COMBINATORICS IN THE EXTERIOR ALGEBRA AND THE BOLLOBÁS TWO FAMILIES THEOREM

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Abstract. We investigate the combinatorial structure of subspaces of the exterior algebra of a finite-dimensional real vector space, working in parallel with the extremal combinatorics of hypergraphs. As an application, we prove a new extension of the Two Families Theorem of Bollobás.

1. Introduction

For several decades there have been useful links between exterior algebra and combinatorics. Constructions exploiting the wedge product have been used in combinatorics to study intersections in hypergraphs, saturation problems, and simplicial complexes; the exterior algebra approach [1, 2, 12, 21, 33] to Bollobás's celebrated Two Families Theorem [7] is a highlight, as is Kalai's method of algebraic shifting [23–25]. Conversely, combinatorial results can be used to elucidate algebraic structures: a central example is the Kruskal-Katona theorem [29, 32, 41], which characterizes f-vectors of simplicial complexes and Hilbert series in certain algebraic structures.

In this paper, we study the combinatorics of linear subspaces of the exterior algebra of a finite dimensional real vector space. Making use of the connections between the two, we prove new results both in the exterior algebra and in extremal set theory. As an application of our results, we prove a new extension of the Two Families Theorem of Bollobás.

The paper is organized as follows. In Section 2, we first recall the basic correspondence between uniform hypergraphs and homogeneous subspaces of the exterior algebra over $\mathbb{R}^n$ (which depends on both a basis for $\mathbb{R}^n$ and a term ordering of the corresponding monomials in $\bigwedge^r \mathbb{R}^n$). We then use the correspondence to prove results about subspaces of the exterior algebra. For example, we determine the maximum dimension of a subspace of $V = \bigwedge^r \mathbb{R}^n$ in which every pair of elements has wedge product 0, and the maximum of $(\dim U)(\dim W)$ over subspaces U of $\bigwedge^r V$ and W of $\bigwedge^s V$ that mutually annihilate. Section 3 considers projections and liftings in the exterior algebra. We prove dimensional fraction bounds for projections and liftings of homogeneous subspaces of the exterior algebra (Lemmas 3.4 and 3.6). In fact the exterior algebra setting allows us more freedom than the combinatorial setting, since a generic choice of basis ensures that images under "random" projections have constant dimension (Corollaries 3.5 and 3.7). Finally,

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in Section 4, we prove (Theorem 4.5 and Corollary 4.6) new extensions of the Bollobás Two Families Theorem for both subspaces and set systems and discuss related examples and questions.

2. Exterior algebra and hypergraphs

> It happens to be rather easy to express the size of an r-graph in terms of exterior powers, but to make use of this expression is a rather different matter. [8, p. 117]

We begin this section by setting up definitions and notation, and defining the connection between hypergraphs and subspaces of the exterior algebra. We then use this connection to prove results about self-annihilating subspaces and pairs of mutually annihilating subspaces of the exterior algebra, and on the dimensional growth when a subspace is wedged with the underlying space.

2.1. **Uniform hypergraphs and homogeneous subspaces.** Given an integer $n > 0$, we write $[n] = \{1, \cdots, n\}$. For $0 \le r \le n$, we write

$$\binom{[n]}{r} = \{A \subseteq [n] \,:\, |A| = r\}.$$

for the collection of r-element subsets of $[n]$. A hypergraph $\mathcal{A}$ with ground set $[n]$ is *r-uniform* if $\mathcal{A} \subseteq \binom{[n]}{r}$; its *density* is $|\mathcal{A}|/\binom{n}{r}$.

For exterior algebra we largely follow the notation and terminology of [9], [4], and [18, Chapter 5], but we emphasize the dependence on a basis. The results included in Section 2 do not depend on the basis; however, some results in Section 3 will require a generic basis.

Let $V = \mathbb{R}^n$, viewed as column vectors, and write

$$\bigwedge V = \bigoplus_{r=0}^{n} \bigwedge^r V$$

for the standard grading of the exterior algebra of V. Let $E = \{e_1, \dots, e_n\}$ be the standard basis for V. For $F \in \mathrm{GL}_n(\mathbb{R})$, denote the columns and the entries of the E-matrix for F by

$$F = (f_1 | \dots | f_n) = (f_{ij}).$$

For $A \in \binom{[n]}{r}$, write $f_A = \bigwedge_{a \in A} f_a \in \bigwedge^r V$, where the elements of A are listed in increasing order. For $A, B \subseteq [n]$, we have

$$f_A \wedge f_B = \begin{cases} 0 & A \cap B \neq \emptyset, \\ \pm f_{A \cup B} & A \cap B = \emptyset, \end{cases} \tag{2.1}$$

where the sign in the non-trivial case is given by the sign of the permutation that sorts the list $a_1, \dots, a_k, b_1, \dots b_l$ into increasing order.

The set $F_r = \left\{ f_A \,:\, A \in \binom{[n]}{r} \right\}$ is a basis for $\bigwedge^r V$ and $\dim \bigwedge^r V = \binom{n}{r}$. For an r-uniform hypergraph $\mathcal{A} \subset \binom{[n]}{r}$, write $F(\mathcal{A}) = \mathrm{span}\{f_A \,:\, A \in \mathcal{A}\}$. Note that $\dim F(\mathcal{A}) = |\mathcal{A}|$, and that f_A and $F(\mathcal{A})$ both depend on our choice of $\mathcal{F}$.

We call a subspace $W \subseteq \bigwedge^r V$ *monomial with respect to* F when $W = F(\mathcal{A})$ for some hypergraph $\mathcal{A} \subseteq \binom{[n]}{r}$. Note that $\mathcal{A} \mapsto F(\mathcal{A})$ forms a bijection between r-uniform hypergraphs on $[n]$ and subspaces of $\bigwedge^r V$ monomial with respect to a fixed basis F; see Lemma 2.1.

Given a non-zero $w \in \bigwedge^r V$, define its *initial set* $\mathrm{ins}_F(w) \in \binom{[n]}{r}$ *with respect to* F as follows: expand w in the basis F_r as $w = \sum_{A \in \binom{[n]}{r}} m_A f_A$. Let

$$\mathrm{ins}(w) = \max \left\{ A \in \binom{[n]}{r} \,:\, m_A \neq 0 \right\}.$$

where the maximum is taken with respect to *reverse colex order* on $\binom{[n]}{r}$: we say $A > B$ exactly when the largest element of the symmetric difference of A and B is an element of B. For example, $\mathrm{ins}(f_1 \wedge f_4 + f_2 \wedge f_3) = \{2, 3\}$. See, for example, [8, Chapter 5] or [3, Chapter 7] for combinatorial treatments of colex order. The corresponding ordering on monomials is sometimes called *reverse lex* in the algebraic combinatorics literature, see for example [18, Section 2.1.2].

The key property of reverse colex we will need is that

$$A > B \text{ if and only if } A \cup C > B \cup C, \text{ whenever } A \cap C = B \cap C = \emptyset. \tag{2.2}$$

It follows immediately that for $C \subseteq [n]$ and $w \in \bigwedge^r V$ satisfying $\mathrm{ins}(w) \cap C = \emptyset$,

$$\mathrm{ins}(w \wedge f_C) = \mathrm{ins}(w) \cup C. \tag{2.3}$$

We define the *initial hypergraph* $\mathcal{H}_F(W) \subseteq \binom{[n]}{r}$ *with respect to* F of a subspace $W \subseteq \bigwedge^r V$ by

$$\mathcal{H}_F(W) = \{\mathrm{ins}(w) \,:\, w \in W, w \neq 0\}.$$

Let us note some basic facts about the correspondence between hypergraphs and subspaces.

Lemma 2.1. *Let $V = \mathbb{R}^n$, $F \in \mathrm{GL}_n(\mathbb{R})$, and $0 \leq r \leq n$. Then*

(i) $\dim W = |\mathcal{H}_F(W)|$ *for any subspace* $W \subseteq \bigwedge^r V$.
(ii) $F(\mathcal{H}_F(W)) = W$ *for* W *monomial with respect to* F.
(iii) $\mathcal{H}_F(F(\mathcal{A})) = \mathcal{A}$ *for any* $\mathcal{A} \subseteq \binom{[n]}{r}$.

