1, q_2 respectively, with the smallest member of $I_{8\delta-e}$ in Q_2;
- disjoint subsets Q_3, Q_4 of $I_{8\delta+e+1}$ of cardinality q_3, q_4 respectively, with the largest member of $I_{8\delta+e+1}$ in Q_4.

Let $r = \lceil (2^{-24\delta} - 2^{-30\delta})W \rceil$. It follows that $r \geq (2^{6\delta} - 1)2^{-30\delta}W \geq (2^{12} - 1)2^{-30\delta}W$ since $\delta \geq 2$.

(1) *There are r copies $L_1, \ldots, L_r$ of $T(\phi, \rho, \delta - \phi, \rho - 1)$, pairwise vertex-disjoint and each $\mathcal{P}'$-left-rainbow with support (P_1, Q_1).*

Since there is a copy of $T(\phi, \rho, \delta - \phi, \rho - 1)$ that is $\mathcal{P}$-left-rainbow, and $\mathcal{P}$ is τ-support-uniform, there is such a copy that is $\mathcal{P}$-left-rainbow with support (P_1, Q_1). Choose $r' \leq r$ maximum such that there are r' copies of $T(\phi, \rho, \delta - \phi, \rho - 1)$, pairwise vertex-disjoint and each $\mathcal{P}'$-left-rainbow with support (P_1, Q_1). Suppose that $r' < r$. By removing the vertices of these copies from the blocks of $\mathcal{P}'$ that contain them, and removing r' vertices arbitrarily from every other block of $\mathcal{P}'$, we obtain a minor of $\mathcal{P}'$, and hence of $\mathcal{P}$, with width $(W' - r', W' - r')$, relative to which there is no left-rainbow copy of $T(\phi, \rho, \delta - \phi, \rho - 1)$ with support (P_1, Q_1). But $\mathcal{P}'$ is $(2^{-30\delta}, \tau)$-support-invariant, and so $W' - r' < 2^{-30\delta}W$. Since $W' \geq 2^{-24\delta}W$, it follows that $r' > (2^{-24\delta} - 2^{-30\delta})W$, contradicting that $r' < r$. This proves (1).

Similarly, we deduce:

- There are r copies $\tilde{L}_1, \ldots, \tilde{L}_r$ of $\tilde{T}(\tilde{\phi}, \tilde{\rho}, \delta - \tilde{\phi}, \tilde{\rho} - 1)$, pairwise vertex-disjoint and each $\mathcal{P}'$-left-rainbow with support (P_2, Q_2).
- There are r copies $M_1, \ldots, M_r$ of $T(\psi, \sigma, \delta - \psi, \sigma - 1)$, pairwise vertex-disjoint and each $\mathcal{P}$-right-rainbow with support (P_3, Q_3).
- There are r copies $\tilde{M}_1, \ldots, \tilde{M}_r$ of $\tilde{T}(\tilde{\psi}, \tilde{\sigma}, \delta - \tilde{\psi}, \tilde{\sigma} - 1)$, pairwise vertex-disjoint and each $\mathcal{P}$-right-rainbow relative to $\mathcal{P}'$ with support (P_4, Q_4).

For $1 \le i \le r$, let H_i be the disjoint union of $L_i, \tilde{L}_i, M_i$ and $\tilde{M}_i$. Thus H_i is $\mathcal{P}'$-rainbow. If h is the smallest member of $I_{8\delta-e}$, then the root of L_i belongs to A'_h and the root of $\tilde{L}_i$ belongs to B'_h; and if h' is the largest member of $I_{8\delta+e+1}$, then the root of M_i belongs to $A'_{h'}$, and the root of $\tilde{M}_i$ belongs to $B'_{h'}$. We call these four roots the *extremes* of H_i. Let $v \in A'_1 \cup B'_1 \cup A'_K \cup B'_K$, and $1 \le i \le r$. If $v \in A'_1 \cup A'_K$ then every vertex of H_i adjacent to v belongs to $V_2(H_i)$, and if $v \in B'_1 \cup B'_K$ then every vertex of H_i adjacent to v belongs to $V_1(H_i)$. In particular, there are at most two extremes of H_i that are adjacent to v. We say

- v *meets* H_i if v is adjacent to some vertex of H_i;
- v meets H_i *internally* if v is adjacent to some vertex of H_i that is not an extreme (and possibly v is also adjacent to one or two extremes);
- v meets H_i *properly* if v is adjacent to one or two extremes, and to no other vertices of H_i, that is, if v meets H_i and does not meet H_i internally.

For $X \subseteq A'_1 \cup B'_1 \cup A'_K \cup B'_K$, let $f(X)$ be the number of $i \in \{1, \dots, r\}$ such that some vertex in X meets H_i, and let $g(X)$ be the number of $i \in \{1, \dots, r\}$ such that some vertex in X meets H_i internally. Choose $D \subseteq A'_1 \cup B'_1 \cup A'_K \cup B'_K$ maximal such that $f(D) \le r/2$ and $g(D) \ge f(D)/8$. For $i = 1, 2$, let $D_i = D \cap V_i(G)$.

(2) *D_1, D_2 both have cardinality less than εn; and $f(D) \le r/2 - \varepsilon n$.*

Let h be the smallest element of $I_{8\delta-e}$. There are at least $r/2$ vertices in A_h with no neighbour in D_2 (the roots of the trees L_i such that no vertex in D meets H_i). Since $r/2 \ge \varepsilon n$ and G is ε-coherent, it follows that $|D_2| < \varepsilon n$; and since $r/2 \ge \lambda W$, we deduce that D_2 λ-misses A_h. Similarly it λ-misses $A_{h'}$, where h' is the largest member of $I_{8\delta+e+1}$; and since $\mathcal{P}$ is λ-concave, D_2 does not λ-cover any of the sets $A_j (h < j < h')$. In particular, since $|I_{8\delta-e}| = |I_{8\delta+e+1}| = 2\tau$, there are at most $\lambda(4\tau)W$ vertices in

$$\bigcup \left(A_j : j \in (I_{8\delta-e} \setminus \{h\}) \cup (I_{8\delta+e+1} \setminus \{h'\})\right)$$

that have neighbours in D_2. It follows that $g(D_2) \le 4\lambda\tau W$. The same holds for D_1; and since $g(D) \le g(D_1) + g(D_2)$, it follows that $g(D) \le 8\lambda\tau W$. But $f(D) \le 8g(D)$, and so

$$f(D) \le 64\lambda\tau W \le 64(2^{-30\delta}\delta^{-1-\eta})(\delta^{\eta+1})((2^{12}-1)^{-1}2^{30\delta}r) = 64(2^{12}-1)^{-1}r \le r/4 \le r/2 - \varepsilon n$$

since $\lambda \le 2^{-30\delta}\delta^{-1-\eta}$, and $r \ge (2^{12}-1)2^{-30\delta}W$, and $r \ge 4\varepsilon n$. This proves (2).

Let F be the set of vertices in $(A'_1 \cup A'_K \cup B'_1 \cup B'_K) \setminus D$ that meet one of $H_1, \dots, H_r$.

