Approximate subgroups with bounded VC-dimension

Logic and Combinatorics Day, Oxford

Anand Pillay

May 12, 2020

Introduction

- This is joint work with Gabriel Conant, and a preprint is available on arXiv.

Introduction

- This is joint work with Gabriel Conant, and a preprint is available on arXiv.
- We study the family of all pairs (G, A) where G is an arbitrary group, and A a finite subset of G, such that

Introduction

- This is joint work with Gabriel Conant, and a preprint is available on arXiv.
- We study the family of all pairs (G, A) where G is an arbitrary group, and A a finite subset of G, such that
- (i) (small tripling) $|A \cdot A \cdot A| \leq k|A|$ (for some fixed k),

Introduction

- This is joint work with Gabriel Conant, and a preprint is available on arXiv.
- We study the family of all pairs (G, A) where G is an arbitrary group, and A a finite subset of G, such that
- (i) (small tripling) $|A \cdot A \cdot A| \leq k|A|$ (for some fixed k),
- (ii) for some fixed finite bipartite graph Γ, the bipartitite graph (G, G, E_A) where $E_A(x, y)$ if $xy \in A$, omits Γ.

Introduction

- This is joint work with Gabriel Conant, and a preprint is available on arXiv.
- We study the family of all pairs (G, A) where G is an arbitrary group, and A a finite subset of G, such that
- (i) (small tripling) $|A \cdot A \cdot A| \leq k|A|$ (for some fixed k),
- (ii) for some fixed finite bipartite graph Γ, the bipartitite graph (G, G, E_A) where $E_A(x, y)$ if $xy \in A$, omits Γ.
- We prove, roughly speaking, that up to a small error, A is a union of a bounded number of translates of a coset nilprogression of bounded rank and step. (Where the terms will be explained later and we will not giving explicit bounds.)

Introduction

- This is joint work with Gabriel Conant, and a preprint is available on arXiv.
- We study the family of all pairs (G, A) where G is an arbitrary group, and A a finite subset of G, such that
- (i) (small tripling) $|A \cdot A \cdot A| \leq k|A|$ (for some fixed k),
- (ii) for some fixed finite bipartite graph Γ, the bipartitite graph (G, G, E_A) where $E_A(x, y)$ if $xy \in A$, omits Γ.
- We prove, roughly speaking, that up to a small error, A is a union of a bounded number of translates of a coset nilprogression of bounded rank and step. (Where the terms will be explained later and we will not giving explicit bounds.)
- The proof makes use of nonstandard methods, related to those in Conant-Pillay-Terry II, as well as existing results on approximate subgroups (Breuillard-Green-Tao). Hrushovski's work on approximate subgroups is also an inspiration.

Background I

- As the audience may be mixed, I would like to start with some background, motivating or explaining the problematic.

Background I

- As the audience may be mixed, I would like to start with some background, motivating or explaining the problematic.
- Let's start with what I will call the arithmetic regularity problem, of trying to say something meaningful about *all* pairs (G, A) where G is a finite group and A a subset of G (in analogy with Szemeredi graph regularity).

Background I

- As the audience may be mixed, I would like to start with some background, motivating or explaining the problematic.
- Let's start with what I will call the arithmetic regularity problem, of trying to say something meaningful about *all* pairs (G, A) where G is a finite group and A a subset of G (in analogy with Szemeredi graph regularity).
- A better not be too small, otherwise there is nothing one can say. But if for example the cardinality of A is $\geq \delta|G|$ for fixed δ then one may hope for some kind of structure (an extreme situation being that A is close to a bounded index subgroup of G).

Background I

- As the audience may be mixed, I would like to start with some background, motivating or explaining the problematic.
- Let's start with what I will call the arithmetic regularity problem, of trying to say something meaningful about *all* pairs (G, A) where G is a finite group and A a subset of G (in analogy with Szemeredi graph regularity).
- A better not be too small, otherwise there is nothing one can say. But if for example the cardinality of A is $\geq \delta|G|$ for fixed δ then one may hope for some kind of structure (an extreme situation being that A is close to a bounded index subgroup of G).
- What about saying something meaningful about all pairs (G, A) where G is a not necessarily finite group, and A an arbitrary finite subset of G?

