Matching Random Points

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Fairness: how to share resources between individuals?

Uneven setting: individuals have different locations/desires/abilities.

Maximize **"overall happiness"**…?

Should we make **X** happier but **Y** less happy?

much

slightly

…even if X already happier than Y?

Maximize $\sum_{\text{individuals}} \boldsymbol{f}(\text{happiness})$.

Given such a metric, what are the consequences?
Can individuals agree on a solution?

Simple but rich mathematical setting: **Matching**

$R = \{\text{red points}\}$
$B = \{\text{blue points}\}$

Independent intensity-1 Poisson point processes in $\mathbb{R}^d$

# of points in A ~ Poisson(vol(A))
Independent #s of points in disjoint sets
Countably infinite # of points

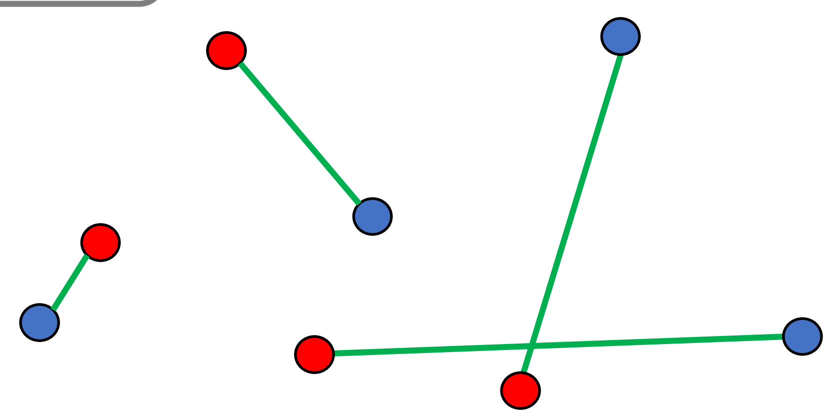
$M = \text{red–blue matching}$

$$\text{Minimize} \sum_{(r,b)\in M} |r-b|^{\gamma}.$$

"fairness" $0 < \gamma < \infty$

Euclidean norm

But then $= \infty$!



Solution: minimize **locally.**

M is **γ-minimal** if $\forall$ *finite* $\{(r_1, b_1), \ldots, (r_m, b_m)\} \subseteq M$,

$$\sum_i |r_i - b_i|^\gamma = \min_{\sigma \in S_m} \sum_i |r_i - b_{\sigma(i)}|^\gamma$$

permutation

Also: $-\infty < \gamma < 0$: replace $|\quad|^\gamma$ with $-|\quad|^\gamma$

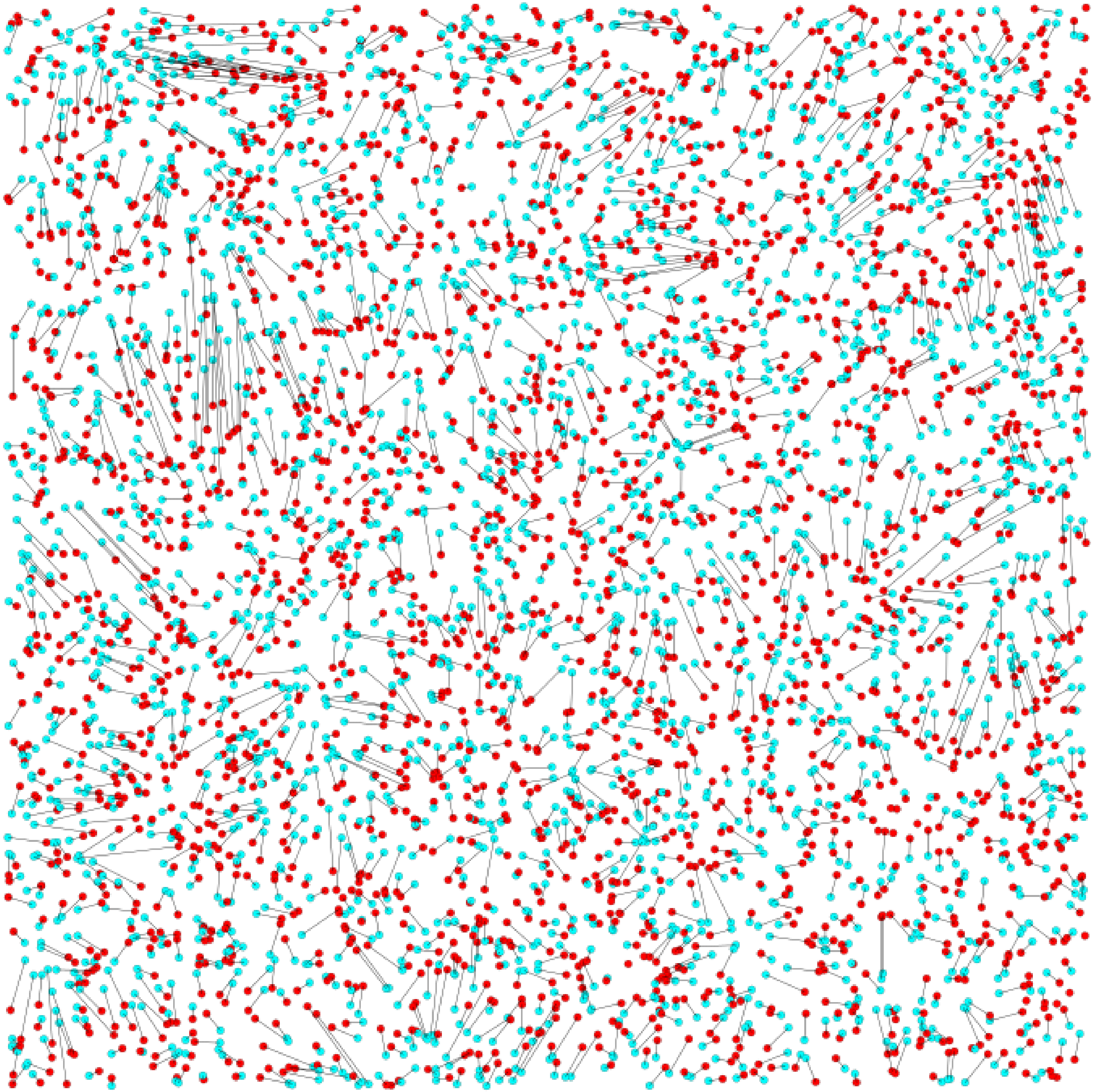
$\gamma = 0$: replace $|\quad|^\gamma$ with $\log|\quad|$