(3) *At most $2\varepsilon n$ vertices in $A'_1 \cup A'_K$ do not belong to F, and the same for $B'_1 \cup B'_K$.*

Since $r \ge \varepsilon n$, and G is ε-coherent, there are fewer than εn vertices in $A'_1 \cup A'_K$ that have no neighbour in any of $H_1, \dots, H_r$. All the other vertices in $A'_1 \cup A'_K$ belong to either D_1 or F, and only at most εn belong to D_1, by (2). Consequently at most $2\varepsilon n$ vertices in $A'_1 \cup A'_K$ do not belong to F, and the same for $B'_1 \cup B'_K$. This proves (3).

Let C be the set of all $i \in \{1, \dots, r\}$ such that D is anticomplete to $V(H_i)$. Thus $|C| = r - f(D)$.

(4) *For each $v \in F$, the number of $i \in C$ such that v meets H_i internally is less than $1/8$ of the number of $i \in C$ such that v meets H_i.*

Since $f(D) \leq r/2 - \varepsilon n$ by (2), it follows that $f(D \cup \{v\}) \leq r/2$, and the maximality of D implies that $g(D \cup \{v\}) < f(D \cup \{v\})/8$. Since $g(D) \geq f(D)/8$, this proves (4).

Let us say $v \in F$ is *happy* if v meets H_i properly, where $i \in C$ is minimum such that v meets H_i.

(5) *We may assume that at least half of the vertices in each of the sets $A'_1 \cap F, A'_K \cap F, B'_1 \cap F, B'_K \cap F$ are happy.*

Let $v \in F$, and take a linear order of C. If we choose the linear order uniformly at random, the probability that v is happy is more than 7/8, by (4); and so the expected number of happy vertices in A'_1 is more than $7/8|A'_1|$. Hence the probability that at least half the vertices in A'_1 are happy is more than 3/4 (because if it were at most 3/4 then the expected number of happy vertices would be at most $(3/4)|A'_1| + (1/4)|A'_1|/2$, which is too small). Similarly the probability that at least half the vertices in A'_K are happy is more than 3/4, and the same for B'_1, B'_K; and so there is positive probability that all four events happen, that is, for some linear order of C, at least half the vertices of each of A'_1, A'_K, B'_1, B'_K are happy. Renumber C in this order; then (5) holds.

Let X be the set of all happy vertices in F. For each $v \in X$, choose $i \in C$ minimum such that v meets H_i. We call i the *happiness* of v. Now $|F \cap A'_1| \geq W - 2\varepsilon n$ by (3), and so $|X \cap A'_1| \geq W'/2 - \varepsilon n \geq 3W'/8$. Thus we may choose $m \leq |C|$ minimum such that one of $A'_1 \cap X, A'_K \cap X, B'_1 \cap X, B'_K \cap X$ contains at least $W'/4$ vertices with happiness at most m. Let $Y' = V(H_1) \cup \dots \cup V(H_m)$ and $I' = I \setminus (I_{8\delta-e} \cup I_{8\delta+e+1})$.

(6) *For each $i \in I' \setminus \{1, K\}$, there is a subset A''_i of A'_i, and a subset B''_i of B'_i, both anticomplete to Y', and both of cardinality $\lceil W'/8 \rceil$.*

Let $i \in I' \setminus \{1, K\}$; we will show that A'_i has a subset with the desired properties. Let $j \in I_{8\delta-e} \cup I_{8\delta+e+1}$. From the choice of m, fewer than $W'/4$ vertices in $X \cap A'_1$ have happiness less than m; and so at most $W'/4 + 2\varepsilon n$ have happiness at most m, since those with happiness exactly m are adjacent to one of the roots of $\tilde{L}_m, \tilde{M}_m$. Since $|X \cap A'_1| \geq 3W'/8$, there are at least $W'/8 - 2\varepsilon n$ vertices in $X \cap A'_1$ that have no neighbour in Y', and in particular have no neighbour in $B'_j \cap Y'$. Since $W'/8 - 2\varepsilon n > \lambda W$, $B'_j \cap Y'$ λ-misses A_1. By the same argument $B'_j \cap Y'$ λ-misses A_K, and so does not λ-cover A_i, since $\mathcal{P}$ is λ-concave. Consequently, for each $j \in I_{8\delta-e} \cup I_{8\delta+e+1}$, there are at most λW vertices in A'_i with a neighbour in $B'_j \cap Y'$; and so there are at most $4\tau\lambda W$ vertices in A'_i with a neighbour in Y'. Since $|A'_i| = W'$ and $W' - 4\tau\lambda W n \geq W'/8$, this proves that A'_i has a subset with the desired properties. From the symmetry under taking transpose, this proves (6).

We chose m such that at least one of $A'_1 \cap X, A'_K \cap X, B'_1 \cap X, B'_K \cap X$ contains at least $W'/4$ vertices with happiness at most m; and from the symmetry, we may assume that $B'_1 \cap X$ contains at least $W'/4$ vertices with happiness at most m. If $v \in B'_1 \cap X$ has happiness at most m, let i be its happiness; then v is adjacent to either the root of L_i or the root of M_i. Choose $Z \subseteq B'_1 \cap X$ with $|Z| \geq W'/8$ such that either every vertex $v \in Z$ is adjacent to the root of L_i, where i is the happiness of v, or every $v \in Z$ is adjacent to the root of M_i, where i is the happiness of v. Choose $B''_1 \subseteq B'_1$ of cardinality $\lceil W'/8 \rceil$ with $B''_1 \subseteq Z$ (this is possible since $|Z| \geq W'/8$). Choose $A''_1 \subseteq A'_1$, $A''_K \subseteq A'_K$, and $B''_K \subseteq B'_K$, all of cardinality $\lceil W'/8 \rceil$. Let $\mathcal{P}'' = (A''_i \ (i \in I''); B''_i \ (i \in I'))$. Then $\mathcal{P}''$ is a balanced minor of $\mathcal{P}$. Its width is at least $W'/8$.

Suppose first that every vertex $v \in Z$ is adjacent to the root of L_i, where i is the happiness of v. Choose $v \in Z$, and let i be its happiness. Let u be the root of L_i. Since $\mathcal{P}'$ is $(a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$-anchored, there is a copy of $\tilde{T}(a, \rho, b, \sigma)$ in $G[Y \cup \{v\}]$ with root v, $\mathcal{P}$-left-rainbow, say S.

Suppose that $a = \delta$. Then S has a rooted subtree S' with root v, isomorphic to $\tilde{T}(\delta, \rho)$. But L_i is a copy of $T(\phi, \rho, \delta - \phi, \rho - 1)$; and so the union of S', L_i, and the edge uv, with root u, is a $\mathcal{P}$-left-rainbow copy of $T(\phi + 1, \rho, \delta - \phi, \rho - 1)$. From the choice of ρ, $\phi + 1 < \delta$; and so there is a $\mathcal{P}$-left-rainbow copy of $T(\phi + 1, \rho, \delta - \phi - 1, \rho - 1)$, contrary to the choice of ϕ. This proves that $a < \delta$.