Proof. For (i), note that the elements of any basis of W whose matrix in F_r is in reduced row echelon form must have distinct initial sets. That F_r is a basis of $\bigwedge^r V$ implies (ii) and (iii). □

We note that taking initial monomials, often with respect to a generic basis, is an important tool in the study of monomial ideals (see e.g. [18]); generally it is applied to ideals, but we will be interested throughout in mere subspaces. It is also easy to describe Kalai's algebraic shifting [25] in this notation: the algebraic shift of a hypergraph $\mathcal{A}$ with ground set $[n]$ is the hypergraph $\mathcal{H}_F(I(\mathcal{A}))$, where the identity matrix I induces the standard basis of $\mathbb{R}^n$, and $F \in \mathrm{GL}_n(\mathbb{R})$ is generic. We will use genericity in a similar spirit, but will need to be able to modify the dimension of the underlying vector spaces; see Sections 3.1 and 3.2.

2.2. **Annihilating subspaces of the exterior algebra.** Define a subspace $W \subseteq \bigwedge V$ to be *self-annihilating* if $v \wedge w = 0$ for all $v, w \in W$. For $r > n/2$, it is clear that $\bigwedge^r V$ is self-annihilating. However, for $r \leq n/2$, the situation is more interesting.

Theorem 2.2. *Let $V = \mathbb{R}^n$ and let W be a self-annihilating subspace of $\bigwedge V$. Then $\dim W \leq 2^{n-1}$. Furthermore, if $W \subseteq \bigwedge^r V$, where $r \leq n/2$, then*

$$\dim W \leq \binom{n-1}{r-1}. \tag{2.4}$$

Theorem 2.2 will follow from the classical Erdős-Ko-Rado Theorem [11], which is both an important tool in extremal combinatorics and the center of a web of generalizations (see, e.g. Godsil and Meager [16]).

We define a hypergraph $\mathcal{A} \subseteq 2^{[n]}$ to be *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$. It is easy to see that if $\mathcal{A}$ is intersecting then $|\mathcal{A}| \leq 2^{n-1}$, as $\mathcal{A}$ can contain at most one set from each pair $\{A, [n] \setminus A\}$. The Erdős-Ko-Rado Theorem gives an optimal bound on the size of an r-uniform intersecting family.

Theorem 2.3 (Erdős, Ko, Rado [11])**.** *Let $\mathcal{A} \subseteq 2^{[n]}$ be an intersecting hypergraph. Then $|\mathcal{A}| \leq 2^{n-1}$. Furthermore, if $\mathcal{A}$ is r-uniform, where $r \leq n/2$, then*

$$|\mathcal{A}| \leq \binom{n-1}{r-1}.$$

We now prove Theorem 2.2.

Proof of Theorem 2.2. Fix $F \in \mathrm{GL}_n(\mathbb{R})$. By the Erdős-Ko-Rado Theorem and Lemma 2.1, it is enough to verify that $\mathcal{H}_F(W)$ is an intersecting family, as $\dim(W) = |\mathcal{H}_F(W)|$ and $\mathcal{H}_F(W)$ is an intersecting hypergraph. Assume, looking for a contradiction, that for some nonzero $u, w \in W$ we have $A \cap B = \emptyset$, where $A = \mathrm{ins}(u)$ and $B = \mathrm{ins}(w)$. Since $u \wedge w = 0$, there must be other sets A', B' in the supports of u, w respectively with $A' \cap B' = \emptyset$ and $A' \cup B' = A \cup B$ (or else $f_{A \cup B}$ will have non-zero coefficient when we expand $u \wedge w$ in the F-monomial basis F_r).

It is impossible that $B' = A$ and $A' = B$ (because A and B are the initial sets of u, w respectively), so we must have $A_0 = A \cap A' \neq \emptyset$ and $B_0 = B \cap B' \neq \emptyset$. This gives disjoint decompositions

$$\begin{aligned} A &= A_0 \cup X, & B &= B_0 \cup Y, \\ A' &= A_0 \cup Y, & B' &= B_0 \cup X, \end{aligned}$$

so by (2.2)

$$A > A' \iff X > Y \iff B' > B,$$

contradicting either $A = \mathrm{ins}_F(u)$ or $B = \mathrm{ins}_F(W)$. $\square$

Both parts of Theorem 2.2 are optimal. For any fixed vector $v \in V$, the space $\{v \wedge z : z \in \bigwedge V\}$ has dimension 2^{n-1}. For $r \leq n/2$ the space $\{v \wedge z : z \in \bigwedge^{r-1} V\}$ has dimension $\binom{n-1}{r-1}$. For $r < n/2$, the extremal cases in Theorem 2.3 have a nice characterization: there is a single element contained in all sets of the family. It is an interesting question to describe the extremal examples for Theorem 2.2. Perhaps, for $r < n/2$, all extremal self-annihilating subspaces take the form just described? (This is trivial for $r = 1$; it is also true for $r = 2$, and follows from the fact that in this case elements of self-annihilating spaces are decomposable.)

We now consider pairs of subspaces. Two subspaces U, W of the exterior algebra are *mutually annihilating* if $u \wedge w = 0$ for all $u \in U$ and $w \in W$. We have the following counterpart to Theorem 2.2 (which implies (2.4) in the special case where we take $U = W$).

Theorem 2.4. *Let $V = \mathbb{R}^n$ and $1 \leq r, s \leq n/2$. Suppose that $U \subseteq \bigwedge^r V$ and $W \subseteq \bigwedge^s V$, and $u \wedge w = 0$ whenever $u \in U$ and $w \in W$. Then*

$$\dim U \dim W \leq \binom{n-1}{r-1}\binom{n-1}{s-1}.$$

Proof. This follows similar lines to the proof of Theorem 2.2: we consider the hypergraphs $\mathcal{A} = \mathcal{H}_F(U)$ and $\mathcal{B} = \mathcal{H}_F(W)$. Then $\mathcal{A}$ is r-uniform, $\mathcal{B}$ is s-uniform, and (arguing as before) we have $A \cap B$ nonempty for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. This means that $\mathcal{A}$ and $\mathcal{B}$ are *cross-intersecting systems*, and so by results of Pyber [40] and Matsumoto and Tokushige [36] we have

$$(\dim U)(\dim W) = |\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{r-1}\binom{n-1}{s-1},$$

as required. $\square$

Note that it is possible to attain equality in Theorem 2.4 by fixing $v \in V$ and setting $U = \{v \wedge z : z \in \bigwedge^{r-1}\}$ and $W = \{v \wedge z : z \in \bigwedge^{s-1}\}$. As with Theorem 2.2, it would be interesting to characterize the extremal examples when $r, s < n/2$.

2.3. **Local LYM in the exterior algebra.** Let $\mathcal{A} \subseteq \binom{[n]}{a}$ be an a-uniform hypergraph. For $1 \leq b \leq n - a$, the *b-th upper shadow* of $\mathcal{A}$ is the hypergraph

$$\partial^b \mathcal{A} = \left\{ B \in \binom{[n]}{a+b} : B \supseteq A \text{ for some } A \in \mathcal{A} \right\}.$$

An elementary result in extremal set theory is the *Local LYM Inequality*, named after the LYM Inequality of Lubell, Meshalkin and Yamamoto [34, 37, 49] (although versions of the local bound go back as far as Sperner [42]).

Lemma 2.5 (Local LYM)**.** *Fix non-negative a, b, n with $a + b \leq n$. Let $\mathcal{A} \subseteq \binom{[n]}{a}$ be an a-uniform hypergraph. Then*

$$\frac{|\partial^b \mathcal{A}|}{\binom{n}{a+b}} \geq \frac{|\mathcal{A}|}{\binom{n}{a}}.$$

The Local LYM Inequality is also known as the *normalized matching property.* Kleitman [31] proved that for finite ranked posets the normalized matching property is equivalent to the LYM bound on the size of an antichain. (We should note that the LYM bound also follows directly from the classical Two Families Theorem: see [8, p. 12], [3, Theorem 2.3.1], and [48, Section 2] for discussion.)