Background I

- As the audience may be mixed, I would like to start with some background, motivating or explaining the problematic.
- Let's start with what I will call the arithmetic regularity problem, of trying to say something meaningful about *all* pairs (G, A) where G is a finite group and A a subset of G (in analogy with Szemeredi graph regularity).
- A better not be too small, otherwise there is nothing one can say. But if for example the cardinality of A is $\geq \delta|G|$ for fixed δ then one may hope for some kind of structure (an extreme situation being that A is close to a bounded index subgroup of G).
- What about saying something meaningful about all pairs (G, A) where G is a not necessarily finite group, and A an arbitrary finite subset of G?
- In general, one is in the first case above, where A is too small, so it is natural to put some additional hypotheses on A.

Background II

- One such hypothesis is "small tripling"; that $|A \cdot A \cdot A| \leq k|A|$ (for fixed k). And we will call this the approximate subgroup problem.

Background II

- One such hypothesis is "small tripling"; that $|A \cdot A \cdot A| \leq k|A|$ (for fixed k). And we will call this the approximate subgroup problem.
- Freiman's theorem that when A is a subset of $\mathbb{Z}$ with small doubling, then A is contained in a "generalized arithmetic progression" of size $\leq C|A|$, is a solution to the specialization of the approximate subgroup problem to $G = \mathbb{Z}$.

Background II

- One such hypothesis is "small tripling"; that $|A \cdot A \cdot A| \leq k|A|$ (for fixed k). And we will call this the approximate subgroup problem.
- Freiman's theorem that when A is a subset of $\mathbb{Z}$ with small doubling, then A is contained in a "generalized arithmetic progression" of size $\leq C|A|$, is a solution to the specialization of the approximate subgroup problem to $G = \mathbb{Z}$.
- Going back to arithmetic regularity, Green has a certain Fourier analytic statement when G is abelian, but the general case of G nonabelian is open.

Background II

- One such hypothesis is "small tripling"; that $|A \cdot A \cdot A| \leq k|A|$ (for fixed k). And we will call this the approximate subgroup problem.
- Freiman's theorem that when A is a subset of $\mathbb{Z}$ with small doubling, then A is contained in a "generalized arithmetic progression" of size $\leq C|A|$, is a solution to the specialization of the approximate subgroup problem to $G = \mathbb{Z}$.
- Going back to arithmetic regularity, Green has a certain Fourier analytic statement when G is abelian, but the general case of G nonabelian is open.
- However there have been a series of results when additional conditions are placed on the (finite) graph (G, G, E_A), where $(x, y) \in E_A$ iff $xy \in A$, such as k-stability, or k-NIP. (Terry-Wolf, Alon-Fox-Zhao, Sisak, Conant-Pillay-Terry I,II)

Background III

- The k-NIP case is more or less the case of some fixed finite graph Γ being omitted by the (G, G, E_A). (See below.)

Background III

- The k-NIP case is more or less the case of some fixed finite graph Γ being omitted by the (G, G, E_A). (See below.)
- Among the above references, only Conant-Pillay-Terry deal with the case where G is not necessarily abelian (but without explicit bounds), and the conclusion is roughly that:

Background III

- The k-NIP case is more or less the case of some fixed finite graph Γ being omitted by the (G, G, E_A). (See below.)
- Among the above references, only Conant-Pillay-Terry deal with the case where G is not necessarily abelian (but without explicit bounds), and the conclusion is roughly that:
- Up to a small error (in various senses) A is a bounded union of translates of a set $\pi^{-1}(B)$ where π is a homomorphism from a bounded index subgroup H of G to a torus $\mathbb{T}$ and B is a nice open neighbourhood of the identity in $\mathbb{T}$.

Background III

- The k-NIP case is more or less the case of some fixed finite graph Γ being omitted by the (G, G, E_A). (See below.)
- Among the above references, only Conant-Pillay-Terry deal with the case where G is not necessarily abelian (but without explicit bounds), and the conclusion is roughly that:
- Up to a small error (in various senses) A is a bounded union of translates of a set $\pi^{-1}(B)$ where π is a homomorphism from a bounded index subgroup H of G to a torus $\mathbb{T}$ and B is a nice open neighbourhood of the identity in $\mathbb{T}$.
- Going back to approximate subgroups, we have the theorem of Breuillard, Green, and Tao, building on Hrushovski, that A is covered by a bounded number of translates of a "coset nilprogession" $P \subseteq (A \cup A^{-1})^8$ (where P has bounded rank, step, and is of bounded normal form).