By taking the union of S and an appropriate subtree of L_i and the edge uv, we obtain a copy of $T(a + 1, \rho, b, \sigma)$ in $G[Y \cup Y' \cup \{v\}]$, with root v. This holds for each $v \in Z$. We claim that $\mathcal{P}''$ is $(a + 1, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$-anchored. Since its width is at least $2^{-3e}(W, W)$, it suffices to check that:

- $Y \cup Y' \subseteq \bigcup \left(\bigcup_{i \in I_j} A_i \cup B_i : 8\delta - (e + 1) + 1 \leq j \leq 8\delta + (e + 1) \right)$, and $Y \cup Y'$ is anticomplete to $A''_i \cup B''_i$ for all $i \in I \setminus \{1, K\}$.
- For every $v \in A''_1$ there is a copy of $T(\tilde{a}, \tilde{\rho}, \tilde{b}, \tilde{\sigma})$ in $G[Y \cup Y' \cup \{v\}]$ with root v, $\mathcal{P}$-left-rainbow.
- For every $v \in B''_1$ there is a copy of $\tilde{T}(a+1, \rho, b, \sigma)$ in $G[Y \cup Y' \cup \{v\}]$ with root v, $\mathcal{P}$-left-rainbow.
- For every $v \in A''_K$ there is a copy of $T(\tilde{c}, \tilde{\rho}, \tilde{d}, \tilde{\sigma})$ in $G[Y \cup Y' \cup \{v\}]$ with root v, $\mathcal{P}$-right-rainbow.
- For every $v \in B''_K$ there is a copy of $\tilde{T}(c, \rho, d, \sigma)$ in $G[Y \cup Y' \cup \{v\}]$ with root v, $\mathcal{P}$-right-rainbow.

The first of these holds from the choice of Y' and of the sets A''_i, B''_i. We have just seen that the third holds; and the other three statements are true because $\mathcal{P}'$ is $(a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$-anchored. But this contradicts the maximality of e.

This completes the case when every vertex $v \in Z$ is adjacent to the root of L_i, where i is the happiness of v; so now we may assume that every vertex $v \in Z$ is adjacent to the root of M_i, where i is the happiness of v. In fact this second case is the same as the first case, as can be seen by reversing the numbering of $A_1, \ldots, A_K$; but checking that this symmetry argument is valid seems more difficult than repeating the argument for the first case, so we will just repeat the argument for the first case.

Choose $v \in Z$, and let i be its happiness. Since $\mathcal{P}'$ is $(a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$-anchored, there is a $\mathcal{P}$-left-rainbow copy of $\tilde{T}(a, \rho, b, \sigma)$ in $G[Y \cup \{v\}]$ with root v, say S. Let u be the root of M_i.

Suppose that $b = \delta$. Then S has a rooted subtree S' with root v isomorphic to $\tilde{T}(\delta, \sigma)$. But M_i is a copy of $T(\psi, \sigma, \delta - \psi, \sigma - 1)$; and so the union of S', M_i, and the edge uv, with root u, is a $\mathcal{P}$-left-rainbow copy of $T(\psi + 1, \sigma, \delta - \psi, \sigma - 1)$. From the choice of σ, $\psi + 1 < \delta$; and so there is a $\mathcal{P}$-left-rainbow copy of $T(\psi + 1, \sigma, \delta - \psi - 1, \sigma - 1)$, contrary to the choice of ψ. This proves that $b < \delta$.

By taking the union of S with an appropriate subtree of M_i and the edge uv, we obtain a copy of $T(a,\rho,b+1,\sigma)$ in $G[Y\cup Y'\cup\{v\}]$. We claim that $\mathcal{P}''$ is is $(a,b+1,c,d,\tilde a,\tilde b,\tilde c,\tilde d)$-anchored, and the argument is as before. But this contradicts the maximality of e.

This contradiction shows that there is a $\mathcal{P}$-rainbow copy of $T(\delta,\eta)$ or its transpose, and since both these contain copies of T, possibly with different root (because all paths in T from root to leaf have length less than η), there is also a $\mathcal{P}$-rainbow copy of T. This proves 3.1. ∎

4 Unbalanced parades

We have finished the difficult part of the paper; now we just have to apply 3.1. One problem is that 3.1 applies only to balanced parades in balanced bigraphs, and it would be easier to use without that restriction. In this section we deduce a version of 3.1 without the balancedness restrictions.

For clarity, in what follows we say "G-adjacent" to mean adjacent in G, and define "G-neighbour", "G-anticomplete" and so on, similarly. Let G be a bigraph, and let $a,b>0$ be integers. For each $u\in V_1(G)$ take a set M_u of a new vertices, and for each $v\in V_2(G)$ take a set M_v of b new vertices. Let H be the bigraph with $V_i(H)=\bigcup_{v\in V_i(G)} M_i$ for $i=1,2$, in which if u,v are G-adjacent then M_u is H-complete to M_v, and if u,v are not G-adjacent then M_u is H-anticomplete to M_v. We say H is obtained from G by (a,b)-*multiplication.* By appropriate multiplication, we can convert an unbalanced parade to a balanced one, and it turns out that all the important properties of the output of 3.1 are preserved under this. That will allow us to prove:

4.1 *Let $\delta\ge 2$ and $\eta\ge 0$ be integers. Let T be a rooted tree bigraph, such that every vertex has degree at most $\delta+1$, and every path from root to leaf has length less than η. Let $\tau=\delta^{\eta+1}$ and $\lambda=2^{-30\delta}\delta^{-1-\eta}$. Let G be a bigraph with a parade $\mathcal{P}$ of length at least (K,K) where $K=(32\delta+4)\tau+2$, such that $\mathcal{P}$ is λ-concave, $(2^{-30\delta},\tau)$-support-invariant and τ-support-uniform. Let $\mathcal{P}$ have width (W_1,W_2). If G is ε-coherent where $\varepsilon\le 2^{-30\delta-1}\min(W_1/|V_1(G)|,W_2/|V_2(G)|)$, then there is a $\mathcal{P}$-rainbow copy of T.*

Proof. By moving to a sub-parade, we may assume that $\mathcal{P}$ has length (K,K). Let $|V_i(G)|=n_i$ for $i=1,2$. Let H be obtained from G by (W_2,W_1)-multiplication, with corresponding sets M_v $(v\in V(G))$. Let $\mathcal{P}=(A_1,\dots,A_K;B_1,\dots,B_K)$, and let $\mathcal{P}'=(A'_1,\dots,A'_K;B'_1,\dots,B'_K)$, where each $A'_i=\bigcup_{v\in A_i}M_v$, and $B'_i=\bigcup_{v\in B_i}M_v$ is defined similarly. Thus $\mathcal{P}'$ is a balanced parade in H.

H is not yet a balanced bigraph, but we remedy that as follows. Let $U_1=A_1\cup\dots\cup A_K$, and $U_2=B_1\cup\dots\cup B_K$. Thus $|U_2|/|U_1|=W_2/W_1$. For $i=1,2$, choose $V_i\subseteq V_i(G)$ with $U_i\subseteq V_i$, such that $|V_2|/|V_1|=|U_2|/|U_1|$, with V_1 maximal. Then $\mathcal{P}$ is a parade in $G[V_1\cup V_2]$, satisfying $|A_i|/|V_1|=|B_j|/|V_2|$ for all i,j. Also, $W_i+|V_i|>n_i$ for some $i\in\{1,2\}$, from the maximality of V_1.