What can we hope for from an exterior version of the Local LYM inequality? For subspaces $U, W \subseteq \bigwedge V$, define

$$U \wedge W = \operatorname{span}\{u \wedge w \,:\, u \in U,\, w \in W\};$$

we also write $U \wedge w = U \wedge \operatorname{span}\{w\}$.

Fix $F \in \mathrm{GL}_n(\mathbb{R})$. For a monomial subspace $F(\mathcal{A}) \subseteq \bigwedge^r V$, equation (2.1) implies

$$F(\mathcal{A}) \wedge \bigwedge^c V = \operatorname{span}\left\{ f_A \wedge f_J \,:\, A \in \mathcal{A},\, J \in \binom{[n]}{c} \right\} = F(\partial^c \mathcal{A}). \tag{2.5}$$

That is, for monomial spaces, wedging with an exterior power of the ground space yields the monomial space generated by the upper shadow of the initial hypergraph. It follows that

$$\frac{\dim\left(F(\mathcal{A}) \wedge \bigwedge^c V\right)}{\binom{n}{r+c}} \geq \frac{\dim(F(\mathcal{A}))}{\binom{n}{r}}.$$

For a general homogeneous subspace of $W \subseteq \bigwedge^r V$, the picture is more complicated. However, we do have the containment

$$\mathcal{H}_F\left(W \wedge \bigwedge^c V\right) \supseteq \left\{ A \cup J \colon\, A \in \mathcal{H}_F(W), J \in \binom{[n] \setminus A}{c} \right\} = \partial^c(\mathcal{H}_F(W)), \tag{2.6}$$

and this suffices to prove a Local LYM bound.

Theorem 2.6 (Local LYM in the exterior algebra)**.** *Let $V = \mathbb{R}^n$ and $W \subseteq \bigwedge^r V$. Then for $0 \le c \le n - r$,*

$$\frac{\dim (W \wedge \bigwedge^c V)}{\dim(\bigwedge^{r+c} V)} \ge \frac{\dim W}{\dim(\bigwedge^r V)}.$$

Proof. Fix $F \in \mathrm{GL}_n(\mathbb{R})$. By (2.6) and Lemmas 2.5 and 2.1,

$$\frac{\dim(W \wedge \bigwedge^c V)}{\binom{n}{r+c}} = \frac{|\mathcal{H}_F(W \wedge \bigwedge^c V)|}{\binom{n}{r+c}} \ge \frac{|\partial^c \mathcal{H}_F(W)|}{\binom{n}{r+c}} \ge \frac{|\mathcal{H}_F(W)|}{\binom{n}{r}} = \frac{\dim W}{\binom{n}{r}}.$$

□

Theorem 2.6 can be viewed as a comparison of the r- and $(r + c)$-entries in the f-vector of the graded $\bigwedge V$-ideal generated by W. The result could also be deduced from a suitable version of the Kruskal-Katona theorem for $\bigwedge V$ (as found, for example, in [4, Theorem 4.1]).

3. Generic linear projections

In this section, we will be interested in the behaviour of subspaces W of $\bigwedge^r V$ under projections and under the operation of wedging with exterior powers of V. In both cases, we will want bounds on the dimension of the resulting subspace. Note that projections change the dimension of the underlying space, while wedging with an exterior power lifts W from $\bigwedge^r V$ to a higher exterior power.

Our proofs will use suitably generic subspaces of V: we show the existence of such subspaces in section 3.1 and prove our bounds on the dimensions of subspaces in section 3.2.

3.1. **Generic projections.** Throughout this section, let $V = \mathbb{R}^N$. We find conditions that guarantee the existence of bases of V that behave generically with respect to projections of given configurations of subspaces. In all cases we find a nonempty Zariski open subset of $\mathrm{GL}_N(\mathbb{R})$ having the desired properties (it makes no significant difference to the final results if we instead use the condition that our sets have complement with Lebesgue measure zero).

Let $F = (f_{ij}) = (f_1|f_2|\dots|f_N) \in \mathrm{GL}_N(\mathbb{R})$; i.e. F is an $N \times N$ matrix with entries f_{ij} and columns f_j. For $J \subseteq [N]$, let $V_J = \mathrm{span}\{f_j : j \in J\}$, and define the linear projection $\pi_J^F : V \to V_J$ by

$$\pi_J^F\left(\sum_{j\in[N]} \alpha_j f_j\right) = \sum_{j\in J} \alpha_j f_j. \tag{3.1}$$

For a subspace C of V and a set $J \subseteq [N]$, we clearly have $\dim(\pi_J^F(C)) \le \min\{\dim C, |J|\}$. We will show that, for typical choices of F, this holds with equality. The proof of Lemma 3.1 follows Frankl and Tokushige [13, L]emma 26.14.

Lemma 3.1. *Let $C_1, \dots, C_m$ be proper linear subspaces of V. Then there exists a non-zero polynomial G in the N^2 variables f_{ij}, $1 \le i, j \le N$, such that $G(F) \ne 0$ implies that $F = (f_{ij}) \in \mathrm{GL}_N(\mathbb{R})$ and*

$$\dim \pi_J^F(C_i) = \min\{\dim C_i, |J|\}$$

for all $1 \le i \le m$ and $J \subseteq [N]$.

Proof. The key idea is to write down a polynomial witnessing that $\pi_J^F(C_i)$ has maximum possible rank. Let $d_i = \dim C_i$. For each $1 \le i \le m$ and $J \subseteq [N]$, let $M_{i,J}$ be an N by $d_i + (N - |J|)$ matrix built by taking d_i columns forming a basis for C_i, together with the $N - |J|$ columns f_j, where $j \in [N] \setminus J$. We choose $G_{i,J}$ to be a minor of $M_{i,J}$ that can witness $M_{i,J}$ having full rank. More precisely:

- If $d_i \ge |J|$, let $G_{i,J}$ be an $N \times N$ minor including all C_i-basis columns, together with any choice of $N - d_i \le N - |J|$ columns f_j.
- Otherwise $d_i < |J|$. In this case there is a collection of d_i rows such that the restriction of the C_i basis to those rows is still linearly independent. Let $G_{i,J}$ be any $d_i + N - |J|$ by $d_i + N - |J|$ minor of $M_{i,J}$ including those d_i rows (and all d_i columns from the basis for C_i).

Note that $V_{[N]\setminus J}$ is the kernel of π_J^F. By construction, $G_{i,J} \neq 0$ implies that $\dim \operatorname{span}\{C_i, V_{[N]\setminus J}\} = \min\{N, d_i + N - |J|\}$, and thus immediately that $\dim(C_i \cap V_{[N]\setminus J}) = \max\{d_i - |J|, 0\}$ and $\dim(\pi_J^F(C_i)) = \min(\dim C_i, |J|)$.

Finally, set $G = (\det F) \displaystyle\prod_{1\le i\le m,\ J\subseteq[N]} G_{i,J}$. We note that G is not the zero polynomial, as for each i and J there are choices of F for which the matrix $M_{i,J}$ has full rank. □

We need an analogous result for subspaces of $\bigwedge^r V$. This is more difficult than for subspaces of V, as the subspace structure of $\bigwedge^r V$ interacts with the exterior algebra structure.

Lemma 3.2. *Let $W \subseteq \bigwedge^r V$ be a linear subspace. For $1 \le m \le N-1$, set*

$$t_m = \max_{J,F} \dim \pi_J^F(W),$$

where the maximum is taken over all $J \in \binom{[N]}{m}$ and $F \in \mathrm{GL}_N(\mathbb{R})$. Then there exists a non-zero polynomial H in the N^2 variables f_{ij}, $1 \le i,j \le N$, such that $H(F) \neq 0$ implies that $F = (f_{ij}) \in \mathrm{GL}_N(\mathbb{R})$ and

$$\dim \pi_J^F(W) = t_m$$

for all $J \in \binom{[N]}{m}$.