Background IV

- Our aim was to combine the approximate subgroup problem and k-NIP arithmetic regularity problem, by considering all pairs (G, A), G an arbitrary group, and A a finite subset with k-tripling, where in addition (G, G, E_A) is d-NIP for some fixed k, d.

Background IV

- Our aim was to combine the approximate subgroup problem and k-NIP arithmetic regularity problem, by considering all pairs (G, A), G an arbitrary group, and A a finite subset with k-tripling, where in addition (G, G, E_A) is d-NIP for some fixed k, d.
- This extends and was motivated by recent work of Martin-Pizarro, Palacin, and Wolf, who assume d-stability instead of d-NIP. Also Sisask has a statement in the d-NIP case when $G = \mathbb{F}_q^n$.

Background IV

- Our aim was to combine the approximate subgroup problem and k-NIP arithmetic regularity problem, by considering all pairs (G, A), G an arbitrary group, and A a finite subset with k-tripling, where in addition (G, G, E_A) is d-NIP for some fixed k, d.
- This extends and was motivated by recent work of Martin-Pizarro, Palacin, and Wolf, who assume d-stability instead of d-NIP. Also Sisask has a statement in the d-NIP case when $G = \mathbb{F}_q^n$.
- The title mentions approximate subgroups: A k-approximate subgroup of a group G is a symmetric subset A such that $A \cdot A$ is covered by k translates of A.

Background IV

- Our aim was to combine the approximate subgroup problem and k-NIP arithmetic regularity problem, by considering all pairs (G, A), G an arbitrary group, and A a finite subset with k-tripling, where in addition (G, G, E_A) is d-NIP for some fixed k, d.
- This extends and was motivated by recent work of Martin-Pizarro, Palacin, and Wolf, who assume d-stability instead of d-NIP. Also Sisask has a statement in the d-NIP case when $G = \mathbb{F}_q^n$.
- The title mentions approximate subgroups: A k-approximate subgroup of a group G is a symmetric subset A such that $A \cdot A$ is covered by k translates of A.
- A finite k-approximate subgroup has k-tripling, and if A is finite with k-tripling then $(A \cup A^{-1})^2$ is $f(k)$-approximate (for an explicit $f = cx^d$). (Tao).

Background IV

- Our aim was to combine the approximate subgroup problem and k-NIP arithmetic regularity problem, by considering all pairs (G, A), G an arbitrary group, and A a finite subset with k-tripling, where in addition (G, G, E_A) is d-NIP for some fixed k, d.
- This extends and was motivated by recent work of Martin-Pizarro, Palacin, and Wolf, who assume d-stability instead of d-NIP. Also Sisask has a statement in the d-NIP case when $G = \mathbb{F}_q^n$.
- The title mentions approximate subgroups: A k-approximate subgroup of a group G is a symmetric subset A such that $A \cdot A$ is covered by k translates of A.
- A finite k-approximate subgroup has k-tripling, and if A is finite with k-tripling then $(A \cup A^{-1})^2$ is $f(k)$-approximate (for an explicit $f = cx^d$). (Tao).
- For G abelian small doubling suffices above.

Coset progressions

- As promised we define some terms. For motivation we start with coset progressions.
- A *generalized arithmetic progression* in a group G is the image of a d-dimensional box $B = \prod_{i=1,..,d}[-L_i, L_i] \subset \mathbb{Z}^d$ under a homomorphism $\pi : \mathbb{Z}^d \to G$.
- Here G is usually abelian, and a properness condition says that π is $1-1$.
- For G abelian a generalized arithmetic progression of dimension d is a 2^d-approximate subgroup.
- Conversely, Green-Ruzsa prove, generalizing Freiman's theorem that if A is a finite subset of an abelian group G, and A has k-doubling, then A is contained in e translates of a *coset progression* $P = P_0 + H$ where P_0 is a generalized arithmetic progression of dimension d, H is a finite subgroup of G, $P \subseteq 2A - 2A$ (and e, d depend on k).