Let G' be the sub-bigraph of H induced on the union of the sets M_v $(v\in V_1\cup V_2)$. Thus G' is a balanced bigraph, and $\mathcal{P}'$ is a balanced parade in it. Let

$$\varepsilon'=2\varepsilon\max\left(\frac{W_1/n_1}{W_2/n_2},\frac{W_2/n_2}{W_1/n_1}\right)\ge 2\varepsilon.$$

(1) *For $i=1,2$, $\varepsilon n_i\le\varepsilon'|V_i|$.*

For this we may assume $i = 1$, from the symmetry. Since $\varepsilon' \geq 2\varepsilon$, we may assume that $n_1 > 2|V_1| \geq |V_1| + W_1$. Since $W_i + |V_i| > n_i$ for some $i \in \{1, 2\}$, it follows that $W_2 + |V_2| > n_2$, and in particular $|V_2| \geq n_2/2$. So $|V_1| = |V_2|(W_1/W_2) \geq W_1 n_2/(2W_2)$, and so $n_1 \leq 2\left((W_2/n_2)/(W_1/n_1)\right)|V_1|$. Hence $\varepsilon n_1 \leq \varepsilon'|V_1|$. This proves (1).

(2) *G' is ε'-coherent.*

Let $w \in V(G')$, with $w \in M_v$ say. From the symmetry we may assume that $w \in V_1(G)$. Then the set of G'-neighbours of w is the union of the sets M_u over all G-neighbours u of v with $u \in V_2$. The number of such u is less than $\varepsilon n_2 \leq \varepsilon'|V_2|$ by (1); and so w has degree less than $\varepsilon'|V_2(G')|$.

Now suppose that there are subsets $Z_i \subseteq V_i(G')$ of cardinality at least $\varepsilon'|V_i(G')|$ for $i = 1, 2$, such that Z_1 is G'-anticomplete to Z_2. For $i = 1, 2$, let Y_i be the set of all $v \in V_i$ such that $M_v \cap Z_i \neq \emptyset$. It follows that $|Y_1| \geq |Z_1|/W_2$, and $|Y_2| \geq |Z_2|/W_1$, and Y_1 is G-anticomplete to Y_2. Consequently $|Y_i| < \varepsilon n_i$ for some $i \in \{1, 2\}$, and from the symmetry we may assume that $i = 1$. Thus $|Z_1| \leq \varepsilon n_1 W_2$, and since $|Z_1| \geq \varepsilon'|V_1(G')|$, it follows that $\varepsilon n_1 W_2 > \varepsilon'|V_1(G')|$. But $|V_1(G')| = W_2|V_1|$, and so $\varepsilon n_1 W_2 > \varepsilon' W_2 |V_1|$, that is, $\varepsilon n_1 > \varepsilon'|V_1|$, contrary to (1). This proves (2).

(3) *$\mathcal{P}'$ is λ-concave, $(2^{-30\delta}, \tau)$-support-invariant and τ-support-uniform.*

To see that $\mathcal{P}'$ is λ-concave, let $Y' \subseteq B'_1 \cup \cdots \cup B'_L$. Let Y be the set of all $v \in B_1 \cup \cdots \cup B_L$ such that $Y' \cap M_v \neq \emptyset$. Thus for $1 \leq i \leq K$, the set of vertices in A'_i with a G'-neighbour in Y' is precisely the union of the sets M_u over all $u \in A_i$ that have a G-neighbour in Y. So Y' λ-covers A'_i if and only if Y λ-covers A_i, and the same for λ-missing; and so $\mathcal{P}'$ is λ-concave.

We observe that for every ordered tree bigraph S with at most τ vertices, the trace of S relative to $\mathcal{P}'$ equals its trace of S relative to $\mathcal{P}$ (because each block of $\mathcal{P}'$ only contains at most one vertex of the tree).

To see that $\mathcal{P}'$ is $(2^{-30\delta}, \tau)$-support-invariant, let $\mathcal{Q}' = (C'_1, \ldots, C'_K; D'_1, \ldots, D'_K)$ be a contraction of $\mathcal{P}$ such that $|C'_i| \geq 2^{-30\delta}|A'_i|$ and $|D'_i| \geq 2^{-30\delta}|B'_i|$ for $i = 1, 2$. For $1 \leq i \leq K$, there are at least $|C'_i|/W_2$ vertices $v \in A_i$ such that $M_v \cap C'_i \neq \emptyset$. Let C_i be a set of $\lceil |C'_i|/W_2 \rceil$ such vertices. Define $D_1, \ldots, D_k$ similarly. Then $\mathcal{Q} = (C_1, \ldots, C_K; D_1, \ldots, D_K)$ is a contraction of $\mathcal{P}$, and $|C_i| \geq 2^{-30\delta}|A_i|$ and $|D_i| \geq 2^{-30\delta}|B_i|$ for $1 \leq i \leq K$. Let S be an ordered tree bigraph with at most τ vertices. We need to show that for every such S, its trace relative to $\mathcal{Q}'$ equals its trace relative to $\mathcal{P}'$. The trace of S relative to $\mathcal{Q}'$ is a subset of its trace relative to $\mathcal{P}'$, and we must show the converse inclusion. Thus, let (H, K) belong to the trace of S relative to $\mathcal{P}'$. Then (H, K) belongs to the trace of S relative to $\mathcal{P}$, as we saw above. Since $\mathcal{P}$ is $(2^{-30\delta}, \tau)$-support-invariant, the trace of S relative to $\mathcal{P}$ equals the trace of S relative to $\mathcal{Q}$; and so (H, K) belongs to the trace of S relative to $\mathcal{Q}$. But then it belongs to the trace relative to $\mathcal{Q}'$. This proves that $\mathcal{P}'$ is $(2^{-30\delta}, \tau)$-support-invariant.

Finally, that $\mathcal{P}'$ is τ-support-uniform is clear, since for every ordered tree bigraph S with at most τ vertices, the trace of S relative to $\mathcal{P}'$ equals its trace relative to $\mathcal{P}$. This proves (3).

Now $\mathcal{P}'$ has width (W, W) where $W = W_1 W_2$. From the symmetry we may assume that $W_1/n_1 \geq W_2/n_2$. By hypothesis, $2\varepsilon \leq 2^{-30\delta} W_2/n_2$; and since $\varepsilon' = 2\varepsilon(W_1/n_1)/(W_2/n_2)$, it follows that

$$\varepsilon' \leq 2^{-30\delta} W_1/n_1 \leq 2^{-30\delta} W_1/|V_1| = 2^{-30\delta} W/|V_1(G')|.$$

Since $|V_1(G')| = |V_2(G')|$, we deduce, from (2), (3) and 3.1 applied to $G', \mathcal{P}'$ and ε', that there is a $\mathcal{P}'$-rainbow copy of T in G', and hence there is a $\mathcal{P}$-rainbow copy of T in G. This proves 4.1. ∎

5 Producing a concave parade

In this section we apply 4.1 to deduce 1.6.

5.1 *Let $\tau \geq 1$ be an integer, and let $0 < \kappa \leq 1$. Let $\mathcal{P} = (A_1, \ldots, A_K; B_1, \ldots, B_L)$ be a parade in a bigraph G. Then there is a contraction $\mathcal{P}' = (A'_1, \ldots, A'_K; B'_1, \ldots, B'_L)$ of $\mathcal{P}$, such that*

- *$|A'_i| \geq \kappa^{2^{K+L}\tau^\tau}|A_i|$ and $|B'_j| \geq \kappa^{2^{K+L}\tau^\tau}|B_j|$ for $1 \leq j \leq L$; and*
- *$\mathcal{P}'$ is (κ, τ)-support-invariant.*

Proof. Let $\mathcal{P} = (A_1, \ldots, A_K; B_1, \ldots, B_L)$ be a parade in a bigraph G. We define the *trace-cost* of a contraction $\mathcal{P}'$ of $\mathcal{P}$ to be the sum of the cardinality of the trace of T, summed over all nonisomorphic ordered tree bigraphs T with at most τ vertices. The cardinality of the trace of any given ordered tree bigraph T is at most 2^{K+L}, and up to isomorphism there are at most τ^τ ordered tree bigraphs T with at most τ vertices. Hence the trace-cost of $\mathcal{P}$ is at most $2^{K+L}\tau^\tau$.