Proof. Fix m. Let $d = \dim(W) \ge t_m$, and choose $J^* \in \binom{[N]}{m}$ and $F^* \in \mathrm{GL}_N(\mathbb{R})$ to realize

$$\dim \pi_{J^*}^{F^*} = t_m. \tag{3.2}$$

Let $\{w_1, \dots, w_d\}$ be a basis of W such that $\pi_{J^*}^{F^*}(w_1), \dots, \pi_{J^*}^{F^*}(w_{t_m})$ are linearly independent in $V_{J^*} = \pi_{J^*}^{F^*}(V)$.

Build an $\binom{N}{r}$ by $\binom{N}{r} - \binom{m}{r} + d$ matrix M_{J^*} by taking the standard coordinates of $w_1, \dots, w_d$ for the first d columns, and the standard Plücker coordinates of the vectors f_K, where $K \in \binom{[N]}{r} \setminus \binom{[J]}{r}$, as the rest of the columns (the entries in these columns are degree-r polynomials in the variables f_{ij}). For any $F \in \mathrm{GL}_N(\mathbb{R})$, the f_K-columns of M_{J^*} form a basis for $\ker \pi_{J^*}^F$. By (3.2), for the specific basis F^* we have

$$\dim \left(W \cap \ker \pi_{J^*}^{F^*} \right) = d - t_m,$$

and thus, when $F = F^*$,

$$\operatorname{rank} M_{J^*} = \binom{N}{r} - \binom{m}{r} + d - (d - t_m) = \binom{N}{r} - \binom{m}{r} + t_m.$$

It follows that there exists a non-zero $\binom{N}{r} - \binom{m}{r} + t_m$ by $\binom{N}{r} - \binom{m}{r} + t_m$ minor of M_{J^*}; call this polynomial H_{J^*}. By our choice of basis for W, we can require that the columns included in that minor are $w_1, \dots, w_{t_m}$, together with all of the f_K^*-columns (note that $\pi_{J^*}^{F^*}(w_1,), \dots, \pi_{J^*}^{F^*}(w_{t_m})$ are linearly independent and the vectors f_K lie in $\ker \pi_{J^*}^{F^*}(w_i)$). Since H_{J^*} is non-zero for the specific basis F^*, it must in fact be a non-zero polynomial in the variables f_{ij}. Furthermore, whenever $H_{J^*}(F) \neq 0$, it is true that $\dim \pi_{J^*}^F(W) = t_m$.

We have found a suitable polynomial witness H_{J^*} for a particular $J^* \in \binom{[N]}{m}$. Let $J \in \binom{[N]}{m}$ be arbitrary, and fix a permutation $\sigma : [N] \to [N]$ with $\sigma(J^*) = J$. If we take σ to act on the columns of F, it induces a permutation of the variables f_{ij} (we set $\sigma(f_{ij}) = f_{i\sigma(j)}$) and thus an automorphism of the polynomial ring generated by the f_{ij}'s.

Consider the matrix $\sigma(M_{J^*})$, by which we mean the matrix resulting when this polynomial automorphism is applied to the entries of M_{J^*}. The w_i-columns are unchanged. For $K = \{k_1, \dots, k_r\} \in \binom{[N]}{r}$ we have

$$\begin{aligned}
\sigma(f_K) &= \sigma(f_{k_1} \wedge \cdots \wedge f_{k_r}) \\
&= \sigma(f_{k_1}) \wedge \cdots \wedge \sigma(f_{k_r}) \\
&= f_{\sigma(k_1)} \wedge \cdots \wedge f_{\sigma(k_r)} = \pm f_{\sigma(K)}.
\end{aligned}$$

By our choice of the permutation σ, we have $K \not\subseteq J^*$ exactly when $\sigma(K) \not\subseteq J$, so the columns of $\sigma(M_{J^*})$ are a basis for $\ker \pi_J^F$. Finally, set $H_J = \sigma(H_{J^*})$. Then H_J is a non-zero polynomial. It is also a $\binom{N}{r} - \binom{m}{r} + t_m$ by $\binom{N}{r} - \binom{m}{r} + t_m$ minor of the matrix $\sigma(M_{J^*})$. When $H_J(F) \neq 0$, then $\dim \pi_J^F(W) \geq t_m$. Since t_m was chosen to be the maximum possible dimension of a projection of W onto an m-dimensional subspace of V, in fact $H_J \neq 0$ implies $\dim \pi_J^F(W) = t_m$.

Finally, take H to be the product of $\det F$ and all the H_J's found by the process described above, as $m = |J|$ varies from 1 to $N - 1$. $\square$

3.2. **Dimensional fractions.** Let $V = \mathbb{R}^n$ and let W be a subspace of $\bigwedge^r V$. We will prove bounds on the size of subspaces obtained from projecting W onto a subspace of V, or wedging with an exterior power of V. Our measure of size will be the *dimensional fraction*

$$\frac{\dim W}{\dim \bigwedge^r V} = \frac{\dim W}{\binom{n}{r}}$$

occupied by a subspace $W \subseteq \bigwedge^r V$.

Let us begin with projections: our first goal will be to show that there exist projections that preserve the dimensional fraction. It will be helpful to consider projections alongside an analogous operation on hypergraphs: for an a-uniform hypergraph $\mathcal{A} \subseteq \binom{[n]}{a}$ and $B \in \binom{[n]}{a+b}$, define the *restriction* $\rho_B(\mathcal{A}) = \{A \in \mathcal{A} : A \subseteq B\}$ (i.e. the subgraph induced by B). The connection between projections and restrictions is given by

$$\pi_B(F(\mathcal{A})) = F(\rho_B(\mathcal{A}));$$

in other words, projecting a monomial space on to the subspace generated by $\{f_i : i \in B\}$ corresponds to taking the restriction of the corresponding hypergraph to B.

The following simple lemma shows that uniform hypergraphs have projections that preserve density.

Lemma 3.3. *Fix non-negative a, b, n with $a \le b \le n$. Let $\mathcal{A} \subseteq \binom{[n]}{a}$ be an a-uniform hypergraph. Then*

$$\max_{B \in \binom{[n]}{b}} \frac{|\pi_B(\mathcal{A})|}{\binom{b}{a}} \ge \frac{|\mathcal{A}|}{\binom{n}{a}}.$$

Proof. Count pairs (A, B) with $A \in \mathcal{A}$, $B \in \binom{[n]}{b}$, and $A \subseteq B$:

$$|\mathcal{A}|\binom{n-a}{b-a} = \sum_{B \in \binom{[n]}{b}} |\pi_B(\mathcal{A})| \le \binom{n}{b} \max_{B \in \binom{[n]}{b}} |\pi_B(\mathcal{A})|.$$

The first expression follows from choosing A first; the second, from choosing B first. Then divide by $\binom{n}{b}\binom{b}{a} = \binom{n}{a}\binom{n-a}{b-a}$. Alternatively, simply note that, choosing a b-set B uniformly at random, the expected number of edges in the restriction $\pi_B(\mathcal{A})$ is $(\binom{b}{a}/\binom{n}{a})|\mathcal{A}|$. □

Let us show that the bound of Lemma 3.3 implies a corresponding bound for dimensional fractions of projections. Fix $F \in \mathrm{GL}_n(V)$. For $J \subseteq [n]$, define the projection $\pi_J^F : V \to V_J$ by (3.1). Abusing notation, we also write $\pi_J^F : \bigwedge^r V \to \bigwedge^r V_J$ for the linear map defined by $\pi_J^F(f_A) = \bigwedge_{a \in A} \pi_J^F(f_a)$. Note that

$$\pi_J^F(f_A) = \begin{cases} f_A & A \subseteq J, \\ 0 & \text{otherwise}. \end{cases} \tag{3.3}$$

Lemma 3.4. *Suppose that $0 < r \le n - d$. Let $W \subseteq \bigwedge^r V$ be a linear subspace and $F \in \mathrm{GL}_n(\mathbb{R})$. Then*

$$\max_{J \in \binom{[n]}{n-d}} \frac{\dim \pi_J^F(W)}{\binom{n-d}{r}} \ge \frac{\dim W}{\binom{n}{r}}.$$