same as $\gamma \to 0$

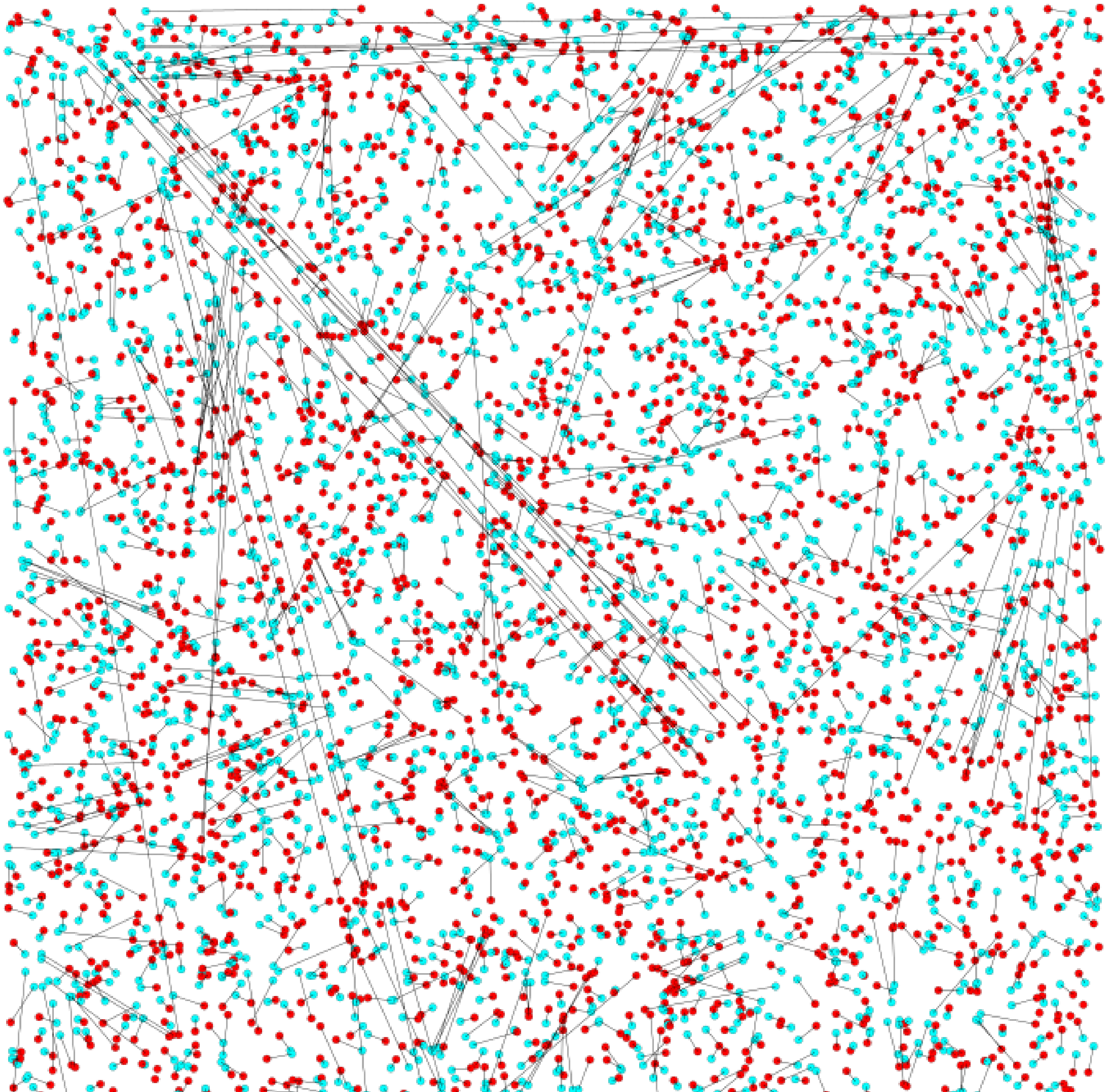
(only scale-invariant choices)

$\gamma = -\infty$ (selfish): lexicographically minimize $\uparrow$ ordering of $|r_1 - b_1|, \ldots, |r_m - b_m|$

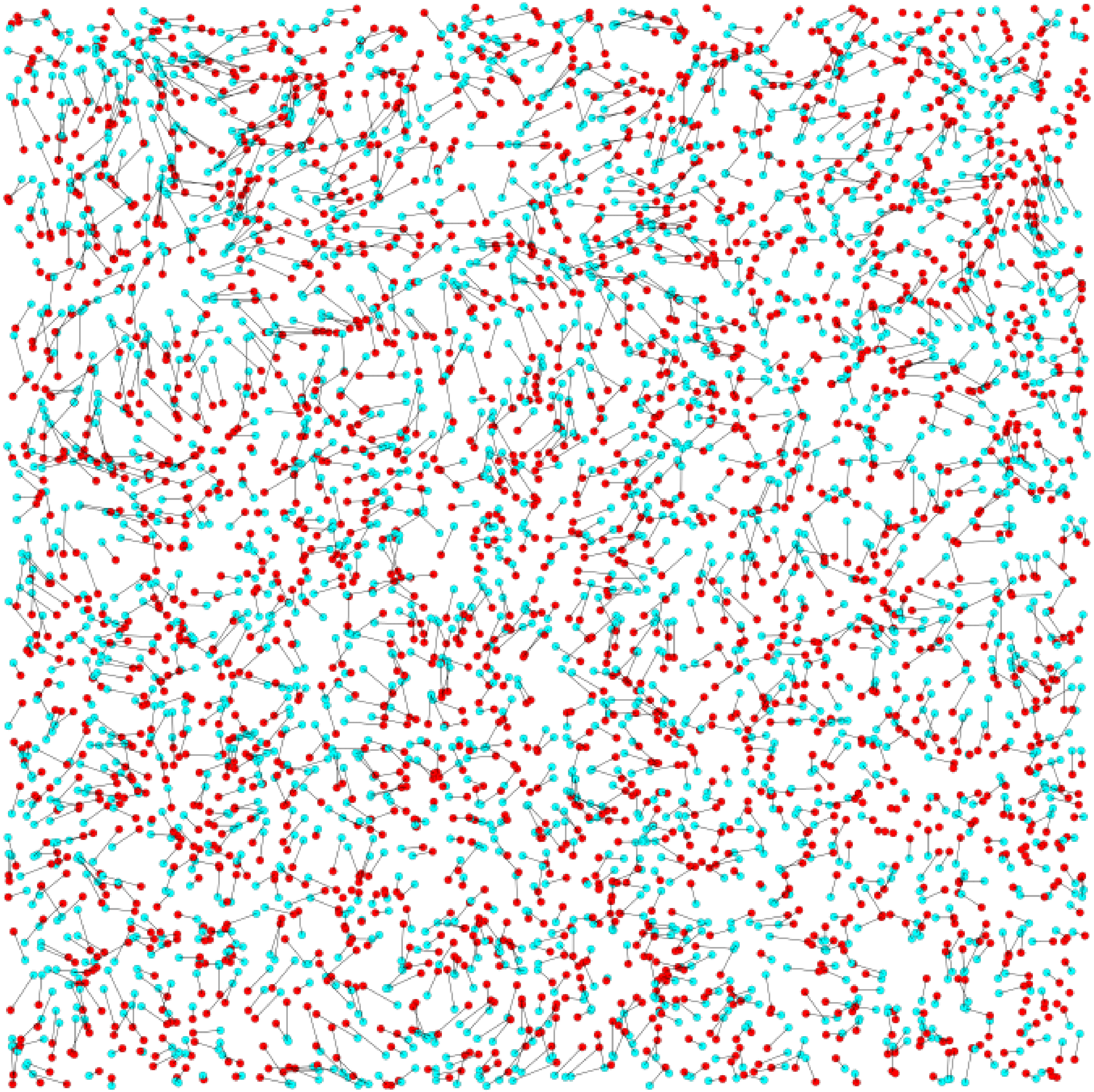
$\gamma = +\infty$ (altruistic): lexicographically minimize $\downarrow$ ordering of $|r_1 - b_1|, \ldots, |r_m - b_m|$