There are integers $t \geq 0$ (for instance $t = 0$) such that there is a contraction

$$\mathcal{P}' = (A'_1, \ldots, A'_K; B'_1, \ldots, B'_L)$$

of $\mathcal{P}$ with $|A'_i| \geq \kappa^t|A_i|$ for $1 \leq i \leq K$, and $|B'_j| \geq \kappa^t|B_j|$ for $1 \leq j \leq L$, and with trace-cost at most $2^{K+L}\tau^\tau - t$. Since trace-cost is nonnegative, it follows that every such t satisfies $t \leq 2^{K+L}\tau^\tau$, and so we can choose t maximum with the stated property. Let $\mathcal{P}'' = (A''_1, \ldots, A''_K; B''_1, \ldots, B''_L)$ be a contraction of $\mathcal{P}'$ such that $|A''_i| \geq \kappa|A'_i|$ for $1 \leq i \leq K$, and $|B''_j| \geq \kappa|B'_j|$ for $1 \leq j \leq L$. For every ordered tree bigraph T, the trace of T relative to $\mathcal{P}''$ is a subset of the trace of T relative to $\mathcal{P}$, and so from the choice of t, equality holds for every T with at most τ vertices, that is, $\mathcal{P}'$ is (κ, τ)-support-invariant. This proves 5.1. ∎

There is a bipartite version of Ramsey's theorem for uniform hypergraphs:

5.2 *For all integers a, b, c there exists N with the following property. Let A, B be two disjoint sets both of cardinality at least N; let $\mathcal{F}$ be the set of all subsets of $A \cup B$ that contain exactly a vertices of A and b vertices of B; and let $\mathcal{H} \subseteq \mathcal{F}$. Then there exists $A' \subseteq A$ and $B' \subseteq B$, with $|A'| = |B'| = c$, such that one of $\mathcal{H}, \mathcal{F} \setminus \mathcal{H}$ contains no subset of $A' \cup B'$.*

By iterated applications of 5.2 (one for each ordered tree bigraph T with at most τ vertices) we deduce:

5.3 *Let $k, \tau \geq 0$ be integers; then there exists an integer $K \geq 0$ with the following property. Let $\mathcal{P}$ be a parade of length (K, K) in a bigraph G, and let $0 < \lambda \leq 1$. Then $\mathcal{P}$ has a sub-parade of length (k, k) which is τ-support-uniform.*

Combining 5.1 and 5.3, we obtain:

5.4 *Let $k, \tau \geq 1$ be integers, and $0 < \kappa \leq 1$; then there exist an integer K with the following property. Let $\mathcal{P}$ be a parade of length at least (K, K) and width (W_1, W_2) in a bigraph G. Then there is a minor $\mathcal{P}'$ of $\mathcal{P}$, with length (k, k) and width at least*

$$(\kappa^{2^{2K}\tau^\tau} W_1, \kappa^{2^{2K}\tau^\tau} W_2),$$

such that $\mathcal{P}'$ is (κ, τ)-support-invariant and τ-support-uniform.

Proof. Let K satisfy 5.3; then we claim it satisfies 5.4. Let $\mathcal{P} = (A_1, \ldots, A_K; B_1, \ldots, B_K)$ be a parade in a bigraph G, of width (W_1, W_2). By 5.1 there is a contraction $\mathcal{P}' = (A_1, \ldots, A_K; B'_1, \ldots, B'_K)$ of $\mathcal{P}$, with width at least

$$(\kappa^{2^{2K}\tau^\tau} W_1, \kappa^{2^{2K}\tau^\tau} W_2),$$

such that $\mathcal{P}'$ is (κ, τ)-support-invariant. By 5.3 applied to $\mathcal{P}'$, the result follows, since being (κ, τ)-support-invariant is inherited by sub-parades. This proves 5.4. ∎

We need the following lemma.

5.5 *Let $K, \ell, t > 0$ be integers, and let $0 < \kappa \leq 1/2$. Let $\lambda = 2\kappa$, let $r = \lceil t/\kappa \rceil$, and let $L = r\ell$. Let $\mathcal{P} = (A_1, \ldots, A_K; B_1, \ldots, B_L)$ be a parade in a bigraph G, of width (W_1, W_2). Suppose that there do not exist $1 < q_0 < \cdots < q_{2t} \leq L$ and a subset $X \subseteq A_1 \cup \cdots \cup A_K$ such that X 2κ-covers B_{q_t}, and X κ-misses B_{q_j} for all $j \in \{0, \ldots, 2t\} \setminus \{t\}$. For $1 \leq j \leq \ell$, let C_j be the union of the sets B_i for all i with $r(j-1) < i \leq rj$. Then the parade $\mathcal{C} = (A_1, \ldots, A_K; C_1, \ldots, C_\ell)$ is λ-bottom-concave. Moreover, for $\tau > 0$, if $\mathcal{P}$ is τ-support-uniform then*

- *so is $\mathcal{C}$;*
- *for every ordered tree bigraph T with $|T| \leq \tau$ and $|V_2(T)| \leq \ell$, if its trace relative to $\mathcal{P}$ is nonempty then its trace relative to $\mathcal{C}$ is nonempty; and*
- *if in addition for some $\kappa' \geq \kappa$, $\mathcal{P}$ is (κ', τ)-support-invariant, then $\mathcal{C}$ is also (κ', τ)-support-invariant.*

Proof. Let $1 \leq h_1 < h_2 < h_3 \leq \ell$, and suppose that there exists $X \subseteq A_1 \cup \cdots \cup A_K$, such that X λ-covers C_{h_2} and λ-misses C_{h_1} and C_{h_3}. Since X λ-covers C_{h_2}, there are at least $\lambda r W_2$ vertices in C_{h_2} with a neighbour in X, and so there exists r_j with $r(h_2 - 1) < r_j \leq rh_2$ such that at least λW_2 vertices in B_{r_j} have a neighbour in X. Hence X λ-covers B_{r_j}. From the hypothesis, either X κ-misses B_i for fewer than t values of i with $r(h_1 - 1) < i \leq rh_1$, or X κ-misses B_i for fewer than t values of i with $r(h_3 - 1) < i \leq rh_3$, and from the symmetry we may assume the former. Consequently there are fewer than $tW_2 + (r-t)\kappa W_2$ vertices in C_{h_1-1} with no neighbour in X. Since X λ-misses C_{h_1}, it follows that

$$tW_2 + (r-t)\kappa W_2 > \lambda |C_{h_1}| = 2\kappa r W_2,$$

and so $t(1-\kappa) > \kappa r$, contrary to the choice of r. This proves that $\mathcal{C}$ is λ-bottom-concave.