Coset nilprogressions

- This is a generalization:

Coset nilprogressions

- This is a generalization:
- In place of the box B above, one considers instead a "box" in the free nilpotent group of step r and rank k, consisting of elements which can be written as a word in the generators $e_1, .., e_k$ where e_i and its inverse appear at most L_i times.

Coset nilprogressions

- This is a generalization:
- In place of the box B above, one considers instead a "box" in the free nilpotent group of step r and rank k, consisting of elements which can be written as a word in the generators $e_1, .., e_k$ where e_i and its inverse appear at most L_i times.
- And a nilprogression (rank k, step r) in a group G is an image of such a box under a homomorphism.

Coset nilprogressions

- This is a generalization:
- In place of the box B above, one considers instead a "box" in the free nilpotent group of step r and rank k, consisting of elements which can be written as a word in the generators $e_1, .., e_k$ where e_i and its inverse appear at most L_i times.
- And a nilprogression (rank k, step r) in a group G is an image of such a box under a homomorphism.
- A *coset nilprogression* in G is a set P of the form $P_0 H$ where P_0 is a nilprogression and H a finite subgroup of G normalized by P_0. P has rank r, step k, if P_0 does.

Coset nilprogressions

- This is a generalization:
- In place of the box B above, one considers instead a "box" in the free nilpotent group of step r and rank k, consisting of elements which can be written as a word in the generators $e_1, .., e_k$ where e_i and its inverse appear at most L_i times.
- And a nilprogression (rank k, step r) in a group G is an image of such a box under a homomorphism.
- A *coset nilprogression* in G is a set P of the form $P_0 H$ where P_0 is a nilprogression and H a finite subgroup of G normalized by P_0. P has rank r, step k, if P_0 does.
- There is an analogue of the properness, or irredundancy condition, which is called c-normal form, and which I will not get into.

NIP and VC-dimension

- Given pairs (G, A), G an arbitrary group, A an arbitrary subset, we say that A is d-NIP, if the graph (G, G, E_A) (mentioned earlier) omits the graph $([d], \mathcal{P}[d], \in)$.

NIP and VC-dimension

- Given pairs (G, A), G an arbitrary group, A an arbitrary subset, we say that A is d-NIP, if the graph (G, G, E_A) (mentioned earlier) omits the graph $([d], \mathcal{P}[d], \in)$.
- This condition is implied by (G, G, E_A) omitting *some* sufficiently large finite subgraph.

NIP and VC-dimension

- Given pairs (G, A), G an arbitrary group, A an arbitrary subset, we say that A is d-NIP, if the graph (G, G, E_A) (mentioned earlier) omits the graph $([d], \mathcal{P}[d], \in)$.
- This condition is implied by (G, G, E_A) omitting *some* sufficiently large finite subgraph.
- Also A being d-NIP is equivalent to the family of left translates of A in G having VC-dimension strictly less than d.

NIP and VC-dimension

- Given pairs (G, A), G an arbitrary group, A an arbitrary subset, we say that A is d-NIP, if the graph (G, G, E_A) (mentioned earlier) omits the graph $([d], \mathcal{P}[d], \in)$.
- This condition is implied by (G, G, E_A) omitting *some* sufficiently large finite subgraph.
- Also A being d-NIP is equivalent to the family of left translates of A in G having VC-dimension strictly less than d.
- Let us mention in passing that if $A \subset G$ is (finite) and d-NIP with k-tripling then already $A \cup A^{-1} \cup \{1\}$ is a $c_d k^e$-approximate subgroup.

Statements

- Recall again the BGT theorem that if A is a finite subset of a group G, which has k-tripling, then there is a coset nilprogression $P \subseteq (A \cup A^{-1})^8$ with rank and step $O_k(1)$ and such that $O_k(1)$ translates of P cover A.

- Recall again the BGT theorem that if A is a finite subset of a group G, which has k-tripling, then there is a coset nilprogression $P \subseteq (A \cup A^{-1})^8$ with rank and step $O_k(1)$ and such that $O_k(1)$ translates of P cover A.