$\gamma = 1$



$\gamma = -\infty$
(stable)



$\gamma = \infty$
(altruistic)

Questions:

Does a γ-minimal matching exist?

Is it unique?

Is every point matched?

What ???

Allow **unmatched** points! Then **γ-minimal** means:

$\forall\ (r_1, b_1), \ldots, (r_m, b_m) \in M$

and unmatched $x_1, \ldots, x_k \in R \cup B$,

$$\left(\#\text{unmatched}\, , \sum_i |r_i - b_i|^\gamma \right)$$

is lexicographically minimized among matchings of $\{r_i, b_i, x_i\}$

(in particular, cannot have both red and blue unmatched points)

Questions:

Does a γ-minimal matching exist?

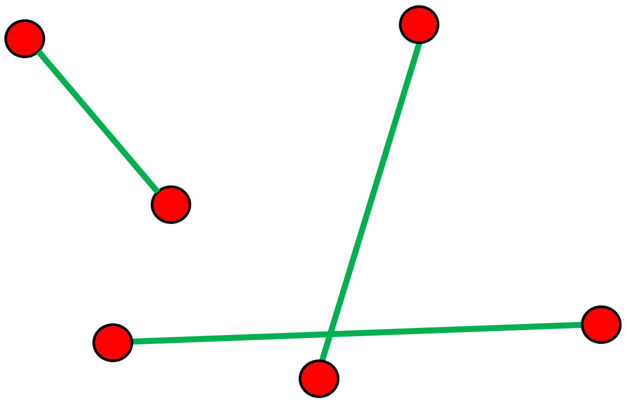
Is it unique?

Is every point matched?

Can we decide on a matching by a local algorithm?

Edge lengths?

Also: **1-colour** matching
(all definitions analogous).



$\gamma = -\infty$: fairly complete picture (especially 1-colour)

$d = 1$: fairly complete picture (especially 2-colour)

fairness makes things harder

$d \geq 2, \;\; \gamma > -\infty$: existence in some cases

Open (e.g.): existence for 2 colours, $d = 2, \;\; \gamma = 1, \infty$?

Case $\gamma = -\infty$: **stable matching**

(Holroyd, Pemantle, Peres, Schramm, 2008)

Theorem: For any $d \geq 1$, and for *R* (and *B*) independent intensity-1 Poisson processes on $\mathbb{R}^d$, a.s. there is a unique $(-\infty)$-minimal 1- (2-)colour matching, and it is perfect (i.e. no unmatched points).

In fact, it is the unique **stable matching (marriage)** in the sense of (Gale, Shapley, 1962):
each point *prefers* a partner as close as possible; matching is **unstable** if there exist a pair (of opposite colours) that both prefer each other over their current situation.

Original formulation (Gale, Shapley, 1962):
n girls, *n* boys have arbitrary preference orders over those of opposite sex.

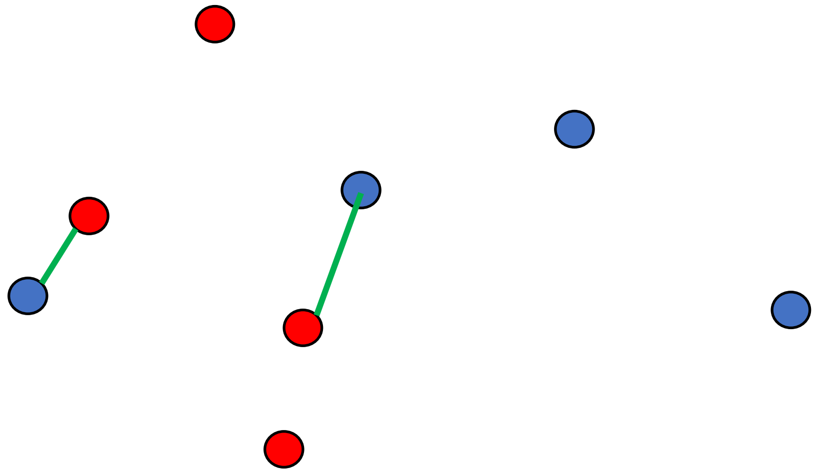
Theorem: there exists a stable set of *n* heterosexual marriages (and an algorithm…)

Not necessarily unique; may not exist in
1-colour / same-sex marriage / “roommates” version.

2012 Nobel memorial prize in Economics:
Roth and Shapley.

Simple algorithm to construct the stable matching in our setting:

match all mutually closest pairs

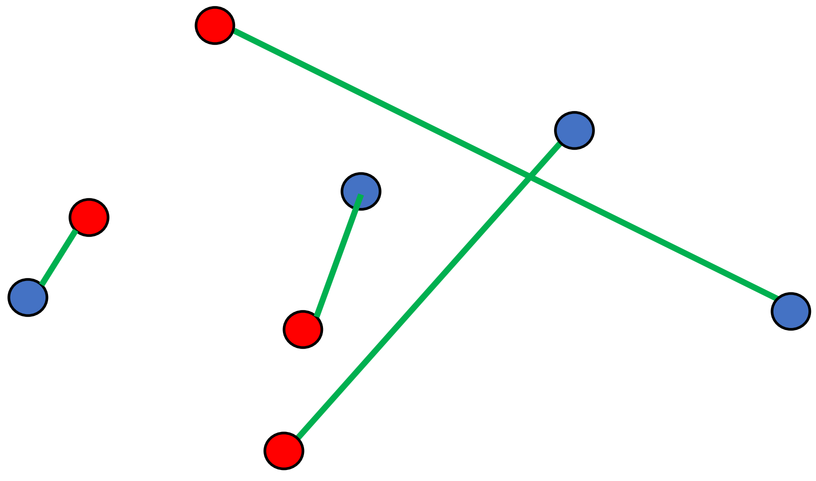


Simple algorithm to construct the stable matching in our setting:

match all mutually closest pairs

remove them

repeat for countably many steps



Theorem: (HPPS 2008) For the stable matching, the matching distance X from a typical point to its partner satisfies $\mathbb{E}\, X^{\alpha-\epsilon} < \infty$ but $\mathbb{E}\, X^{\beta} = \infty$, where:

	α	β
1-colour	d	d
2-colour, $d = 1$	1/2	1/2
2-colour, $d = 2$	0.496	1
2-colour, $d \geq 3$	$\Theta(1 / \log d)$	d

Theorem: (Eccles, Holroyd, Liggett, 2020+) For the 1-colour stable matching in $d = 1$,

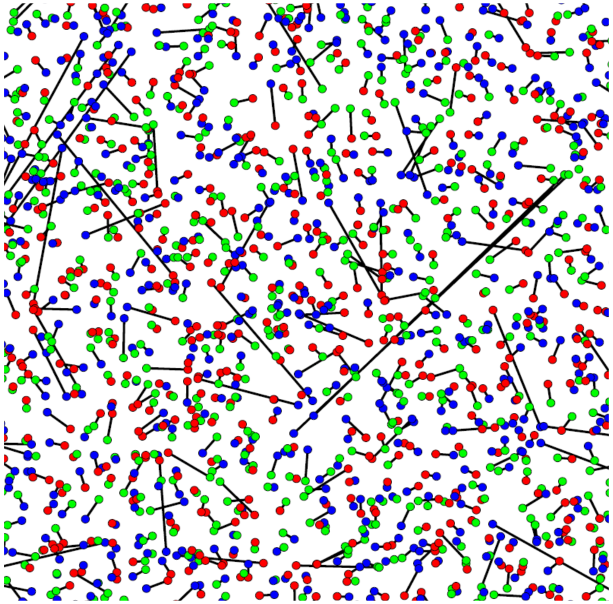
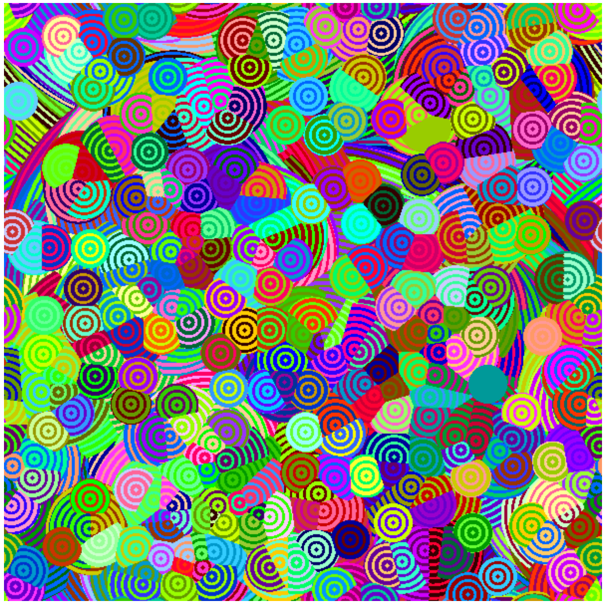
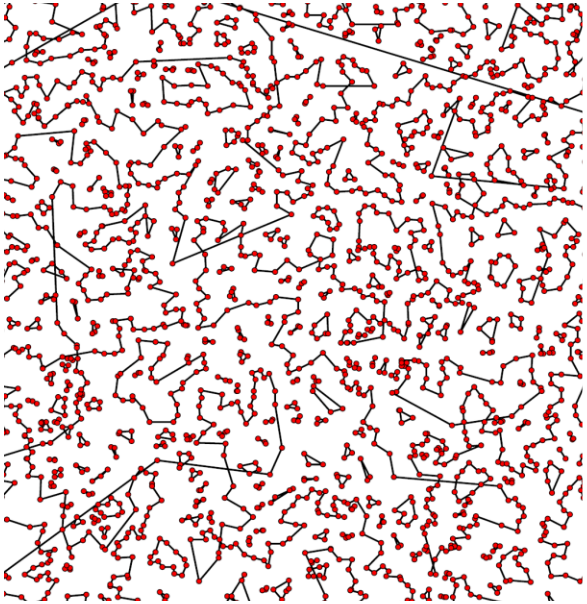
$$\mathbb{P}(X > r) \sim c/X$$