If $\mathcal{P}$ is τ-support-uniform then clearly so is $\mathcal{C}$. Let T be an ordered tree bigraph, with $|T| \leq \tau$ and $|V_2(T)| \leq \ell$ that has nonempty trace relative to $\mathcal{P}$. Let $|V_1(T)| = s$ and $|V_2(T)| = t$. Thus $t \leq \ell$. Since $\mathcal{P}$ is τ-support-uniform, the trace of T consists of all pairs (I, J) where $I \subseteq \{1, \ldots, K\}$ with

$|I| = s$ and $J \subseteq \{1, \dots, L\}$ with $|J| = t$. In particular, $(\{1, \dots, s\}, \{r, 2r, \dots, tr\})$ belongs to the trace of T relative to $\mathcal{P}$; and so $(\{1, \dots, s\}, \{1, 2, \dots, t\})$ belongs to the trace of T relative to $\mathcal{C}$. This proves the second bullet.

It remains to show that if in addition $\mathcal{P}$ is (κ', τ)-support-invariant then so is $\mathcal{C}$. Let T be an ordered tree bigraph, with $|T| \le \tau$. Let $|V_1(T)| = s$ and $|V_2(T)| = t$ say. Let $1 \le p_1 < \cdots < p_s \le K$ and $1 \le q_1 < \cdots < q_t \le \ell$, such that there is a $(A_{p_1}, \dots, A_{p_s}; C_{q_1}, \dots, C_{q_t})$-rainbow copy of T. For $1 \le i \le s$ let $A'_{p_i} \subseteq A_{p_i}$, and for $1 \le j \le t$ let $C'_{q_j} \subseteq C_{q_j}$, where $(A'_{p_1}, \dots, A'_{p_s}; C'_{q_1}, \dots, C'_{q_t})$ is a parade with width at least $(\kappa' W_1, \kappa' r W_2)$. We must show that there is an $(A'_{p_1}, \dots, A'_{p_s}; C'_{q_1}, \dots, C'_{q_t})$-rainbow copy of T.

For $1 \le j \le t$, since $|C'_{q_j}| \ge \kappa' r W_2$, there exists g_j with $r(q_j - 1) < g_j \le r q_j$ such that $|C'_{q_j} \cap B'_{g_j}| \ge \kappa' W_2 \ge \kappa W_2$. Choose $D_{q_j} \subseteq C'_{q_j} \cap B'_{g_j}$ of cardinality $\lceil \kappa W_2 \rceil$ for $1 \le j \le t$. Since $\mathcal{P}$ is τ-support-uniform and (κ, τ)-support-invariant, it follows that there is a copy of T that is $(A'_{p_1}, \dots, A'_{p_s}; D_{q_1}, \dots, D_{q_s})$-rainbow and hence $(A'_{p_1}, \dots, A'_{p_s}; C'_{q_1}, \dots, C'_{q_t})$-rainbow. This proves 5.5. ▮

5.6 *For every tree bigraph T, there exist $d > 0$ and an integer K, such that, for every bigraph G with a parade $\mathcal{P}$ of length at least (K, K), if for some $\varepsilon > 0$, G is ε-coherent and $\mathcal{P}$ has width at least $(\varepsilon d |V_1(G)|, \varepsilon d |V_2(G)|)$, then there is a $\mathcal{P}$-rainbow copy of T in G.*

Proof. We proceed by induction on $|T|$, and may assume that $|T| \ge 2$. Choose $\delta \ge 2$ and $\eta \ge 0$ such that T is a sub-bigraph of $T(\delta, \eta)$ and of $\tilde{T}(\delta, \eta)$. (The latter were defined within the proof of 3.1.) Let $\lambda = 2^{-9\delta - 1} \delta^{-1-\eta}$, and $\kappa = \lambda/2$. Let $r = \lceil (|T| - 1)/\kappa \rceil$. From the inductive hypothesis, there exist K', d' such that for every tree bigraph T' with $|T'| < |T|$, the theorem is satisfied with T', K', d' replacing T, K, d. By increasing K', we may assume that $K' \ge 6r\delta^{\eta+2}$, and $K' > 2|T| + 1$, and K' is a multiple of $4r$. Let $\ell = K'/(4r)$. Let $\tau = \delta^{\eta+1}$. Let K satisfy 5.4 with k replaced by K'. Let

$$d = \kappa^{-2^K \tau^\tau} \max(d', 2^{9\delta}/r).$$

We claim that K, d satisfy 5.6. Let $\mathcal{P}$ be a parade in a bigraph G, of length (K, K) and width (W_1, W_2), where $W_i \ge \varepsilon d |V_i(G)|$ for $i = 1, 2$, and G is ε-coherent. We assume (for a contradiction) that there is no $\mathcal{P}$-rainbow copy of T. By 5.4, there is a minor $\mathcal{P}'$ of $\mathcal{P}$ of length (K', K') and width at least $\kappa^{2^{2K} \tau^\tau} (W_1, W_2)$, such that $\mathcal{P}'$ is τ-support-uniform and (κ, τ)-support-invariant. Let $\mathcal{P}' = (A_1, \dots, A_{K'}; B_1, \dots, B_{K'})$, and let its width be (w_1, w_2). Let $t = |T|$. From the symmetry we may assume that some vertex of $V_1(T)$ has degree one in T.

(1) *There do not exist $1 \le r_0 < \cdots < r_{2t} \le K'$, such that for some $X \subseteq A_1 \cup \cdots \cup A_{K'}$, X 2κ-covers B_{r_t}, and X κ-misses B_{r_i} for all $i \in \{0, \dots, 2t\} \setminus \{t\}$.*

Suppose that such X and $r_0, \dots, r_{2t}$ exist. Let B'_{r_t} be a set of $\lceil \kappa w_2 \rceil$ vertices in B_{r_t} that have a neighbour in X; for $0 \le i \le 2t$ with $i \ne t$, let B'_{r_i} be a set of $\lceil \kappa w_2 \rceil$ vertices in B_{r_i} that do not have a neighbour in X; and for $i \in \{1, \dots, K'\}$ with $i \ne r_0, \dots, r_{2t}$, let B'_i be a subset of B_i of cardinality $\lceil \kappa w_2 \rceil$. Partition $\{1, \dots, K'\}$ into two sets I_1, I_2, both of cardinality $K'/2 \ge |T|$. Let X_1 be the intersection of X with the union of the sets $A_i (i \in I_1)$, and define X_2 similarly. At least one of X_1, X_2 κ-covers B_{r_t}, since their union 2κ-covers B_{r_t}, and from the symmetry we may assume that X_2 κ-covers B_{r_t}. Since $|I_1| \geq K'/2 \geq t$, Let $J = \{r_0, r_1, \ldots, r_{2t}\}$. Then the parade $\mathcal{P}'' = (A_i\ (i \in I_1); B'_j\ (j \in J))$ has width at least $(w_1, \kappa w_2)$.

Let $v_0 \in V_1(T)$ be a vertex of T with degree one, and let u_0 be its neighbour. Let $T' = T \setminus \{v_0\}$. From the inductive hypothesis, there is a $\mathcal{P}'$-rainbow copy of T', since for $i = 1, 2$,

$$\kappa^{2^{2K}\tau^\tau} W_i \geq \kappa^{2^{2K}\tau^\tau} (\varepsilon d|V_i(G)|) \geq d'\varepsilon|V_i(G)|.$$

Hence there is an ordered bigraph S, obtained from T' by ordering $V_1(T')$ and $V_2(T')$, with nonempty trace relative to $\mathcal{P}'$.

