The threshold bias of the Clique-factor game

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The threshold bias of the Clique-factor game

\[ \frac{n}{r} \] vertex-disjoint copies of \( K_r \)

- spamming
- every vertex contained in \( \Delta \)
- \( r \) constant, \( n \) very large
$b$ - biased Maker-Breaker games — the rules

- $(X, F)$ board & family of winning sets
- in round $i$: Maker claims 1 element of $X$, Breaker claims $b$ elements
- winner: Maker if Maker occupies all elements of some $F \in F$
  Otherwise Breaker wins
biased Maker-Breaker games — the rules

- \((X, F)\) board & family of winning sets
- in round \(i\): Maker claims \(1\) element of \(X\), Breaker claims \(b\) elements
- Winner: Maker if Maker occupies all elements of some \(F \in F\)
  Otherwise Breaker wins

Examples: \(X = E(K_n)\)
- \(F = \{\text{all copies of } \Delta\}\)
- \(F = \{\text{all PM's}\}\)
- \(F = \{\text{all spanning trees}\}\)
- \(F = \{\text{all HAM cycles}\}\)
**b - biased Maker-Breaker games**

- \((X, \mathcal{F})\) board & family of winning sets
- in round \(i\): **Maker** claims 1 element of \(X\), **Breaker** claims \(b\) elements
- **Winner:** **Maker** if Maker occupies all elements of some \(F \in \mathcal{F}\)
  
  Otherwise **Breaker** wins

**Examples:**
- \(X = E(K_n)\)
- \(\mathcal{F} = \{\text{all copies of } \Delta\}\)
- \(\mathcal{F} = \{\text{all PM's}\}\)
- \(\mathcal{F} = \{\text{all spanning trees}\}\)
- \(\mathcal{F} = \{\text{all HAM cycles}\}\)

**Threshold bias**

\[ b^* = b^*(F, n) \]

Smallest \(b\) s.t. Breaker wins

\((F \neq \emptyset \land |A| \geq 2 \forall A \in \mathcal{F})\)
Threshold biases for some natural games on $K_n$

Connectivity game \( (1+o(1)) \frac{n}{\ln n} \)

Perfect Matching game \( (1+o(1)) \frac{n}{\ln n} \)

Hamiltonicity game \( (1+o(1)) \frac{n}{\ln n} \)

\( \Delta \)-game \( \Theta(\sqrt{n}) \)

- Chvátal & Erdős '78, Beck '82, Gebauer & Szabó '09
- C&E '78, Beck '85
- Bollobás & Papaiannou '82, Krivelevich & Szabó '08, Krivelevich '11
- C&E '78, Balogh & Samotij '11
- Glažič & Srivastav '18+
Threshold biases for some natural games on $K_n$

Connectivity game
$(1+o(1)) \frac{n}{\ln n} = (1+o(1)) \frac{n}{\ln n}$

Perfect Matching game
$(1+o(1)) \frac{n}{\ln n} = (1+o(1)) \frac{n}{\ln n}$

Hamiltonicity game
$(1+o(1)) \frac{n}{\ln n} = (1+o(1)) \frac{n}{\ln n}$

$\Delta$-game
$\Theta(\sqrt{n}) \neq \Theta(n)$

$M \sim G(n,m)$
$m = \frac{1}{b+1}(\frac{n^2}{2})$
Threshold biases for some natural games on $K_n$

Connectivity game  
$(1 + o(1)) \frac{n}{\ln n} = (1 + o(1)) \frac{n}{\ln n}$

Perfect Matching game  
$(1 + o(1)) \frac{n}{\ln n} = (1 + o(1)) \frac{n}{\ln n}$

Hamiltonicity game  
$(1 + o(1)) \frac{n}{\ln n} = (1 + o(1)) \frac{n}{\ln n}$

$\Delta$-game  
$\Theta(\sqrt{n}) 
eq \Theta(n)$

Random Graph Intuition — Erdős paradigm
Theorem: $H$-game ($\nu(H) \geq 3$) 