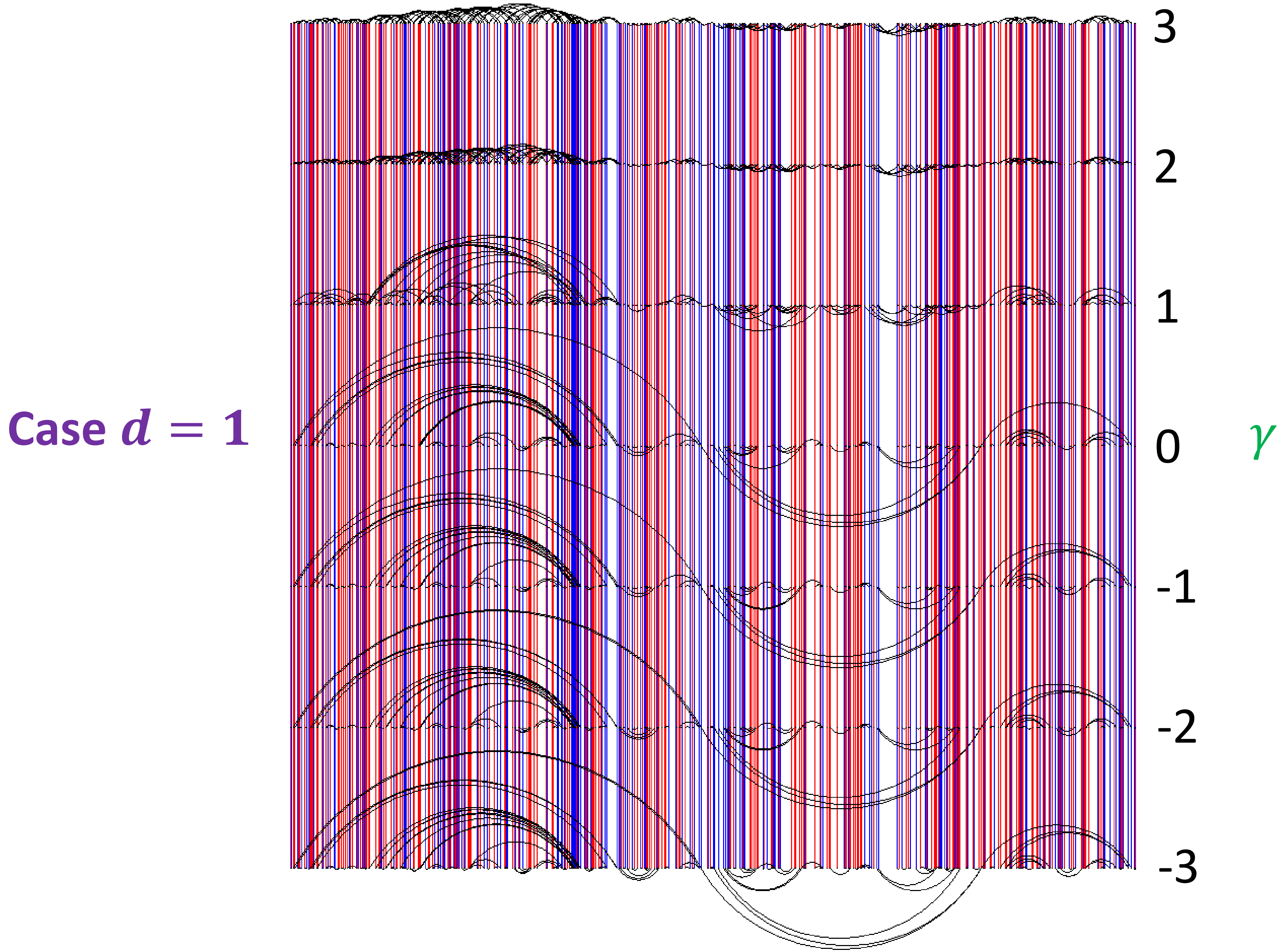
where $c = e^{2\gamma}$

Euler-Mascheroni constant

Thomas M. Liggett, 1944 - 2020

Many variants…


Case $d = 1$
3
2
1
0
-1
-2
-3
γ

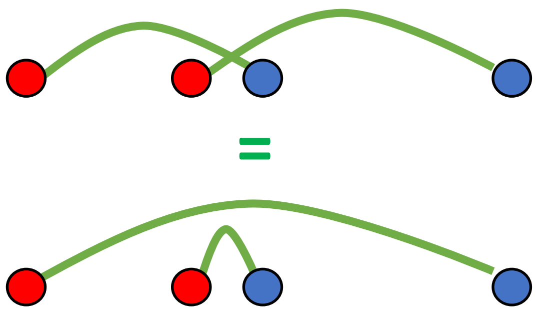
Theorem (Janson, Holroyd, Wästlund, 2020+)
For $d = 1$, any γ,
a.s. every γ-minimal 1- or 2-colour matching is perfect.
Similarly on the strip $\mathbb{R} \times [0,1]$.

Note: Much stronger conclusion than:

Theorem: for any d, any *stationary* γ-minimal matching is perfect.
Proof:
1-colour: ≤ 1 unmatched point.
2-colour: all unmatched points same colour
+ ergodic decomposition.

The picture for $d = 1$

$\gamma = 1$ is special:



Introduce:

$\gamma = 1^+$

$\gamma = 1^-$

break ties

The picture for $d = 1$: classification

Theorem (JHW 2020+) For d=1, a.s. the set of 2-colour γ-minimal matchings is:

$\gamma \geq 1^+$: countable family $\left(M^k\right)_{k\in\mathbb{Z}}$; no stationary matching

locally finite

$\gamma = 1$: uncountable; uncountably many stationary matchings

locally finite and locally infinite

locally finite locally infinite

$\gamma = 1^-$: $(M_k)_{k\in\mathbb{Z}}$, M_∞, $M_{-\infty}$;

only stationary matchings are mixtures of M_∞, $M_{-\infty}$

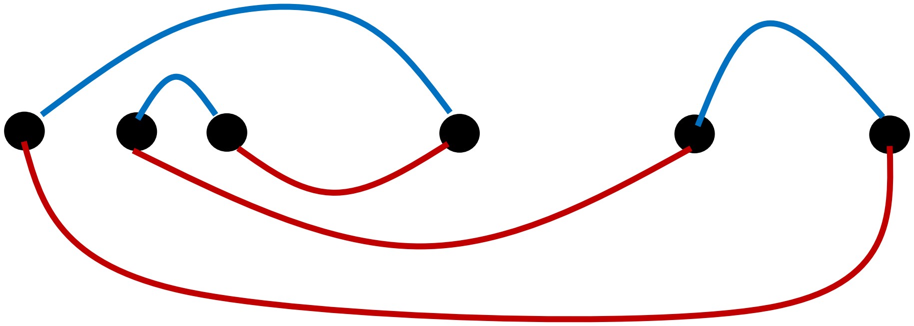
locally infinite

$\gamma < 1^-$: unique M; $\therefore$ stationary.

Theorem (JHW 2020+): *d*=1, 2-colour.

For all $\gamma \geq 1^+$, the matchings are the same.

For any $\gamma < 1^-, \gamma' \leq 1^-$,
M, M' have finite differences.



Theorem (JHW 2020+):

d=1, 2-colour, $\gamma < 1^-$ (or M_∞ or $M_{-\infty}$)

Matching distance X satisfies $\mathbb{E}\, X^\alpha < \infty$ iff $\alpha < \frac{1}{2}$.

M is a *finitary factor* of *(R,B)* with coding radius L satisfying

$$\mathbb{E}\, L^\beta < \infty$$

Can determine partner of a point by looking at points within (random) radius L

d=1, 1-color:

similar, but some proofs missing

- in particular, uniqueness for $\gamma < 1$

Higher dimensions:

Theorem (JHW 2020+):

$\exists$ a stationary (hence perfect) γ-minimal matching for:

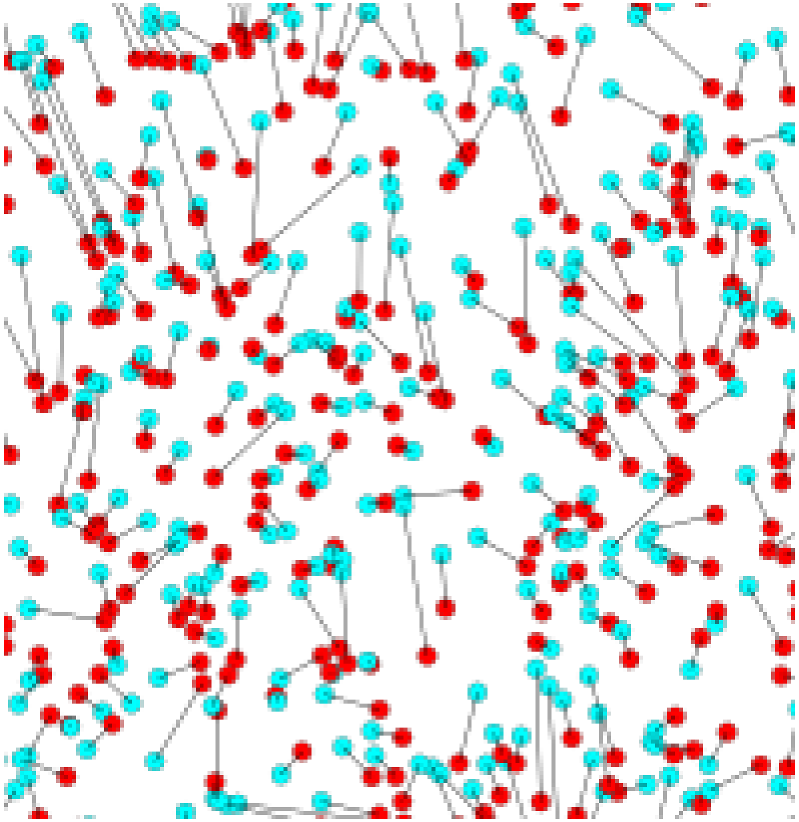
2-colour, $d \leq 2,\ \gamma < 1$;
2-colour, $d \geq 3,\ \gamma < \infty$;
1-colour, $d \geq 2,\ \gamma < \infty$.