Our theorem is:

Theorem 0.1

Suppose A is a finite subset of a group G, and A has k-tripling and d-NIP. Given $\epsilon > 0$, there is a coset nilprogression $P \subseteq G$, and a subset $Z \subseteq AP$ with $|Z| < \epsilon|A|$ (the error set) such that
(i) $P \subseteq AA^{-1} \cap A^{-1}A$ and $A \subseteq CP$ for some $C \subseteq A$,
(ii) For some $D \subseteq C$, $|(A \Delta DP) \setminus Z| < \epsilon|P|$.
(iii) For $g \in G \setminus Z$, $|gP \cap A| < \epsilon|P|$ or $|gP \cap A| > (1-\epsilon)|P|$.
Moreover rank and step and normal form of P, and the cardinality of C, are bounded by constants depending only on d, k, ϵ. And if G is abelian we can take P to be a (proper) coset progression.

Comments

- Note first that there is no ϵ in BGT. The dependence on ϵ is a typical feature of regularity statements.

Comments

- Note first that there is no ϵ in BGT. The dependence on ϵ is a typical feature of regularity statements.
- Item (i) includes (and improves) BGT (of course under the additional NIP hypothesis).

Comments

- Note first that there is no ϵ in BGT. The dependence on ϵ is a typical feature of regularity statements.
- Item (i) includes (and improves) BGT (of course under the additional NIP hypothesis).
- The rest says that up to the error set Z, A is a union of a bounded number of translates of the coset nilprogresson P, which is a fairly tight structure theorem for A.

Comments

- Note first that there is no ϵ in BGT. The dependence on ϵ is a typical feature of regularity statements.
- Item (i) includes (and improves) BGT (of course under the additional NIP hypothesis).
- The rest says that up to the error set Z, A is a union of a bounded number of translates of the coset nilprogresson P, which is a fairly tight structure theorem for A.
- Note that (iii) follows from (ii).

Comments

- Note first that there is no ϵ in BGT. The dependence on ϵ is a typical feature of regularity statements.
- Item (i) includes (and improves) BGT (of course under the additional NIP hypothesis).
- The rest says that up to the error set Z, A is a union of a bounded number of translates of the coset nilprogresson P, which is a fairly tight structure theorem for A.
- Note that (iii) follows from (ii).
- When G (or even $AA^{-1} \cap A^{-1}A$) has exponent at most r, then the coset nilprogression can be replace by a finite subgroup H, so that after throwing away the error set Z, A is a bounded union of left cosets of H up to a set of size at most $\epsilon|H|$.

Comments

- Note first that there is no ϵ in BGT. The dependence on ϵ is a typical feature of regularity statements.
- Item (i) includes (and improves) BGT (of course under the additional NIP hypothesis).
- The rest says that up to the error set Z, A is a union of a bounded number of translates of the coset nilprogresson P, which is a fairly tight structure theorem for A.
- Note that (iii) follows from (ii).
- When G (or even $AA^{-1} \cap A^{-1}A$) has exponent at most r, then the coset nilprogression can be replace by a finite subgroup H, so that after throwing away the error set Z, A is a bounded union of left cosets of H up to a set of size at most $\epsilon|H|$.
- In the rest of this talk, I will discuss aspects of the proof of the Theorem (which is more than just superimposing CPTII on BGT, although the srtategy is similar).

Remarks on proofs I

- The proof is essentially a nonstandard proof. Namely prove a single statement in a nonstandard environment; a group G equipped with a pseudofinite subset A which has k-tripling (with respect to the pseudofinite counting measure) and A is d-NIP.

Remarks on proofs I

- The proof is essentially a nonstandard proof. Namely prove a single statement in a nonstandard environment; a group G equipped with a pseudofinite subset A which has k-tripling (with respect to the pseudofinite counting measure) and A is d-NIP.
- (If one wants, consider G, A as living in a nonstandard model of set theory which thinks A is finite.)