Proof. Let $J \in \binom{[n]}{n-d}$. By equations (2.2) and (3.3), the restriction $\rho_J(\mathcal{H}(W))$ of the initial hypergraph of W is contained in the initial hypergraph of the projection $\pi_J^F(W)$. That is, $\rho_J(\mathcal{H}(W)) \subseteq \mathcal{H}(\pi_J^F(W))$. By Lemmas 2.1 and 3.3,

$$\begin{aligned} \max_{J \in \binom{[n]}{n-d}} \frac{\dim \pi_J^F(W)}{\binom{n-d}{r}} &= \max_{J \in \binom{[n]}{n-d}} \frac{|\mathcal{H}(\pi_J^F(W))|}{\binom{n-d}{r}} \\ &\ge \max_{J \in \binom{[n]}{n-d}} \frac{|\rho_J(\mathcal{H}(W))|}{\binom{n-d}{r}} \ge \frac{\mathcal{H}(W)}{\binom{n}{r}} = \frac{\dim W}{\binom{n}{r}}. \end{aligned}$$

□

The existence of generic subspaces implies that a typical projection achieves the bound of Lemma 3.4:

Corollary 3.5. *Fix $0 < d \le n$. Let $W \subseteq \bigwedge^r V$ be a linear subspace and $F \in \mathrm{GL}_n(\mathbb{R})$. Then there exists a nonempty Zariski open set of $F \in \mathrm{GL}_n(\mathbb{R})$ satisfying*

$$\frac{\dim \pi_{[n-d]}^F(W)}{\binom{n-d}{r}} \ge \frac{\dim W}{\binom{n}{r}}.$$

Proof. By Lemma 3.2, for all F outside of the zero set of a particular polynomial, the dimension of $\pi_J^F(W)$ depends only on $|J|$, and thus $\dim \pi_{[n-d]}^F(W) = \max_{J\in\binom{[n]}{n-d}} \dim \pi_J^F(W)$. □

We now turn to the behavious of W under wedging with exterior powers of V. We know from Theorem 2.6 that wedging with exterior powers of V preserves the dimensional fraction. However, for our application we will need a stronger bound. We will show (Corollary 3.7) that if W has a *projection* with large dimensional fraction then wedging with a suitable exterior power of V gives a subspace achieving at least the same dimensional fraction.

We first bound the dimensional fraction of $W \wedge \bigwedge^d V$ in terms of the average dimensional fraction of a projection onto an $(n-d)$-dimensional subspace.

Lemma 3.6. *Suppose that $0 < r \le n - d \le n$. Let $W \subseteq \bigwedge^r V$ and $F \in \mathrm{GL}_n(\mathbb{R})$. Then*

$$\frac{\dim\left(W \wedge \bigwedge^d V\right)}{\binom{n}{r+d}} \ge \frac{1}{\binom{n}{n-d}} \sum_{J\in\binom{[n]}{n-d}} \frac{\dim \pi_J^F(W)}{\binom{n-d}{r}}.$$

Proof. Recall that the columns $\{f_1, \dots, f_n\}$ of F form a basis for V and that we write $f_K = f_{k_1} \wedge \cdots \wedge f_{k_d}$ when $K = \{k_1, \dots, k_d\}$ with the elements listed in increasing order. We know that $W \wedge \bigwedge^d V = \operatorname{span}\left\{W \wedge f_K \,:\, K \in \binom{[n]}{d}\right\}$, and so

$$\mathcal{H}\left(W \wedge \bigwedge^d V\right) \supseteq \bigcup_{K\in\binom{[n]}{d}} \mathcal{H}(W \wedge f_K).$$

For $K \in \binom{[n]}{d}$, we have $v \wedge f_K = \pi_{[n]\setminus K}^F(v) \wedge f_K$ for all $v \in V$, and so $\dim(W \wedge f_K) = \dim \pi_{[n]\setminus K}^F(W)$. Furthermore,

$$\mathcal{H}(W \wedge f_K) = \mathcal{H}\left(\pi_{[n]\setminus K}^F(W) \wedge f_K\right) = \left\{J \cup K : J \in \mathcal{H}\left(\pi_{[n]\setminus K}^F(W)\right)\right\}.$$

Each set $S \in \mathcal{H}(W \wedge \bigwedge^d W)$ has size $r + d$, and can occur in at most $\binom{r+d}{r}$ distinct families $\mathcal{H}(W \wedge f_K)$ (as there are only $\binom{r+d}{r}$ sets $K \subseteq S$ of size d). Thus

$$\begin{aligned}\left|\mathcal{H}\left(W \wedge \bigwedge^d V\right)\right| &\ge \frac{1}{\binom{r+d}{r}} \sum_{K\in\binom{[n]}{d}} \left|\mathcal{H}\left(\pi_{[n]\setminus K}^F(W) \wedge f_K\right)\right| \\ &= \frac{1}{\binom{r+d}{r}} \sum_{K\in\binom{[n]}{d}} \dim\left(\pi_{[n]\setminus K}^F(W)\right).\end{aligned}$$

By Lemma 2.1, $\dim\left(W \wedge \bigwedge^d V\right) = \left|\mathcal{H}(W \wedge \bigwedge^d V)\right|$. Since $\binom{n}{n+d}\binom{n+d}{r} = \binom{n}{n-d}\binom{n-d}{r}$, the result now follows. □

Once again, we use the existence of generic subspaces to obtain the desired bound.

Corollary 3.7. *Let $V = \mathbb{R}^n$ and fix $0 < r < r + d \le n$. Let $W \subseteq \bigwedge^r V$ be a linear subspace. Then there exists a nonempty Zariski open set of $F \in \mathrm{GL}_n(\mathbb{R})$ satisfying*

$$\frac{\dim\left(W \wedge \bigwedge^d V\right)}{\binom{n}{r+d}} \ge \frac{\dim \pi_{[n-d]}^F(W)}{\binom{n-d}{r}} = \max_{J^*,F^*} \frac{\dim \pi_{J^*}^{F^*}(W)}{\binom{n-d}{r}},$$

where the maximum is taken over all $J^* \in \binom{[N]}{n-d}$ *and* $F^* \in \mathrm{GL}_N(\mathbb{R})$.

Proof. By Lemma 3.2, for all F outside of the zero set of a particular polynomial, the dimension of $\pi_J^F(W)$ depends only on $|J|$, and thus $\dim \pi_{[n-d]}^F(W) = \frac{1}{\binom{n}{n-d}} \sum_{J \in \binom{[n]}{n-d}} \dim \pi_J^F(W)$. The inequality then follows from Lemma 3.6. □

4. Two Families Theorems

4.1. **Context and consequences.** Bollobás's Two Families Theorem [7] has been rediscovered in different forms and proved in several different ways (see [1, 2, 6, 17, 19, 21, 22, 28, 33, 44], Tuza's surveys [46, 47] of applications, and the expository discussions in Bollobás [8, Chapters 9 and 15], Anderson [3, Section 1.3], Babai and Frankl [5, Sections 5.1 and 6.2], Kalai [26], Matoušek [35, Miniature 33], Jukna [20, Section 9.2.2], Frankl and Tokushige [13, Sections 26.2–4], and Gerbner and Patkós [15, Section 1.1]). The simplest version of the Two Families Theorem is perhaps the following:

Theorem 4.1 (Uniform Two Families)**.** *Let* $(A_1, B_1), \dots, (A_m, B_m)$ *be a sequence of pairs of sets with* $|A_i| = a$ *and* $|B_i| = b$ *for every* i. *Suppose that*

(i) $A_i \cap B_i = \emptyset$ *for* $1 \le i \le m$, *and*
(ii) $A_i \cap B_j \neq \emptyset$ *for* $i \neq j$.

Then $m \le \binom{a+b}{a}$. *Furthermore, if* $m = \binom{a+b}{a}$ *then there is some set* S *of cardinality* $a + b$ *such that the* A_i *are all subsets of* S *of size* a, *and* $B_i = S \setminus A_i$ *for each* i.

A striking feature of this theorem is that the upper bound depends only on a and b, and not on the size of the ground set (compare Theorem 2.3).