For such $H$, exist $c, C > 0$ s.t. for all $n$:

$$c \cdot n^{\nu_2(H)} \leq b^*(H\text{-game}, n) \leq C \cdot n^{\nu_2(H)}$$

- $m_2(H) = \max \left\{ \frac{e(H')-1}{\nu(H')-2} : H' \leq H, \nu(H') \geq 3 \right\}$
Theorem: $H$-game ($v(H) \geq 3$)

\[
\forall \text{ such } H \exists c, C > 0 \text{ s.t. for all } n : \\
c n^{\sqrt[m_2(H)]{m_2(H)}} \leq b^*(H\text{-game}, n) \leq C \cdot n^{\sqrt[m_2(H)]{m_2(H)}}
\]

- $m_2(H) = \max\left\{ \frac{e(H'')-1}{v(H')-2} : H' \leq H, v(H') \geq 3 \right\}$

Winning strategy for Breaker
Theorem: \( H\)-game \((v(H) \geq 3)\)

\[
\forall \text{ such } H \exists c, C > 0 \text{ s.t. for all } n:
\]

\[
c \cdot n^{\frac{1}{m_2(H)}} \leq b^*(H\text{-game}, n) \leq C \cdot n^{\frac{1}{m_2(H)}}
\]

- \( m_2(H) = \max \left\{ \frac{e(H') - 1}{v(H') - 2} : H' \leq H, v(H') \geq 3 \right\} \)
- Winning strategy for Maker → random strategy
- \( G(n, m) \) robustly contains \( H \):
  \[
  \forall F \subseteq G(n, m) \text{ s.t. } |F| \leq \varepsilon \cdot m : H \subseteq G(n, m) \setminus F
  \]
Bednarska & Łuczak 2000

Theorem: H-game \( (v(H) \geq 3) \)

\[ \forall \text{such } H \exists C, \epsilon \geq 0 \text{ s.t. for all } n : \]

\[ c \cdot n^{m_2(H)} \leq b^*(H, n) \leq C \cdot n^{m_2(H)} \]

- \( m_2(H) = \max \left\{ \frac{e(H')-1}{v(H')-2} : H' \leq H, v(H') \geq 3 \right\} \)

- Winning strategy for Maker \( \rightarrow \) random strategy

- \( G(n,m) \) robustly contains \( H \):

\[ \forall F \subseteq G(n,m) \text{ s.t. } |F| \leq \epsilon \cdot m : H \subseteq G(n,m) \setminus F \]

- Problem: if winning structure is spanning
For every fixed strategy of Breaker Maker draws a random graph $\Gamma \sim G(n, p)$.

If $c \cdot \frac{\ln n}{n} \leq p \leq \frac{1}{b}$ then w.h.p. $\Gamma$ is such that Maker can claim a subgraph $M \subseteq \Gamma$ s.t. $\Delta(M) \geq (1-\epsilon)p \cdot n$

$\rightarrow M = G(np) \setminus B$
Local Resilience

For every fixed strategy of Breaker Maker draws a random graph \( G \sim G(n, p) \).

If \( C \cdot \frac{\ln n}{n} \leq p \leq c \cdot \frac{1}{b} \) then w.h.p. \( \Gamma \) is such that Maker can claim a subgraph \( M \subseteq \Gamma \) s.t. \( \delta(M) \geq (1-\varepsilon)p \cdot n \)

\[ M = G(np) \setminus B \]

Example: PM, HAM, Connectivity for \( b = c \cdot \frac{n}{\ln n} \)
"Local Resilience"

For every fixed strategy of Breaker Maker draws a random graph \( \Gamma \sim G(n,p) \).

