Factors in randomly perturbed graphs

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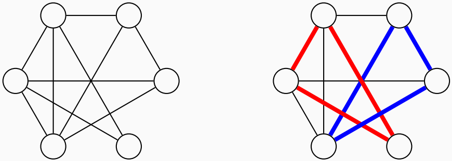
Factors

A *H*-factor in a graph G is a collection of pairwise vertex-disjoint copies of H in G that covers all vertices of G.

Example: The graph below has a K_3-factor.



Problem

Determine conditions on G which guarantee a H-factor in G.

Minimum degree thresholds in graphs

Problem

Determine the minimum degree threshold that ensures a graph G contains a H-factor.

K_3-factor (Corrádi, Hajnal 1963)

$$\delta(G) \geq 2n/3$$

K_r-factor (Hajnal and Szemerédi, 1970)

$$\delta(G) \geq (1 - 1/r)n$$

C_ℓ-factor (conjectured by El-Zahar and solved by Abbasi, 1998)

$$\delta(G) \geq \begin{cases} \frac{n}{2} & \text{if } \ell \text{ is even,} \\ \frac{\ell+1}{2\ell} n & \text{if } \ell \text{ is odd.} \end{cases}$$

Kühn and Osthus (2009) determined up to an additive constant the minimum degree threshold for H-factor, for any graph H.

Thresholds in random graph

We consider the binomial random graph $G(n, p)$.

Problem

Determine the threshold $t = t(n, H)$ at which $G(n, p)$ a.a.s. contains a H-factor.

- If $p = \omega(t)$, a.a.s. $G(n, p)$ has a H-factor, and
- if $p = o(t)$, a.a.s. $G(n, p)$ does not have a H-factor.

K_r-factor (Johansson, Kahn and Vu, 2008)

$$t(n) = n^{-2/r}(\log n)^{2/(r^2-r)}$$

C_ℓ-factor (Johansson, Kahn and Vu, 2008)

$$t(n) = n^{-(\ell-1)/\ell}(\log n)^{1/\ell}$$

Their result is more general, but the problem for general H-factor is still open for some graphs H.

Thresholds in randomly perturbed graphs

Let $\alpha, p \in [0,1]$, $n \in \mathbb{N}$ and G_α be a n-vertex graph with minimum degree at least αn. We call $G_\alpha \cup G(n,p)$ a randomly perturbed graph. (Bohman, Frieze and Martin, 2003)

Problem

Given α, determine the threshold $t = t(n, \alpha, H)$ at which $G_\alpha \cup G(n,p)$ a.a.s. contains a H-factor.

- If $p = \omega(t)$, for any G_α a.a.s. $G_\alpha \cup G(n,p)$ has a H-factor, and
- if $p = o(t)$, for one G_α, a.a.s. $G_\alpha \cup G(n,p)$ does not have a H-factor.

Small α: the dense graph 'helps' $G(n,p)$ to have the spanning structure.

$\longleftrightarrow$

Small p: the random edges 'help' the dense graph to have the spanning structure.

Clique factors

Threshold for K_r-factor in $G_\alpha \cup G(n,p)$:

- Balogh, Treglown and Wagner (2019)
  $$t(n) = n^{-2/r} \text{ for } \alpha \in \left(0, \tfrac{1}{r}\right)$$
  They gave the threshold for H-factors for small α, for any H.
- Han, Morris and Treglown (2021)
  $$t(n) = n^{-2/(r-k)} \text{ for } \alpha \in \left(\tfrac{k}{r}, \tfrac{k+1}{r}\right)$$

The threshold is constant within each interval and jumps at the endpoints. For K_3-factor:

$\alpha = 0$	$0 < \alpha < 1/3$	$1/3 < \alpha < 2/3$	$2/3 \leq \alpha$
$n^{-2/3}(\log n)^{1/3}$ Johansson, Kahn and Vu, 2008	$n^{-2/3}$ Balogh, Treglown, and Wagner, 2019	n^{-1} Han, Morris, and Treglown, 2021	0 Corrádi, Hajnal, 1963

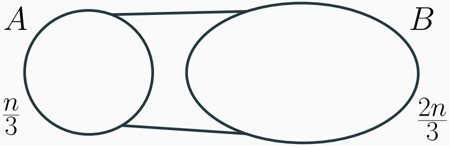
Question

What about the boundary case(s)?

$\alpha = 0$	$0 < \alpha < 1/3$	$\alpha = 1/3$	$1/3 < \alpha < 2/3$	$2/3 \leq \alpha$
$n^{-2/3}(\log n)^{1/3}$	$n^{-2/3}$	$n^{-1}(\log n)$	n^{-1}	0

For $\alpha = 1/3$, $\omega(n^{-1})$ is not enough and $\omega(n^{-1}\log n)$ is needed:

$K_{\frac{n}{3},\frac{2n}{3}}$



Theorem (Böttcher, Parczyk, S. and Skokan, 2020+)

There exists $C > 0$ such that with $p \geq C \log n/n$ the following holds. $G_{1/3} \cup G(n,p)$ a.a.s. contains a triangle factor.