There are two standard approaches to proving the Two Families Theorem, each of which exemplifies important methods in the field and leads to a different generalization. One approach is combinatorial (see Bollobás [7], or the elegant counting argument due to Katona [28]). With this approach, the assumption that the sets in each pair have the same sizes can be relaxed. When $|A| = a$ and $|B| = b$, we will say that the pair (A, B) has *profile* (a, b) and *profile sum* $a + b$. Note that when $|X| = a + b$, there are $\binom{a+b}{b}$ complementary pairs (A, B) with profile (a, b) and $A, B \subseteq X$. Bollobás's original result [7] is equivalent to Theorem 4.2, which weights each pair of sets by the nominal fraction of the set of pairs with matching union and profile that it occupies.

Theorem 4.2 (Weighted Two Families)**.** *Let* $(A_1, B_1), \dots, (A_m, B_m)$ *be a finite collection of pairs of finite sets. Let* $a_i = |A_i|$ *and* $b_i = |B_i|$ *for* $1 \le i \le m$. *Suppose that*

(i) $A_i \cap B_i = \emptyset$ *for* $1 \le i \le m$, *and*
(ii) $A_i \cap B_j \neq \emptyset$ *for* $i \neq j$.

Then

$$\sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \le 1. \tag{4.1}$$

Furthermore, if equality is achieved, then there is some finite set S *and* $0 \le a_0 \le |S|$ *such that the* A_i *are the subsets of* S *of size* a_0 *and* $B_i = S \setminus A_i$ *for each* i.

A second approach, introduced by Lovász [33], uses exterior algebra methods. This method gives an elegant argument that naturally extends to subspaces of a finite dimensional vector space; a set system version of Two Families follows immediately (using the standard construction illustrated in Corollary 4.6). Frankl [12] used a similar approach and noted that this method also allows the relaxation of condition (ii): instead of requiring A_i and B_j to intersect for all pairs with $i \neq j$, we insist only that the intersection is non-trivial when $i < j$. Proofs of this form of the Two Families Theorem also appeared in [1, 2, 21].

Theorem 4.3 (Uniform Skew Subspace Two Families)**.** *Let* $(A_1, B_1), \ldots, (A_m, B_m)$ *be pairs of non-trivial subspaces of* $V = \mathbb{R}^N$*. Suppose that* $\dim A_i \leq a$ *and* $\dim B_i \leq b$ *for* $1 \leq i \leq m$*, and*

(i) $\dim(A_i \cap B_i) = 0$ *for* $1 \leq i \leq m$*, and*
(ii) $\dim(A_i \cap B_j) > 0$ *for* $1 \leq i < j \leq m$*.*

Then $m \leq \binom{a+b}{a}$*.*

A version for hypergraphs follows immediately.[1]

Corollary 4.4 (Uniform Skew Two Families)**.** *Let* $(A_1, B_1), \ldots, (A_m, B_m)$ *be a sequence of pairs of sets with* $|A_i| = a$ *and* $|B_i| = b$ *for every* i*. Suppose that*

(i) $A_i \cap B_i = \emptyset$ *for* $1 \leq i \leq m$*, and*
(ii) $A_i \cap B_j \neq \emptyset$ *for* $1 \leq i < j \leq m$*.*

Then $m \leq \binom{a+b}{b}$*.*

Thus there are two completely different extensions of the Two Families Theorem: in one case, the set pairs are weighted according to their size; and in the other, the intersection condition is weakened to a skew intersection condition. It is natural to wonder if the Two Families Theorem can be extended in both these directions at once. In other words, is there a Two Families Theorem that has both weights and a skew hypothesis? For example, Tuza [46, Question 12] asked whether linear algebra techniques can be used to prove Two Families theorems in cases where the two families are not of constant profile.

The main result of this section is the following, which shows that under suitable conditions it is indeed possible to combine the two directions of generalization. We first state the result for subspaces.

Theorem 4.5. *Let* $(A_1, B_1), \ldots, (A_m, B_m)$ *be pairs of non-trivial subspaces of a finite-dimensional real vector space. Write* $a_i = \dim A_i$ *and* $b_i = \dim B_i$ *for* $1 \leq i \leq m$*. Suppose that*

(i) $\dim(A_i \cap B_i) = 0$ *for* $1 \leq i \leq m$*,*
(ii) $\dim(A_i \cap B_j) > 0$ *for* $1 \leq i < j \leq m$*, and*
(iii) $a_1 \leq a_2 \leq \cdots \leq a_m$ *and* $b_1 \geq b_2 \geq \cdots \geq b_m$*.*

Then

$$\sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \leq 1. \tag{4.2}$$

[1]Note that there is not a unique extremal hypergraph for Corollary 4.4: for example, B_1 can be any b-element set disjoint from A_1.

We prove this in the next subsection. The proof works in varying levels of the exterior algebra and over vector spaces of varying dimension. For this, we will use the Local LYM inequalities of Section 2 and the projection and wedging bounds of Section 3.

A combinatorial version of Theorem 4.5 follows immediately via a standard construction:

Corollary 4.6 (Weighted Skew Two Families)**.** *Let $(A_1, B_1), \ldots, (A_m, B_m)$ be pairs of finite non-empty sets. Write $a_i = |A_i|$ and $b_i = |B_i|$ for $1 \le i \le m$. Suppose that*

(i) $A_i \cap B_i = \emptyset$ *for* $1 \le i \le m$,
(ii) $A_i \cap B_j \ne \emptyset$ *for* $1 \le i < j \le m$, *and*
(iii) $a_1 \le a_2 \le \cdots \le a_m$ *and* $b_1 \ge b_2 \ge \cdots \ge b_m$.

Then

$$\sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \le 1. \tag{4.3}$$

Proof. Let $N \in \mathbb{N}$ be large enough that we may assume $A_i, B_i \subseteq [N]$ for $1 \le i \le m$. Let $\{e_1, \ldots, e_N\}$ be the standard basis of $\mathbb{R}^N$. Map each set A_i to the subspace $A_i' = \operatorname{span}\{e_a : a \in A_i\} \subseteq \mathbb{R}^N$ and each B_i to the subspace $B_i' = \operatorname{span}\{e_b : b \in B_i\} \subseteq \mathbb{R}_N$. Then $\dim A_i' = a_i$, $\dim B_i' = b_i$, and the hypotheses of Theorem 4.5 are satisfied by these subspaces. □

A bound of form (4.2) does not hold for arbitrary families of pairs satisfying a skew intersection condition without adding some restriction on the set sizes, as the following examples show.

Example 4.7 (Babai and Frankl [5, Exercise 5.1.1])**.** List all pairs (A, A^C) with $A \in 2^{[n]}$, sorted by decreasing cardinality of the first element. This "death" example, in which the a_i's decrease as the b_i's increase, satisfies (i) and (ii), but

$$\sum_{1=1}^{2^n} \frac{1}{\binom{n_i}{a_i}} = \sum_{j=0}^{n} \frac{\binom{n}{j}}{\binom{n}{j}} = n+1.$$

Example 4.8. Keeping one family of sets of constant size is also insufficient. Set $(A_i, B_i) = (\{i\}, [i-1])$ for $1 \le i \le n$. Now

$$\sum_{i=1}^{n} \frac{1}{\binom{n_i}{a_i}} = \sum_{i=1}^{n} \frac{1}{i} \sim \log n.$$

Returning to subspaces, it is natural to wonder whether a weighted Two Families Theorem holds under the full symmetric cross-intersecting hypothesis. Theorem 4.5 allows some progress:

Corollary 4.9. *Let $n \ge 2$, and suppose that $(A_1, B_1), \ldots, (A_m, B_m)$ are pairs of non-trivial subspaces of $V = \mathbb{R}^N$ such that $a_i + b_i = n$ for $1 \le i \le m$, where $a_i = \dim A_i$ and $b_i = \dim B_i$. Suppose that*

(i) $\dim(A_i \cap B_i) = 0$ *for* $1 \le i \le m$, *and*
(ii) $\dim(A_i \cap B_j) > 0$ *for* $1 \le i, j \le m$ *with* $i \ne j$.

Then

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{a_i}} \leq 1.$$

Proof. Permute the subscripts of the pairs (A_i, B_i) so that the A_i's are listed in increasing order of dimension; since our cross-intersecting hypothesis (ii) is symmetric, we can do so. Because the profile sums $a_i + b_i = n$ are constant, the resulting system satisfies all hypotheses of Theorem 4.5. □

The following also follows straighforwardly from Theorem 4.5.