If \( c \cdot \frac{\ln n}{n} \leq p \leq c \cdot \frac{1}{b} \) then w.h.p. \( \Gamma \) is such that Maker can claim a subgraph \( M \subseteq \Gamma \) s.t. \( \delta(M) \geq (1-\varepsilon)p \cdot n \).

\[ M = G(np) \setminus B \]

- Example: PM, HAM, Connectivity for \( b = c \cdot \frac{n}{\ln n} \)
- Problem: If every vertex is in \( \Delta \) in winning structure
Theorem

Let $n, \varepsilon$ be given. Let $p \geq 10^8 \varepsilon^{-2} n^{-1/2}$ and $p \leq 10^{-24} \varepsilon^6 n^{-1}$.

In the $b$-biased game on $E(K_n)$, Maker can claim a subgraph $M$ of $G \sim G(n,p)$ such that for all $v \in V$:

- $d_M(v) \geq (1-\varepsilon)np$
- $e(M[N_G(v)]) \geq (1-\varepsilon)\frac{p^3n^2}{2}$.
Corollary

For all $D$ there exists $c$ such that for all $n$ & for all $H$ on $n$ vertices with $\Delta(H) \leq D$:

Maker wins the $b$-biased $H$-game on $K_n$ if

$$b \leq c \cdot \left(\frac{n}{\log n}\right)^{1/2D}$$

Proof:

- $M \subseteq G(n, p)$ s.t. $\delta(M) \geq (1 - \varepsilon)np$ & $e(M \cap N_c(H)) \geq (1 - 3\varepsilon)\frac{n^2p^3}{2}$

- "Sparse Blow-up lemma" for $G(n, p)$
Corollary

For all $D$ there exists $c$ such that for all $n$ & for all $H$, $H$ on $n$ vertices with $\Delta(H) \leq D$:

**Maker** wins the $b$-biased $H$-game on $K_n$ if

$$b \leq c \cdot \left(\frac{n}{\log n}\right)^{1/\sqrt{D}}$$

Remarks: (1) universal result

(2) more general version for "bounded degree & bounded degeneracy"
**Corollary**

For all $D$ there exists $c$ such that for all $n \&$ for all $H$ on $n$ vertices with $\Delta(H) \leq D$:

Maker wins the $b$-biased $H$-game on $K_n$ if

$$b \leq c \cdot \left(\frac{n}{\log n}\right)^{\frac{1}{6}}$$

Example: If $b \leq c \cdot \left(\frac{n}{\log n}\right)^{\frac{1}{2}}$ then Maker wins $\Delta$-factor game.

Remember: Breaker wins $\Delta$-game for $b \geq 2\sqrt{n}$. 
Corollary

For all $D$ there exists $c$ such that for all $n$ & for all $H$ on $n$ vertices with $\Delta(H) \leq D$:

**Maker** wins the $b$-biased $H$-game on $K_n$ if

$$b \leq c \cdot \left( \frac{n}{\log n} \right)^{1/3}$$

**Example 2:** If $b \leq c \cdot \left( \frac{n}{\log n} \right)^{1/3}$ then **Maker** wins $K_4$-factor game.

They prove: **Breaker** wins $K_4$-factor game for $b \geq Cn^{1/3}$.
Corollary

For all $D$ there exists $c$ such that for all $n$ and for all $H$ on $n$ vertices with $\Delta(H) \leq D$:

**Maker** wins the $b$-biased $H$-game on $K_n$ if

$$b \leq c \cdot \left( \frac{n}{\log n} \right)^{1/D}$$

$\blacktriangleright$ $b^*(K_{D+1} \text{-factor, } n) = n^{1/D} + o(1)$ for $D = 2, 3$

$\blacktriangleright$ "$\frac{1}{D}$ probably not correct for larger $D$"
Theorem
For all $D \geq 3$ exists $c$ such that for every $n \in (D+1) \mathbb{Z}$ in the $b$-biased MB-game on $K_n$