Remarks on proofs I

- The proof is essentially a nonstandard proof. Namely prove a single statement in a nonstandard environment; a group G equipped with a pseudofinite subset A which has k-tripling (with respect to the pseudofinite counting measure) and A is d-NIP.
- (If one wants, consider G, A as living in a nonstandard model of set theory which thinks A is finite.)
- The use of model theory or logic has two aspects: (a) proving the relevant statement in the nonstandard (pseudofinite) environment and (b) pulling it down suitably to obtain the theorem.

Remarks on proofs I

- The proof is essentially a nonstandard proof. Namely prove a single statement in a nonstandard environment; a group G equipped with a pseudofinite subset A which has k-tripling (with respect to the pseudofinite counting measure) and A is d-NIP.
- (If one wants, consider G, A as living in a nonstandard model of set theory which thinks A is finite.)
- The use of model theory or logic has two aspects: (a) proving the relevant statement in the nonstandard (pseudofinite) environment and (b) pulling it down suitably to obtain the theorem.
- Part (b) is essentially routine. Part (a) is the main thing although the current proof still involves going down here and there and appealing to BGT. In any case, from here on it is model theory.

Remarks on proofs II

- So we have (G, A) as above which some might want to think of as an ultraproduct of (G_i, A_i) with the A_i finite, with k-tripling and d-NIP. No harm in assuming G saturated.

Remarks on proofs II

- So we have (G, A) as above which some might want to think of as an ultraproduct of (G_i, A_i) with the A_i finite, with k-tripling and d-NIP. No harm in assuming G saturated.
- Let H be the subgroup of G generated by A. H is what is called $\vee$- or ind-definable, being a union of the definable sets $A^{\pm m} = A^m \cup A^{-m} \cup \{1\}$, where for $m \geq 2$, $A^{\pm m}$ is covered by finitely many left (right) translates of A (using the assumptions).

Remarks on proofs II

- So we have (G, A) as above which some might want to think of as an ultraproduct of (G_i, A_i) with the A_i finite, with k-tripling and d-NIP. No harm in assuming G saturated.
- Let H be the subgroup of G generated by A. H is what is called $\vee$- or ind-definable, being a union of the definable sets $A^{\pm m} = A^m \cup A^{-m} \cup \{1\}$, where for $m \geq 2$, $A^{\pm m}$ is covered by finitely many left (right) translates of A (using the assumptions).
- Let μ be the pseudofinite counting measure, normalized such that $\mu(A) = 1$.

Remarks on proofs II

- So we have (G, A) as above which some might want to think of as an ultraproduct of (G_i, A_i) with the A_i finite, with k-tripling and d-NIP. No harm in assuming G saturated.
- Let H be the subgroup of G generated by A. H is what is called $\vee$- or ind-definable, being a union of the definable sets $A^{\pm m} = A^m \cup A^{-m} \cup \{1\}$, where for $m \geq 2$, $A^{\pm m}$ is covered by finitely many left (right) translates of A (using the assumptions).
- Let μ be the pseudofinite counting measure, normalized such that $\mu(A) = 1$.
- Let $\mathcal{R}$ be the "ring" generated by the left-right translates of A by elements of H.

Remarks on proofs II

- So we have (G, A) as above which some might want to think of as an ultraproduct of (G_i, A_i) with the A_i finite, with k-tripling and d-NIP. No harm in assuming G saturated.
- Let H be the subgroup of G generated by A. H is what is called $\vee$- or ind-definable, being a union of the definable sets $A^{\pm m} = A^m \cup A^{-m} \cup \{1\}$, where for $m \geq 2$, $A^{\pm m}$ is covered by finitely many left (right) translates of A (using the assumptions).
- Let μ be the pseudofinite counting measure, normalized such that $\mu(A) = 1$.
- Let $\mathcal{R}$ be the "ring" generated by the left-right translates of A by elements of H.
- Then μ is $< \infty$-valued on elements of $\mathcal{R}$ and is both left and right H-invariant.

Remarks on proofs III

- I will state the basic steps, with some simplifications, and where A being NIP (and pseudofinite) is essential.