Corollary 4.10. *Let $n \geq 2$ and suppose that $(A_1, B_1), \ldots, (A_m, B_m)$ are pairs of non-trivial subspaces of $V = \mathbb{R}^N$. Write $a_i = \dim A_i$ for $1 \leq i \leq m$, and let $b = \max_i \dim(B_i)$. Suppose that*

(i) $\dim(A_i \cap B_i) = 0$ *for* $1 \leq i \leq m$, *and*
(ii) $\dim(A_i \cap B_j) > 0$ *for* $1 \leq i, j \leq m$ *with* $i \neq j$.

Then

$$\sum_{i=1}^{m} \frac{1}{\binom{a_i+b}{a_i}} \leq 1.$$

Proof. First, permute the subscripts of the pairs (A_i, B_i) of spaces so that the A_i's are listed in increasing order of dimension; since our cross-intersecting hypothesis (ii) is symmetric, we can do so.

Let $a = \max_i a_i$, and embed the entire system in $\mathbb{R}^{a+b}$. For each $b_i < b$, extend B_i by including in it $b - b_i$ linearly independent vectors outside A_i. The resulting system, in which $a_1 \leq \cdots \leq a_m$ and $b_i = b$ for $1 \leq i \leq m$, satisfies the hypotheses of Theorem 4.5. □

Note that the proof of Corollary 4.10 does not use the full symmetric cross-intersecting condition: the argument goes through as long as the pairs of spaces are fully cross-intersecting between distinct profiles, but possibly only weakly cross-intersecting (with respect to some ordering) within the collections of pairs with the same profile.

4.2. **Proof of Theorem 4.5.** First, a definition: for a subspace $C \subseteq V = \mathbb{R}^N$ with basis $\{c_1, \ldots, c_d\}$, we define the *d-blade*

$$v_C = c_1 \wedge \cdots \wedge c_d \in \bigwedge^d V.$$

Although v_C is only determined up to a non-zero constant, $\operatorname{span}\{v_C\}$ is a well-defined one-dimensional subspace of $\bigwedge^d V$.

We now sketch our strategy. The hypotheses of Theorem 4.5 allow both the a_i's and the profile sums $n_i = a_i + b_i$ to vary in i. Because the profile sums can vary, we will want to vary the dimension of the underlying vector space. Because the a_i's can vary, we will want to vary the exterior degree as well. We will deal with this by inductively constructing a sequence of subspaces Z_i, where Z_i lies in $\bigwedge^{a_i} \mathbb{R}^{n_i}$. The space Z_i encodes the intersection structure of the pairs $(A_1, B_1), \ldots, (A_i, B_i)$ and will satisfy

$$\frac{\dim Z_i}{\binom{n_i}{a_i}} \geq \sum_{j=1}^{i} \frac{1}{\binom{n_j}{a_j}}. \tag{4.4}$$

Proof of Theorem 4.5. The main step in the proof lies in associating to the space $Z_i \subseteq \bigwedge^{a_i} \mathbb{R}^{n_i}$ a suitable space $Y_i \subseteq \bigwedge^{a_{i+1}} \mathbb{R}^{n_{i+1}}$ such that

$$\frac{\dim Y_i}{\binom{n_{i+1}}{a_{i+1}}} \geq \frac{\dim Z_i}{\binom{n_i}{a_i}},$$

and Y_i does not contain the a_{i+1}-blade corresponding to the space A_{i+1}. We then extend Y_i by the a_{i+1}-blade, increasing its dimension by 1, to obtain Z_{i+1} satisfying inequality (4.4) for $i+1$. Continuing through to $i = m$ and noting that $\dim Z_m \leq \binom{n_m}{a_m}$ gives the desired inequality.

Rather than defining spaces Z_i, Y_i directly, we define them as projections of a sequence of spaces W_i sitting in appropriate exterior powers of the ground space V. We recursively construct the sequence $W_i \subseteq \bigwedge^{a_i} V$ by setting $W_0 = \{0\}$ and, for $0 \leq i \leq m-1$,

$$W_{i+1} = \operatorname{span}\left\{W_i \wedge \bigwedge^{a_{i+1}-a_i} V,\, v_{A_{i+1}}\right\}. \tag{4.5}$$

We will fix a suitable basis F for V and use it to define a sequence of subspaces $V_{n_i} = \pi^F_{[n_i]}(V)$ of V. Since V_{n_i} is generated by the first n_i basis elements of F, we have $\dim V_{n_i} = n_i$. We then define Z_i as the projection of W_i on $\bigwedge^{a_i} V_{n_i}$, and take Y_i to be the projection of W_i onto $\bigwedge^{a_i} V_{n_{i+1}}$, wedged with $\bigwedge^{a_{i+1}-a_i} V_{n_{i+1}}$. That is, Y_i is a subspace of $\bigwedge^{a_i+1} V_{n_{i+1}}$, as is Z_{i+1}. As we prove our chain of inequalities, we will need to relate the dimensions of Z_i and Z_{i+1}; the space Y_i provides an intermediate step.

Let us give precise definitions of the spaces described above. Let $C_i = \operatorname{span}\{A_i, B_i\}$, and let $n_i = \dim C_i = a_i + b_i$ (by hypothesis (i)). By Lemmas 3.1 and 3.2, there is a Zariski open set of bases $\{f_1, \dots, f_N\}$ for V that satisfy the following: for every $J \subseteq [N]$ and all $1 \leq i < j \leq m$,

$$\dim(\pi^F_J(C_i)) = \min\{n_i, |J|\}, \tag{4.6a}$$

$$\dim(\pi^F_J(A_i \cap B_j)) = \min\{\dim(A_i \cap B_j), |J|\}, \tag{4.6b}$$

$$\dim \pi^F_J(W_i) = t_{i,|J|}, \tag{4.6c}$$

where $t_{i,|J|}$ is the maximum dimension of $\pi^F_{J^*}(W_i)$ over all choices of F and J^* with $|J^*| = |J|$. Fix one such generic basis F, and note that it will satisfy Corollaries 3.5 and 3.7. Let

$$Z_i = \pi^F_{[n_i]}(W_i),\ X_i = \pi^F_{[n_{i+1}]}(W_i) \ \text{ and } \ Y_i = X_i \wedge \bigwedge^{a_{i+1}-a_i} V_{[n_{i+1}]}. \tag{4.7}$$

Thus Z_i is a subspace of $\bigwedge^{a_i} V_{n_i}$, while X_i is a subspace of $\bigwedge^{a_i} V_{n_{i+1}}$ and Y_i is a subspace of $\bigwedge^{a_{i+1}} V_{n_{i+1}}$.

We will verify that for $0 \leq i \leq m-1$

$$\dim Z_{i+1} = \dim Y_i + 1 \tag{4.8}$$

and

$$\frac{\dim Y_i}{\binom{n_{i+1}}{a_{i+1}}} \geq \frac{\dim Z_i}{\binom{n_i}{a_i}}. \tag{4.9}$$

We then complete the proof by applying (4.8) and (4.9) in alternation until the final result is reached:

$$1 \geq \frac{\dim Z_m}{\binom{n_m}{a_m}} = \frac{1+\dim Y_{m-1}}{\binom{n_m}{a_m}} \geq \frac{1}{\binom{n_m}{a_m}} + \frac{\dim Z_{m-1}}{\binom{n_{m-1}}{a_{m-1}}} = \cdots \geq \sum_{i=1}^{m} \frac{1}{\binom{n_i}{a_i}}.$$

Proof of (4.8): By the definitions (4.5) and (4.7) of W_{i+1} and Z_{i+1},

$$\begin{aligned} Z_{i+1} &= \pi^F_{[n_{i+1}]}(W_{i+1}) \\ &= \operatorname{span}\left(\pi^F_{[n_{i+1}]}(v_{A_{i+1}}), \pi^F_{[n_{i+1}]}\left(W_i \wedge \bigwedge^{a_{i+1}-a_i} V_{[n_{i+1}]}\right)\right) \\ &= \operatorname{span}\left(\pi^F_{[n_{i+1}]}(v_{A_{i+1}}), \pi^F_{[n_{i+1}]}(W_i) \wedge \bigwedge^{a_{i+1}-a_i} V_{[n_{i+1}]}\right) \\ &= \operatorname{span}(\pi^F_{[n_{i+1}]}(v_{A_{i+1}}), Y_i). \end{aligned}$$