Triangle factors - a bit more

Theorem (Böttcher, Parczyk, S. and Skokan, 2020+)

For $0 < \beta < 1/12$ there exist $\gamma > 0$ and $C > 0$ such that for any α with $4\beta \leq \alpha \leq 1/3$ and $p \geq C/n$ the following holds. If G

- has minimum degree at least $(\alpha - \gamma)\, n$, and
- is not 'β-close' to the extremal graph,

then a.a.s. $G \cup G(n,p)$ contains at least $\min\{\alpha n, \lfloor n/3 \rfloor\}$ pairwise vertex-disjoint triangles.

Theorem (Böttcher, Parczyk, S., and Skokan, 2020+)

There exists $C > 0$ such that with $p \geq C \log n/n$ the following holds. $G \cup G(n,p)$ a.a.s. contains at least $\min\{\delta(G), \lfloor n/3 \rfloor\}$ pairwise vertex-disjoint triangles.

Our results extend to cycle factors.

$\alpha = 0$	$0 < \alpha < 1/\ell$	$\alpha = 1/\ell$	$1/\ell < \alpha < \alpha^\star$	$\alpha^\star \leq \alpha$
$n^{-(\ell-1)/\ell}(\log n)^{1/\ell}$	$n^{-(\ell-1)/\ell}$	$n^{-1}\log n$	n^{-1}	0

Proof idea - Extremal case 1/2

Let G be a graph with $\delta(G) \geq n/3$ and assume G is β-close to the extremal graph.

Proof idea - Extremal case 2/2

Proof idea - Non extremal case 1/2

Let G be a graph with $\delta(G) \geq (1/3 - \gamma)n$ and assume G is not β-close to the extremal graph.

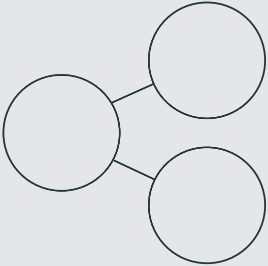
We apply the regularity lemma.

Stability tool (Balogh, Mousset, and Skokan)

The reduced graph R has a matching with $(1/3 + 4\gamma)v(R)$ edges.

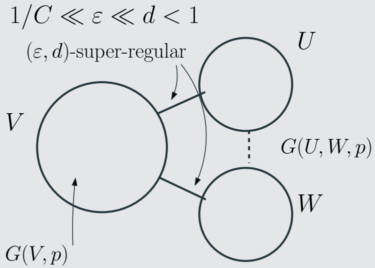
Proof idea - Non extremal case 2/2

Lemma (Böttcher, Parczyk, S., and Skokan, 2020+)



Proof idea - Non extremal case 2/2

Lemma (Böttcher, Parczyk, S., and Skokan, 2020+)



The leaf clusters have equal sizes but are slightly smaller then the centre. Moreover the total number of vertices is divisible by 3.

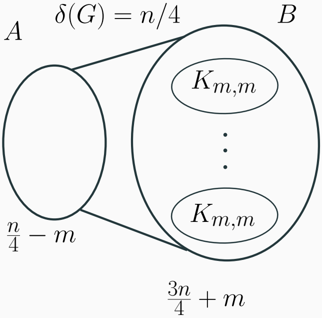
With $p \geq C/n$, a.a.s. the cherry has a triangle factor.

Larger clique factors

K_4-factor in $G \cup G(n,p)$:

$\alpha = 0$	$0 < \alpha < 1/4$	$\alpha = 1/4$	$1/4 < \alpha < 2/4$	$\dots$
$n^{-1/2}(\log n)^{1/6}$	$n^{-1/2}$	$n^{-2/3}(\log n)^{1/3}$?	$n^{-2/3}$	$\dots$

False!



- Assume $G \cup G(n,p)$ has a K_4-factor.
- Since $|A| = n/4 - m$, there must be at least m disjoint K_4's with all vertices in B.
- For small $\varepsilon > 0$ and $n^{7\varepsilon} \le m \le n^{1/7}$, $\mathbb{E}[\#K_4's \text{ in } B] < m$.
- The same calculation shows that even $p = n^{-2/3+\varepsilon}$ does not suffice.

Open problems

Other boundary cases

- Investigate the perturbed threshold for clique factors for other boundary cases.

General H-factors

- Balogh, Treglown and Wagner gave the perturbed thresholds for perfect H-factor for $\alpha < 1/|H|$. The problem is still wide open for larger values of α.

Thank you!