Since $\mathcal{P}'$ is τ-support-uniform, and (κ, τ)-support-invariant, and the trace of S relative to $\mathcal{P}'$ is nonempty, and $|I_1| \geq |V_1(T)|$, there is an isomorphism from S to a $\mathcal{P}''$-rainbow induced sub-bigraph H of G, with $\mathcal{P}''$-ordering isomorphic to S, where u_0 is mapped to a vertex of H in B'_{r_t}, say u. Choose $v \in X$ adjacent to u; such a vertex exists since PP X covers B'_{r_t}. But then v has no other neighbour in $V(H)$, since X is anticomplete to B'_{r_i} for all $\in I_2 \setminus \{t\}$; and $v \in A_h$ where $h \in I_2$. Thus adding v to H gives a $\mathcal{P}$-rainbow copy of T, a contradiction. This proves (1).

We recall that $K' = 4r\ell$. For $1 \leq i \leq 4\ell$, let C_i be the union of the sets B_j for all j with $r(i-1) < j \leq ri$. Then $\mathcal{C} = (A_1, \ldots, A_{K'}; C_1, \ldots, C_{4\ell})$ is a parade, of width (w_1, rw_2), and $rw_2 \geq r\kappa^{2^{2K}\tau^\tau} W_2$. By 5.5, $\mathcal{C}$ is 2κ-bottom-concave, τ-support-uniform and (κ, τ)-support-invariant.

Let $1 \leq i \leq K'$, and let $X \subseteq A_i$ be such that X λ-covers each of $C_1, \ldots, C_{4\ell}$. This is possible, because A_i λ-covers each of $C_1, \ldots, C_{4\ell}$, since $\lambda \leq 1/2$ and $w_1 \geq \varepsilon|V_1(G)|$, and $rw_2 \geq \varepsilon|V_2(G)|$. Let us say $j \in \{1, \ldots, 4\ell\}$ is a *pit* for X if X λ-misses C_j. Since $\mathcal{C}$ is λ-bottom-concave, there are at most two pits for X, and if there are two then they are consecutive. On the other hand, if we choose X minimal, then it follows by deleting one vertex from X that for some $j \in \{1, \ldots, 4\ell\}$, fewer than $\lambda|C_j| + \varepsilon|V_2(G)|$ vertices in C_j have a neighbour in X, and since $\lambda|C_j| + \varepsilon|V_2(G)| \leq (1-\lambda)|C_j|$, it follows that j is a pit for X. Thus there is either exactly one pit, or exactly two consecutive pits for X. Let us say A_i is *left-first* if there is a choice of $X \subseteq A_i$ such that X λ-covers each of $C_1, \ldots, C_{4\ell}$, and X has a pit j with $j > 2\ell$; and *right-first* if there is a choice of $X \subseteq A_i$ such that X λ-covers each of $C_1, \ldots, C_{4\ell}$, and X has a pit j with $j \leq 2\ell$. Each A_i is therefore either left-first or right-first, possibly both; and so we may assume (by reversing the order of $C_1, \ldots, C_\ell$ if necessary) that at least half of them are left-first. Let $I \subseteq \{1, \ldots, K'\}$ with $|I| = K'/2 = 2r\ell$ such that A_i is left-first for each $i \in I$.

Let $i \in I$, and choose $X \subseteq A_i$ such that X λ-covers each of $C_1, \ldots, C_{4\ell}$, and X has a pit j with $j > 2\ell$. Choose $X^{2\ell} \subseteq X$ minimal such that $X^{2\ell}$ λ-covers $C_{2\ell}$, and for $2\ell > i \geq 1$ in turn, inductively choose $X^i \subseteq X^{i+1}$ minimal such that X^i λ-covers C_i. To show that this is possible, we will prove inductively that for $2\ell \geq i \geq 1$:

- X^i does not λ-miss any of $C_1, \ldots, C_{i-1}$;
- X^i both λ-covers and λ-misses C_i;
- X^i does not λ-cover any of $C_{i+1}, \ldots, C_{2\ell}$; and
- fewer than $\lambda|C_i| + \varepsilon|V_2(G)|$ vertices in C_i have a neighbour in X^i.

Suppose then that either $i = 2\ell$ or X^{i+1} satisfies the four bullets; then the choice of X^i is possible, and it remains to show that X^i satisfies the four bullets. Certainly X^i λ-covers C_i from its definition; and from the minimality of X^i, fewer than $\lambda|C_i| + \varepsilon|V_2(G)| \le (1-\lambda)|C_i|$ vertices in C_i have a neighbour in X^i, and so X^i λ-misses C_i. Since it also λ-misses C_j, because j is a pit for X and $X^i \subseteq X$, and $j > 2\ell$, we deduce from concavity that X^i does not λ-cover any of $C_{i+1}, \dots, C_{2\ell}$. Thus the second, third and fourth bullets hold. But also, since X^i λ-covers C_i and λ-misses C_j, concavity implies that X^i does not λ-miss any of $C_1, \dots, C_{i-1}$. Thus all four bullets hold. This completes the inductive definition of $X^1, \dots, X^{2\ell}$. Let us call the sequence $(X^1, \dots, X^{2\ell})$ a λ-*ladder* in A_i.

For each $i \in I$, choose a λ-ladder $(X_i^1, \dots, X_i^{2\ell})$ in A_i. For $1 \le j \le 2\ell$, let us say that $v \in C_j$ is *unwanted* if for some $i \in I$, either v has a neighbour in X_i^j, or v has no neighbour in $X_i^{j'}$ for some j' with $j < j' \le 2\ell$ (and hence $j < 2\ell$, and v has no neighbour in X_i^{j+1}). We say v is *wanted* if it is not unwanted. (Note that if $v \in C_j$ is wanted, then for each $i \in I$, v has no neighbour in any of $X_1^i, \dots, X_j^i$.) It follows that the total number of unwanted vertices in C_j is at most $|I|(2\lambda|C_j| + \varepsilon|V_2(G)|)$, since for each $i \in I$, there are at most $\lambda|C_j| + \varepsilon|V_2(G)|$ vertices in C_j with a neighbour in X_i^j, and at most $\lambda|C_j|$ vertices in C_j that have no neighbour in X_i^{j+1} (when $j < 2\ell$). Since $|I|(2\lambda|C_j| + \varepsilon|V_2(G)|) \le |C_j|/2$ it follows that at least $|C_j|/2$ vertices in C_j are wanted. For $1 \le j \le 2\ell$, let $C'_j \subseteq C_j$ be a set of vertices that are wanted, with $|C'_j| = \lceil |C_j|/2 \rceil$.

(2) *There do not exist $q_0, \dots, q_{2t} \in I$ with $q_0 < q_1 < \cdots < q_{2t}$, such that for some $Y \subseteq C'_1 \cup \cdots \cup C'_\ell$, Y 2κ-covers A_{q_t}, and Y κ-misses A_{q_i} for all $i \in \{0, \dots, 2t\} \setminus \{t\}$.*

Suppose that such $q_0, \dots, q_{2t}, j, Y$ exist. Choose $u \in V_2(T)$ such that all its T-neighbours except possibly one have degree one in T. (This is possible, because if $V_2(T)$ contains a leaf of T, let u be that leaf, and if all leaves belong to $V_1(T)$, let u be a leaf of the tree obtained from T by deleting all leaves.) Let v be a neighbour of u such that all its other neighbours are leaves; and let u have s neighbours different from v. Let T' be obtained from T by deleting u and all its neighbours except v. Let A'_{q_t} be a subset of $\lceil \kappa W_1 \rceil$ vertices in A_{q_t} that have a neighbour in Y, and for $0 \le i \le 2t$ with $i \ne t$, let A'_{q_i} be a subset of $\lceil \kappa W_1 \rceil$ vertices in A_{q_i} that have no neighbour in Y. Let $I_1 = \{q_0, q_1, \dots, q_{2t}\}$.