So it will suffice to check that $\pi^F_{[n_{i+1}]}\left(v_{A_{i+1}}\right) \notin Y_i$. By hypothesis (i), we have $v_{A_{i+1}} \wedge v_{B_{i+1}} \neq 0$. Since $n_{i+1} = a_{i+1} + b_{i+1}$, it follows from (4.6a) that

$$\pi^F_{[n_{i+1}]}(v_{A_{i+1}}) \wedge \pi^F_{[n_{i+1}]}(v_{B_{i+1}}) \neq 0. \tag{4.10}$$

Now consider $y \in W_i \wedge \bigwedge^{a_{i+1}-a_i} V$. For $h < i+1$, hypothesis (ii) implies that $v_{A_h} \wedge v_{B_{i+1}} = 0$. Since (by (4.5)), y is a linear combination of elements $\{v_{A_h} \wedge \bigwedge^{a_{i+1}-a_h} V : h \leq i\}$, it follows that $y \wedge v_{B_{i+1}} = 0$. Thus

$$\pi^F_{[n_{i+1}]}(y) \wedge \pi^F_{[n_{i+1}]}(v_{B_{i+1}}) = 0. \tag{4.11}$$

Equation (4.8) now follows from (4.10) and (4.11).

Proof of (4.9): Our argument depends on how (a_{i+1}, b_{i+1}) is related to (a_i, b_i).

- **Profile unchanged.** When $(a_{i+1}, b_{i+1}) = (a_i, b_i)$, we also know $n_{i+1} = n_i$ and $Y_i = Z_i$, so (4.9) follows immediately.
- **Profile sum constant.** When $(a_{i+1}, b_{i+1}) = (a_i + c, b_i - c)$ for some $c > 0$, we have $n_{i+1} = n_i$ and $Y_i = Z_i \wedge \left(\bigwedge^c V_{[n_i]}\right)$, so Lemma 2.6 gives (4.9).
- **B_i's shrink faster.** When $(a_{i+1}, b_{i+1}) = (a_i + c, b_i - c - d)$ for some $c \geq 0$ and $d > 0$, we have $n_{i+1} = n_i - d$. By Lemma 2.6,

$$\frac{\dim Y_i}{\binom{n_{i+1}}{a_{i+1}}} = \frac{\dim\left(\pi^F_{[n_i-d]}(W_i) \wedge \bigwedge^c V_{[n_i-d]}\right)}{\binom{n_i-d}{a_i+c}} \geq \frac{\dim \pi^F_{[n_i-d]}(W_i)}{\binom{n_i-d}{a_i}}.$$

Since $\pi^F_{[n_i-d]}(W_i) = \pi^F_{[n_i-d]}\left(\pi^F_{[n_i]}(W_i)\right)$, Corollary 3.5 and our generic choice of F imply

$$\frac{\dim \pi^F_{[n_i-d]}(W_i)}{\binom{n_i-d}{a_i}} \geq \frac{\dim \pi^F_{[n_i]}(W_i)}{\binom{n_i}{a_i}} = \frac{\dim Z_i}{\binom{n_i}{a_i}},$$

Thus (4.9) holds.

- **A_i's grow faster.** When $(a_{i+1}, b_{i+1}) = (a_i + c + d, b_i - c)$ for some $c \geq 0$ and $d > 0$, we have $n_{i+1} = n_i + d$. By Lemma 2.6,

$$\begin{aligned} \frac{\dim Y_i}{\binom{n_{i+1}}{a_{i+1}}} &= \frac{\dim\left(\pi^F_{[n_i+d]}(W_i) \wedge \bigwedge^{d+c} V_{[n_i+d]}\right)}{\binom{n_i+d}{a_i+d+c}} \\ &\geq \frac{\dim\left(\pi^F_{[n_i+d]}(W_i) \wedge \bigwedge^{d} V_{[n_i+d]}\right)}{\binom{n_i+d}{a_i+d}}. \end{aligned}$$

Since $\pi^F_{[n_i]}(W_i) = \pi^F_{[n_i]}\left(\pi^F_{[n_i+d]}(W_i)\right)$ we can apply Corollary 3.7 and equation (4.6c) to obtain

$$\frac{\dim\left(\pi^F_{[n_i+d]}(W_i) \wedge \bigwedge^{d} V_{[n_i+d]}\right)}{\binom{n_i+d}{a_i+d}} \geq \frac{\dim\left(\pi^F_{[n_i]}(W_i)\right)}{\binom{n_i}{a_i}} = \frac{\dim Z_i}{\binom{n_i}{a_i}}.$$

Thus (4.9) holds. □

4.3. **Limiting Examples and Questions.** It is not clear that we can hope to further relax condition (iii) of Theorem 4.5 and Corollary 4.6, which requires that

$$a_1 \leq a_2 \leq \cdots \leq a_m \text{ and } b_1 \geq b_2 \geq \cdots \geq b_m.$$

Examples 4.7 and 4.8 both violate condition (iii) for many values of i, and both examples satisfy

$$\sum_{i=1}^{m} \frac{1}{\binom{n_i}{a_i}} = \Omega(\log m).$$

However, there are examples achieving a weighted sum greater than 1 that violate condition (iii) for just one value of i.

Example 4.11. For $a, b, c > 0$, set $n = a+b$. Build a pair of families by first listing all profile-(a, b) complementary pairs of subsets of $[n]$. Choose $S \in \binom{[n]}{b+1}$. Any such S intersects non-trivially with each A_i so far. Now append a pair (A^*, B^*) to the list, where $|A^*| = a$, $|B^*| = b + c$, and $S \subseteq B^*$ (the elements of A^* and B^* can otherwise be chosen arbitrarily). The weighted sum is

$$\frac{\binom{n}{a}}{\binom{n}{a}} + \frac{1}{\binom{n+c}{a}} > 1.$$

Example 4.12. Fix $a, b, c, d > 0$ and $a > c$. Let $n = a + b$. Build a system by first listing all profile-(a, b) complementary pairs of subsets of $[n]$, then all profile-$(a - c, b + c + d)$ complementary pairs of subsets of $[n + d]$. This pair of families is skew cross-intersecting; note that when $d = 0$ it is two "levels" of Example 4.7. However, the weighted sum is

$$\frac{\binom{n}{a}}{\binom{n}{a}} + \frac{\binom{n+d}{a-c}}{\binom{n+d}{a-c}} = 2.$$

What about other relaxations of the cross-intersecting condition? For example, would it be enough to require the full cross-intersecting condition for pairs with distinct profiles, but only skew for pairs with the same profile?

Conjecture 4.13. *Let $(A_1, B_1), \ldots, (A_m, B_m)$ be pairs of finite non-empty subsets of $\mathbb{N}$. Write $a_i = |A_i|$ and $b_i = |B_i|$ for $1 \le i \le m$. Suppose that*

(i) $A_i \cap B_i = \emptyset$ *for* $1 \le i \le m$,
(ii) $A_i \cap B_j \ne \emptyset$ *for* $1 \le i < j \le m$, *and*
(iii) $A_i \cap B_j \ne \emptyset$ *if* $|A_i| \ne |A_j|$ *or* $|B_i| \ne |B_j|$.

Then

$$\sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \le 1. \tag{4.12}$$

Finally, we note that several other directions of generalization have been studied. For example, Tuza [45] further weakened the skew condition (ii) to require only that at least one of $A_i \cap B_j$ and $A_j \cap B_i$ be non-trivial for each $1 \le i, j, \le m$, $i \ne j$, a version considered further by Király, Nagy, Pálvölgyi, and Visontai [30]. Füredi [14], Talbot [43], and Kang, Kim, and Kim [27], considered t-intersecting pairs of families, while Einstein [10] (corrected in Oum and Wee [39]) and O'Neill and Verstraete [38] look at more than two families of sets. Can any of these variations be further addressed with exterior algebra methods?

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