Uniqueness open. Perfectness in general open.
Existence open for other cases (note $d \leq 2,\ \gamma \in \{1, \infty\}$!)

Is there a case with no existence?

Theorem (Holroyd 2010): $\exists$ no 1-minimal 2-colour matching on the strip.

Note: 1-minimal matchings in d=2 have *no crossings*

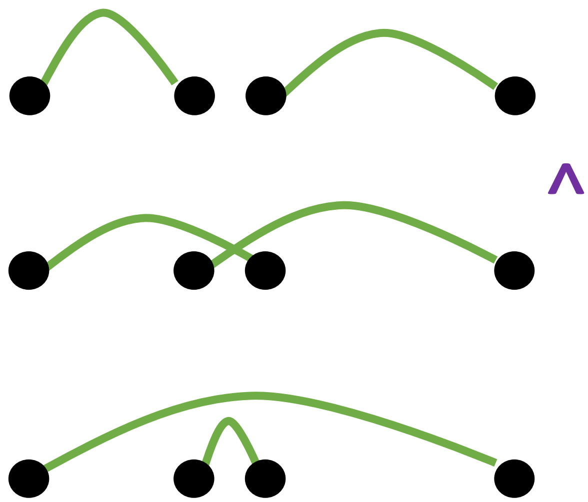


Open: is there a stationary perfect 2-colour matching of independent Poisson processes with no crossings?

Theorem (Holroyd 2010): Yes if we drop stationarity.