As in the proof of (1), from the inductive hypothesis, there is a $\mathcal{P}'$-rainbow copy of T'. Hence there is an ordered bigraph S, obtained from T' by ordering $V_1(T')$ and $V_2(T')$, with nonempty trace relative to $\mathcal{P}'$, and so with nonempty trace relative to $\mathcal{C}$, by the second bullet of 5.5.

Let $\mathcal{C}' = (A'_i \ (i \in I_1); C'_j \ (\ell+1 \le j \le 2\ell))$. Since $\mathcal{C}$ is τ-support-uniform, and (κ, τ)-support-invariant, and $\kappa \le 1/2$ (and so each $|C'_j| \ge \kappa|C_j|$), and the trace of S relative to $\mathcal{C}$ is nonempty, it follows that there is a $\mathcal{C}'$-rainbow induced sub-bigraph H of G, with $\mathcal{P}$-ordering isomorphic to S, where some vertex $v' \in V(H) \cap A_{q_t}$ is mapped by the isomorphism to v. Choose $u' \in Y$ adjacent to v'; such a vertex exists since Y covers A'_{q_t}. But then u' has no other neighbour in $V(H)$, since Y misses A'_{q_i} for all $i \in \{0, \dots, 2t\} \setminus \{t\}$. Choose $p_1, \dots, p_s \in I \setminus \{q_0, \dots, q_{2t}\}$. (This is possible since $|I| \ge 2t + 1 + s$.) Since $u' \in Y \subseteq C'_1 \cup \cdots \cup C'_\ell$, there exists j with $1 \le j \le \ell$ such that $u' \in C'_j$. For $1 \le i \le s$, choose $x_i \in X_j^{p_i}$ adjacent to u'. (This is possible since u' is wanted.) Then x_i is nonadjacent to all vertices in $V_2(H)$, since all of these vertices are wanted and belong to $C'_{\ell+1} \cup \cdots \cup C'_{2\ell}$. Thus adding u' and $x_1, \dots, x_s$ to H gives a $\mathcal{P}$-rainbow copy of T, a contradiction. This proves (2).

Since $|I| = 2r\ell$, we may partition I into ℓ "intervals" each containing $2r$ elements of I, say $I_1, \dots, I_\ell$. More exactly, we partition I into $I_1, \dots, I_\ell$, where each of these sets has cardinality $2r$, and for $1 \le h < i \le \ell$, every element of I_h is less than every element of I_i. For $1 \le h \le \ell$ let D_h

be the union of the sets A_i $(i \in I_h)$. Then $\mathcal{D} = (D_1, \dots, D_\ell; C'_1, \dots, C'_\ell)$ is a parade. By (2) and 5.5, $\mathcal{D}$ is λ-top-concave. Now $\mathcal{C}$ is τ-support-uniform, and (κ, τ)-support-invariant, and λ-bottom-concave; and so $\mathcal{C}' = (A_i \ (i \in I); C'_1, \dots, C'_\ell)$ is τ-support-uniform, $(2\kappa, \tau)$-support-invariant, and 2λ-bottom-concave, since $|C'_i| \geq |C_i|/2$ for each i. Thus $\mathcal{D}$ is 2λ-bottom-concave; and by 5.5, $\mathcal{D}$ is also τ-support-uniform and $(2\kappa, \tau)$-support-invariant. In summary then, $\mathcal{D}$ is τ-support-uniform, $(2\kappa, \tau)$-support-invariant and 2λ-concave. Hence $\mathcal{D}$ satisfies the hypotheses of 4.1, and so there is a $\mathcal{D}$-rainbow, and hence $\mathcal{P}$-rainbow, copy of T, a contradiction. This proves 5.6. ∎

Finally we can prove 1.6, which we restate:

5.7 *For every forest bigraph H, there exists $\varepsilon > 0$, such that every ε-coherent bigraph contains H.*

Proof. We may assume that H is a tree bigraph. Let K, d satisfy 5.6. We may assume by increasing d that $d \geq 1$. Choose $\varepsilon > 0$ such that $2K\varepsilon d \leq 1$. We claim that every ε-coherent bigraph contains T. Let G be an ε-coherent bigraph. It follows that $|V_1(G)|, |V_2(G)| \geq \varepsilon^{-1} \geq 2Kd \geq 2K$. Hence for $i = 1, 2$, $|V_i(G)|/K \geq \lceil |V_i(G)|/(2K) \rceil$, and so we may choose K subsets of $V_i(G)$, pairwise disjoint and each of cardinality $\lceil |V_i(G)|/(2K) \rceil \geq \varepsilon d |V_i(G)|$. These sets, in order, form a parade of length (K, K) and width at least $\varepsilon d(|V_1(G)|, |V_2(G)|)$, and so by 5.6, G contains H. This proves 5.7. ∎

References

[1] B. Alecu, A. Atminas, V. Lozin and V. Zamaraev, "Graph classes with linear Ramsey numbers", *Discrete Math.* **344** (2021), 112307, `arXiv:1910.12109`.

[2] N. Alon, J. Pach, R. Pinchasi, R. Radoičić and M. Sharir, "Crossing patterns of semi-algebraic sets", *J. Combinatorial Theory, Ser. A*, **111** (2005), 31–326.

[3] M. Axenovich, C. Tompkins and L. Weber, "Large homogeneous subgraphs in bipartite graphs with forbidden induced subgraphs", *J. Graph Theory* **97** (2021), 34–46 `arXiv:1903.09725`.

[4] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, "Pure pairs. I. Trees and linear anticomplete pairs", *Advances in Math.*, **375** (2020), 107396, `arXiv:1809.00919`.

[5] P. Erdős, "Graph theory and probability", *Canadian J. Math.* **11** (1959), 34–38.

[6] P. Erdős and A. Hajnal, "On spanned subgraphs of graphs", *Graphentheorie und Ihre Anwendungen* (Oberhof, 1977).

[7] P. Erdős and A. Hajnal, "Ramsey-type theorems", *Discrete Applied Math.* **25** (1989), 37–52.

[8] P. Erdős, A. Hajnal and J. Pach, "A Ramsey-type theorem for bipartite graphs", *Geombinatorics* 10 (2000), 64–68.

[9] J. Fox and J. Pach, "Erdős-Hajnal-type results on intersection patterns of geometric objects", in *Horizon of Combinatorics* (G.O.H. Katona et al., eds.), Bolyai Society Studies in Mathematics, Springer, 79–103, 2008.

[10] D. Korándi, J. Pach and I. Tomon, "Large homogeneous submatrices", *SIAM J. Discrete Math.* **34** (2020), 2532–2552, `arXiv:1903.06608`.

[11] V. Rödl, "On universality of graphs with uniformly distributed edges", *Discrete Math.* **59** (1986), 125–134.

[12] A. Scott, P. Seymour and S. Spirkl, "Pure pairs. VII. Homogeneous submatrices in 0/1-matrices with a forbidden submatrix", submitted for publication, `arXiv:2101.03537`.