Remarks on proofs III

- I will state the basic steps, with some simplifications, and where A being NIP (and pseudofinite) is essential.
- **Step I: The NIP stabilizer theorem**:

Remarks on proofs III

- I will state the basic steps, with some simplifications, and where A being NIP (and pseudofinite) is essential.
- **Step I: The NIP stabilizer theorem**:
- $\Gamma = \{g \in G : \mu(gA\Delta A) = 0\} = \{g \in G : \mu(Ag\Delta A) = 0\}$ is a countably $\mathcal{R}$-type-definable normal subgroup of H contained in $AA^{-1} \cap A^{-1}A$ and of "bounded index" in H. (In fact Γ is $H^{00}_{\mathcal{R}}$.)

Remarks on proofs III

- I will state the basic steps, with some simplifications, and where A being NIP (and pseudofinite) is essential.
- **Step I: The NIP stabilizer theorem**:
- $\Gamma = \{g \in G : \mu(gA\Delta A) = 0\} = \{g \in G : \mu(Ag\Delta A) = 0\}$ is a countably $\mathcal{R}$-type-definable normal subgroup of H contained in $AA^{-1} \cap A^{-1}A$ and of "bounded index" in H. (In fact Γ is $H^{00}_{\mathcal{R}}$.)
- The quotient H/Γ is a locally compact group (logic topology) $\mathbb{G}$ say, let $\pi : H \to \mathbb{G}$ be the canonical surjective homomorphism.

Remarks on proofs III

- I will state the basic steps, with some simplifications, and where A being NIP (and pseudofinite) is essential.
- **Step I: The NIP stabilizer theorem**:
- $\Gamma = \{g \in G : \mu(gA\Delta A) = 0\} = \{g \in G : \mu(Ag\Delta A) = 0\}$ is a countably $\mathcal{R}$-type-definable normal subgroup of H contained in $AA^{-1} \cap A^{-1}A$ and of "bounded index" in H. (In fact Γ is $H^{00}_{\mathcal{R}}$.)
- The quotient H/Γ is a locally compact group (logic topology) $\mathbb{G}$ say, let $\pi : H \to \mathbb{G}$ be the canonical surjective homomorphism.
- **Step II: Uniqueness of measure**:

Remarks on proofs III

- I will state the basic steps, with some simplifications, and where A being NIP (and pseudofinite) is essential.
- **Step I: The NIP stabilizer theorem**:
- $\Gamma = \{g \in G : \mu(gA\Delta A) = 0\} = \{g \in G : \mu(Ag\Delta A) = 0\}$ is a countably $\mathcal{R}$-type-definable normal subgroup of H contained in $AA^{-1} \cap A^{-1}A$ and of "bounded index" in H. (In fact Γ is $H^{00}_{\mathcal{R}}$.)
- The quotient H/Γ is a locally compact group (logic topology) $\mathbb{G}$ say, let $\pi : H \to \mathbb{G}$ be the canonical surjective homomorphism.
- **Step II: Uniqueness of measure**:
- For any left-H-invariant nontrivial measure ν on $\mathcal{R}$ the 0-ideal of ν corresponds to the 0-ideal of μ, which will be the "non-generics". (In fact ν will equal μ up to scaling.)

Remarks on proofs IV

- **Step III: Generic locally compact domination**:

Remarks on proofs IV

- **Step III: Generic locally compact domination**:
- Let λ be a Haar measure on $\mathbb{G}$, let X be a set in $\mathcal{R}$. Then there is a closed set $E \subset \mathbb{G}$ of λ-measure 0 such that for all $c \in \mathbb{G} \setminus Z$, exactly one of $\pi^{-1}(c) \cap X$, $\pi^{-1}(c) \setminus X$ is "μ-wide" (is not contained in a definable set of μ-measure 0).

Remarks on proofs IV

- **Step III: Generic locally compact domination**:
- Let λ be a Haar measure on $\mathbb{G}$, let X be a set in $\mathcal{R}$. Then there is a closed set $E \subset \mathbb{G}$ of λ-measure 0 such that for all $c \in \mathbb{G} \setminus Z$, exactly one of $\pi^{-1}(c) \cap X$, $\pi^{-1}(c) \setminus X$ is "μ-wide" (is not contained in a definable set of μ-measure 0).
- A compactness argument and trick from CPTII yields:

Remarks on proofs IV

- **Step III: Generic locally compact domination**:
- Let λ be a Haar measure on $\mathbb{G}$, let X be a set in $\mathcal{R}$. Then there is a closed set $E \subset \mathbb{G}$ of λ-measure 0 such that for all $c \in \mathbb{G} \setminus Z$, exactly one of $\pi^{-1}(c) \cap X$, $\pi^{-1}(c) \setminus X$ is "μ-wide" (is not contained in a definable set of μ-measure 0).
- A compactness argument and trick from CPTII yields:
- **Step IV**: Let $\Gamma = \cap W_n$, where $(W_n)_n$ are decreasing sets in $\mathcal{R}$. Then for every $\epsilon > 0$, there is $Z \in \mathcal{R}$ with $\mu(Z) < \epsilon$, and n, such for all $g \in G \setminus Z$ either $\mu(gW_n \cap A) = 0$ or $\mu(gW_n \setminus A) = 0$.

Remarks on proofs IV

- **Step III: Generic locally compact domination**:
- Let λ be a Haar measure on $\mathbb{G}$, let X be a set in $\mathcal{R}$. Then there is a closed set $E \subset \mathbb{G}$ of λ-measure 0 such that for all $c \in \mathbb{G} \setminus Z$, exactly one of $\pi^{-1}(c) \cap X$, $\pi^{-1}(c) \setminus X$ is "μ-wide" (is not contained in a definable set of μ-measure 0).
- A compactness argument and trick from CPTII yields:
- **Step IV**: Let $\Gamma = \cap W_n$, where $(W_n)_n$ are decreasing sets in $\mathcal{R}$. Then for every $\epsilon > 0$, there is $Z \in \mathcal{R}$ with $\mu(Z) < \epsilon$, and n, such for all $g \in G \setminus Z$ either $\mu(gW_n \cap A) = 0$ or $\mu(gW_n \setminus A) = 0$.
- We can write W_n as W^4 for some $W \in \mathcal{R}$ containing Γ which is an approximate subgroup. Appealing to (ultra) BGT there is an internal coset nilprogression $P \subseteq G$ with $P \subseteq W_n$ and finitely many translates of P covering W_n.

Remarks on proofs V

- So with one replaces W_n in the conclusion of Step 4, by P, to obtain the desired statement in the nonstandard environment:

Remarks on proofs V

- So with one replaces W_n in the conclusion of Step 4, by P, to obtain the desired statement in the nonstandard environment:
- **Step V**: For any $\epsilon > 0$ there is an internal coset nilprogression P in normal form, and $Z \subseteq AP$ with $Z \in \mathcal{R}$ and $\mu(Z) < \epsilon$, such that $P \subseteq AA^{-1} \cap A^{-1}A$, A is covered by finitely many translates of P, for each $g \in G \setminus Z$, $\mu(gP \cap A) = 0$ or $\mu(gP \setminus A) = 0$. We conclude that $A \setminus Z$ is a finite union of translates gP of P, up to measure 0.

Final comment

- A natural question, and one asked by possibly Pierre Simon in an earlier talk by Conant on this topic, is whether there is a pure measure-theoretic (model-theoretic) statement in the background.

Final comment

- A natural question, and one asked by possibly Pierre Simon in an earlier talk by Conant on this topic, is whether there is a pure measure-theoretic (model-theoretic) statement in the background.
- The answer should be yes.

Final comment

- A natural question, and one asked by possibly Pierre Simon in an earlier talk by Conant on this topic, is whether there is a pure measure-theoretic (model-theoretic) statement in the background.
- The answer should be yes.
- There is an account of generically stable ϕ-measures for $\phi(x, y)$ an NIP formula, and k-tripling generalizes to the presence of an invariant measure.

Final comment

- A natural question, and one asked by possibly Pierre Simon in an earlier talk by Conant on this topic, is whether there is a pure measure-theoretic (model-theoretic) statement in the background.
- The answer should be yes.
- There is an account of generically stable ϕ-measures for $\phi(x, y)$ an NIP formula, and k-tripling generalizes to the presence of an invariant measure.
- And one should make an assumption on the relevant locally compact group H/Γ, namely that it is an inverse limit of Lie groups, each of whose connected components is nilpotent (which is something proved in BGT when A is a pseudofinite approximate subgroup).