Proofs

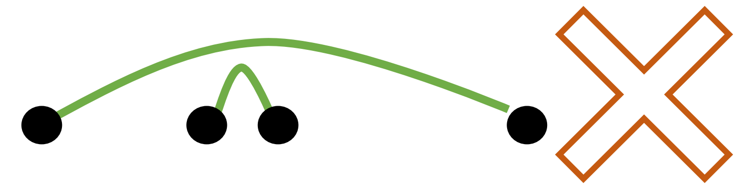
d=1 classification



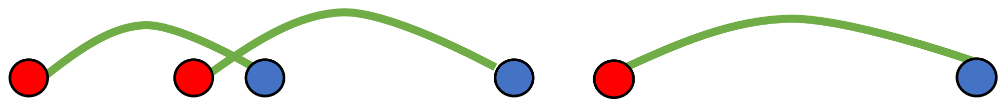
$\gamma > 1$

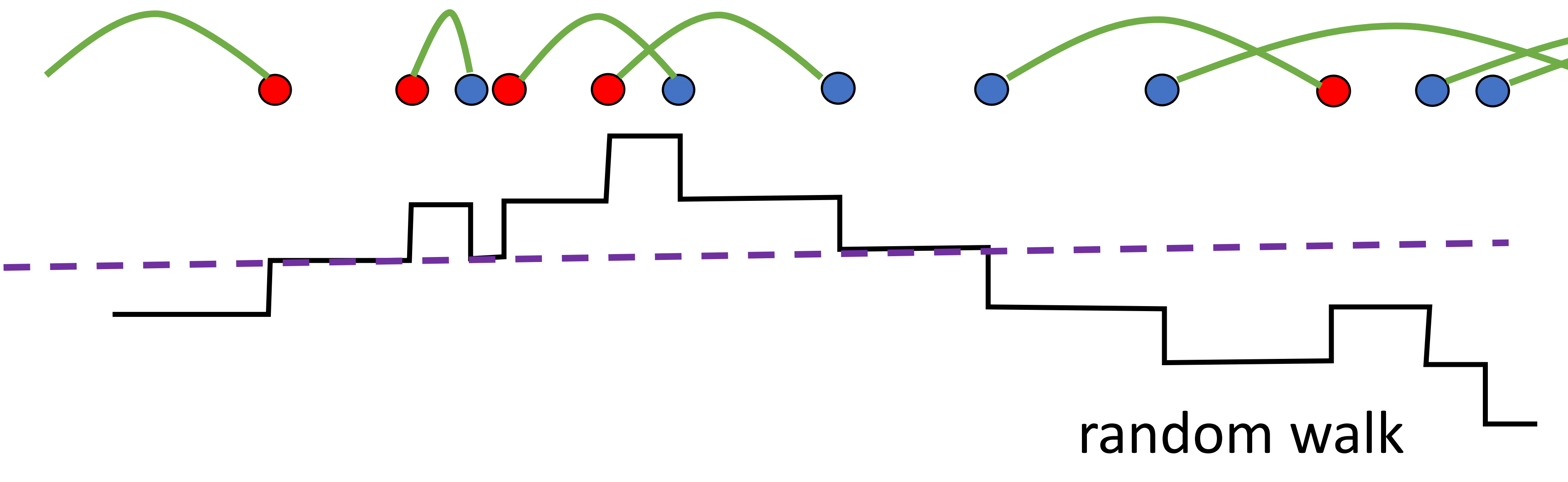
$\gamma < 1$

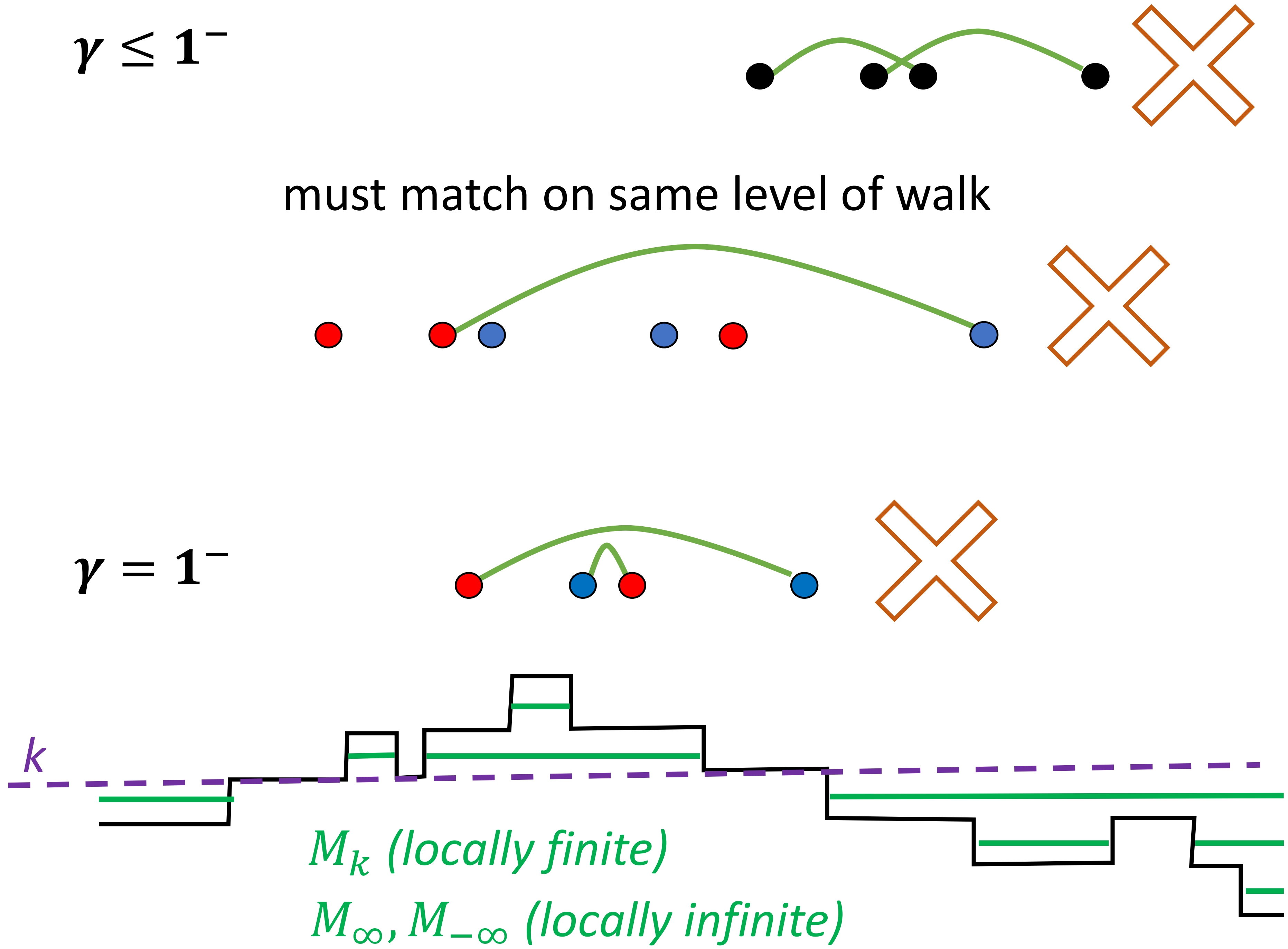
$\gamma \geq 1^{+}$



ordering of red points = ordering of blue partners





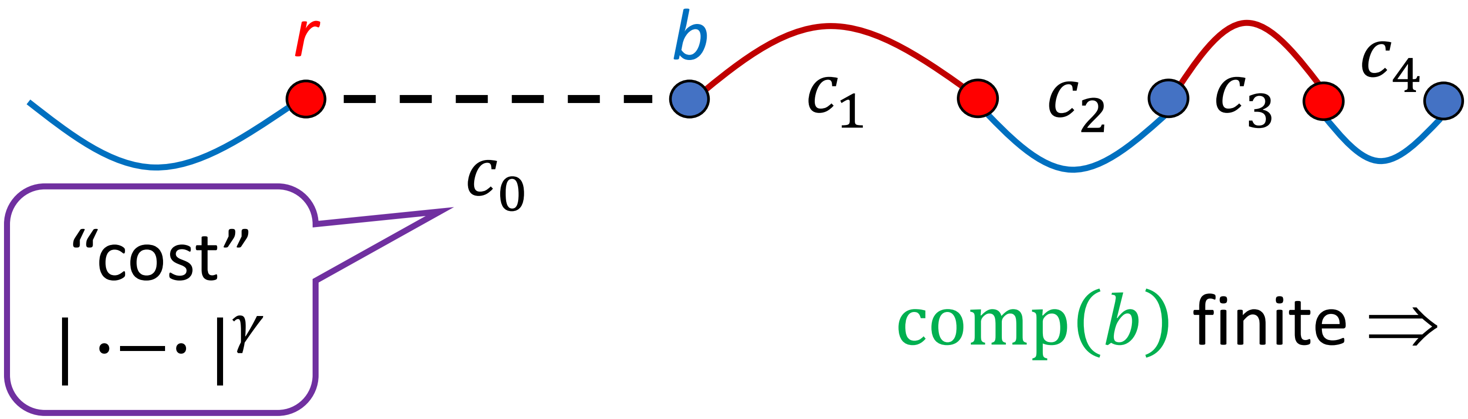


$\gamma \leq 1^-$
must match on same level of walk
$\gamma = 1^-$
k
M_k (locally finite)
$M_\infty, M_{-\infty}$ (locally infinite)

Perfectness: 2-colour, $d = 1$ (or strip), any γ.

potentially unmatched = unmatched in some γ-minimal M

a.s. exist red and blue potentially unmatched points *r*, *b*



$\mathrm{comp}(b)$ finite $\Rightarrow$ ⨳

$\mathrm{comp}(b)$ infinite:

$$c_0 \geq c_1$$
$$c_1 \geq c_2$$
$$c_0 + c_2 \geq c_1 + c_3$$
$$c_1 + c_3 \geq c_2 + c_4$$
$$\cdots$$

$\Rightarrow c_0 \geq c_i \;\; \forall i$

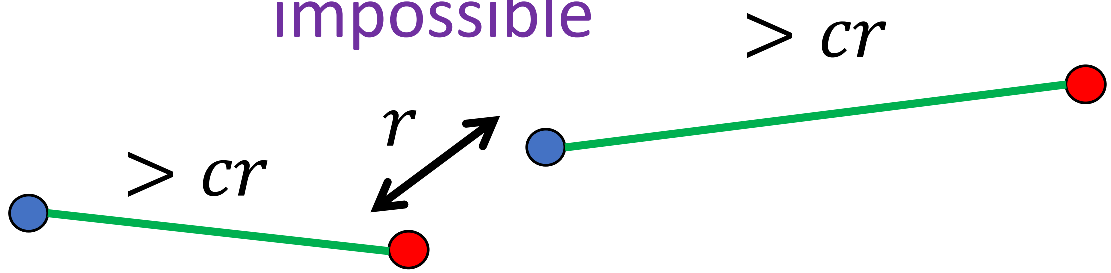
⨳ (percolation)

Existence: $\gamma < 1$, all *d*.

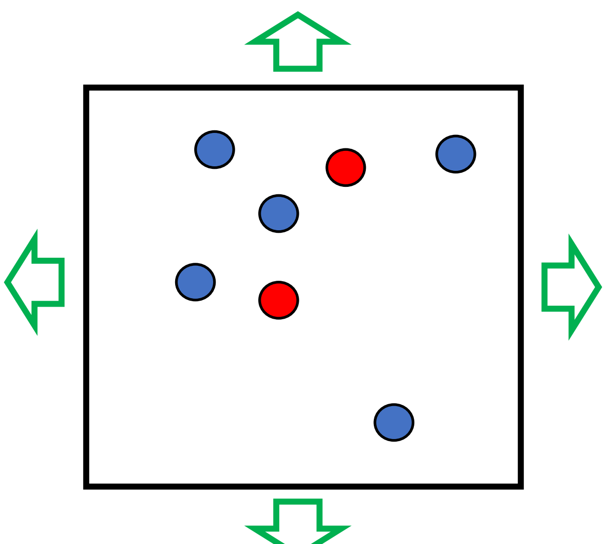
Quasi-stability:

$\exists\ c(\gamma) \geq 1$:

impossible



stationary
subsequential
limit



All unmatched
points same
colour...

Existence: provided $0 < \gamma < \infty$ and there exists a stationary (not necessarily minimal) M with finite "average cost":

$$\mathbb{E}\, X^{\gamma} < \infty$$

subsequential limit of matchings M_n with

$$\mathbb{E}\, X_n^{\gamma} \searrow \inf_{M} \mathbb{E}\, X^{\gamma}$$

is γ-minimal.

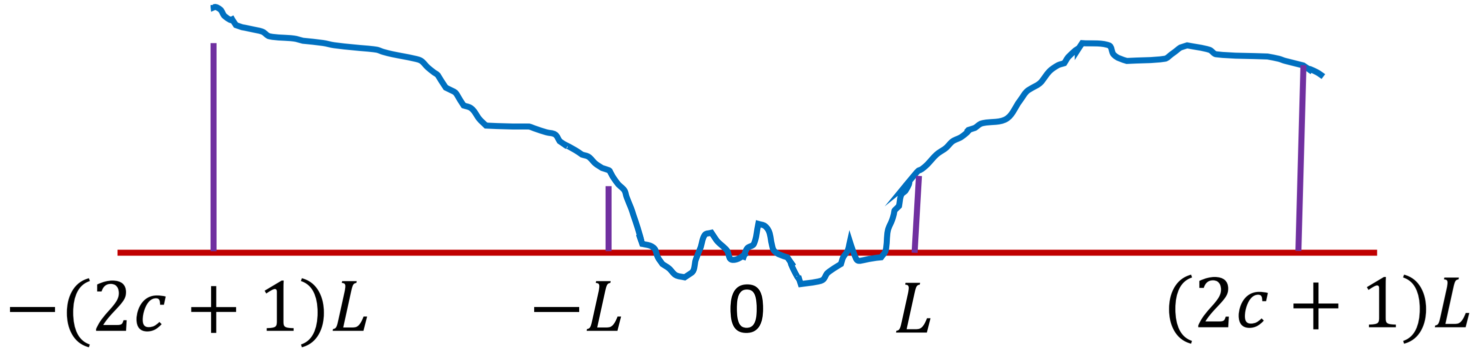
E.g. true for 2 colours:

$d \leq 2,\quad \gamma < \frac{d}{2}$

$d \geq 3,\quad \gamma < \infty$

Uniqueness, finite differences, finitariness:
$d = 1,\ \gamma < 1.$

Random walk / Brownian motion estimates $\Rightarrow$
$\exists\ L$ (random, with $\mathbb{E}\, L^{\beta} < \infty$)

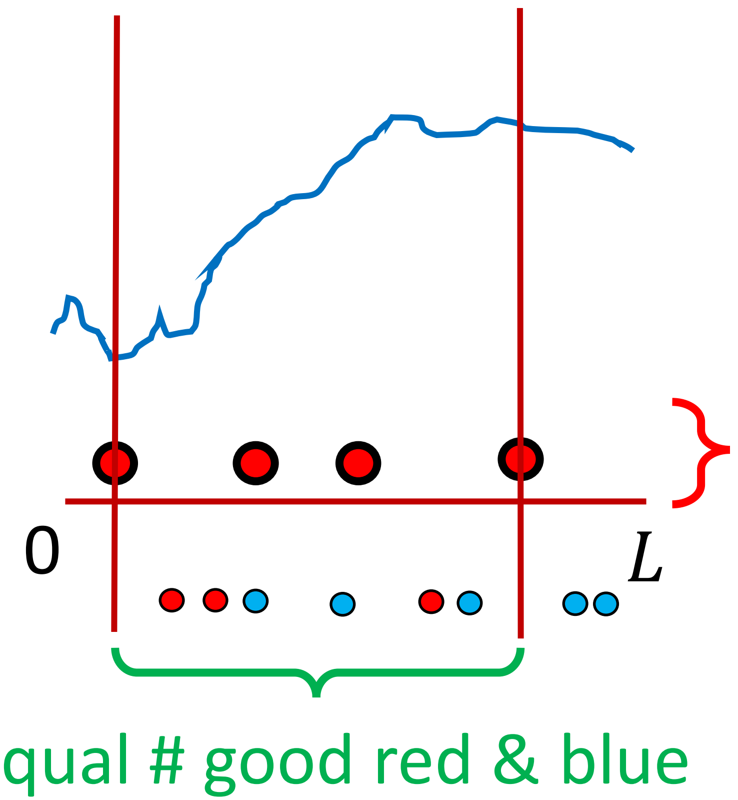


Quasi-stability + same-level matching $\Rightarrow$
matching “trapped” in $[-L, L]$

Tail bound:

$d = 1,\ \gamma < 1.$

Bad point: distance to partner > $(2c+1)L$



Bad points of ≤ 1 colour, say red.

$\mathbb{E}$ # bad red in [0,*L*] $\leq \mathbb{E}$ range of random walk in [0,*L*]

$$L\,\mathbb{P}[X > (2c+1)L] \leq C/\sqrt{L} \qquad \mathbb{E}X^{\frac{1}{2}-\epsilon} < \infty$$

Open Questions

Any setting (colours, d, γ) with non-existence ?

(e.g. 2 colours, $d = 2,\ \gamma = 1, \infty$)

Uniqueness for $d = 1,\ \gamma < 1,\ 2$ colours ?

Perfectness for $d \geq 2$?

Better tail bounds for $d = 1$?

Tail bounds, uniqueness, phase transitions … for $d \geq 2$?