Maker wins the $K_{D+1}$-factor game if $b \leq c \cdot n^{\frac{2}{5D+3}}$
For all $D \geq 3$ there exist $c, C$ such that for every $n \in (D+1) \mathbb{Z}$ in the $b$-biased MB-game on $K_n$

(a) Maker wins the $K_{D+1}$-factor game if $b \leq c \cdot n^{\frac{2}{D+3}}$

(b) Breaker wins the $K_{D+1}$-factor game if $b \geq C \cdot n^{\frac{2}{D+3}}$
Threshold bias $b^* \left( \frac{c}{f, n} \right)$

Connectivity, PM, HAM

Δ-game

Δ-factor game

$K_{D+1}$ - factor game

Clever Game

$\frac{2}{D+3}$

$\Theta(\sqrt{n}) \neq \Theta(n)$

$\frac{n^{\frac{1}{2} + o(1)}}{D+1}$

our result

Random Game

$\frac{n}{\ln n}$

$\frac{n^{\frac{2}{3} + o(1)}}{2(D+1) + o(1)}$

Johansson, Kahn, Vu 2008
Why \( n \geq \frac{2}{D+3} \) ?

\[ \text{every } v \text{ is in a copy of } K_{D+1} \]

\[ \text{For which } b \text{ can } \text{Breaker achieve} \]

\[ \exists v \text{ that is not in a copy of } K_{D+1} \]
Why \( n^{\frac{2}{D+3}} \)?

For which \( b \) can Breaker achieve "\( \exists v \) that is not in a copy of \( K_{D+1} \)"?

- Fix \( v \)
- always play \( b \) edges at \( v \)
- prevent \( K_D \) in \( N_M(v) \)

\[ \Rightarrow \] use Bednarska-Łuczak

\[ \text{works if } b \geq C \cdot \left( \frac{n}{b} \right)^{\frac{1}{m_2(K_D)}} \iff b \geq C' n^{\frac{1}{m_2(K_D)+1}} = C' n^{\frac{2}{D+3}} \]
Maker's strategy for $b \leq cn^{\frac{2}{D+3}}$

Build a "pseudo-random" graph

(P1) $\delta(M) \geq \frac{np}{2}$ & for all $X,Y$ $|X| \geq \frac{\log n}{p}$, $|Y| \geq \alpha n$ $\exists v \in X$ s.t. $|N_H(v) \cap Y| \geq \frac{|Y|p}{2}$

(P2) $\forall v \forall X \subseteq N(v)$ $\exists |X| \geq \alpha np : K_D \subseteq M[X]$}

(P3) $\forall V_1, \ldots, V_{D+2}$, $|V_i| \geq n^{\frac{1}{2b}}$ $\exists K_{D+2}'$
Maker's strategy for $b \leq cn^{2/D+3}$

Build a "pseudo-random" graph

Apply ABKNP 2017

$\triangleright M_i \subseteq G(n, p)$ s.t. $\delta(M_i) \geq (1-\varepsilon)n^p$ \quad $\Delta e_{M_i}(N_G(v)) \geq (1-\varepsilon)p^3 n^2$

$\varepsilon$ - small constant, $\rho = c_\varepsilon b^{-\varepsilon} = kn^{-D+3}$
Maker's strategy for \( b \leq cn^{\frac{2D}{D+3}} \)

Build a "pseudo-random" graph

Apply ABKNP 2017

- \( M_1 \subseteq G(n_{1}, p) \) s.t. \( \delta(M_1) \geq (1-\epsilon)n_{1} \) \& \( e_{M_1}(N_{G}(v)) \geq (1-\epsilon)^{\frac{p_{2}^{3}}{2} n_{1}^{2}} \)

\( \epsilon \) - small constant \quad \rho = c_{\epsilon} b^{-1} = Kn^{-\frac{2}{D+3}}

- \( M_2 \subseteq G(n_{2}, q) \) s.t. \( \delta(M_2) \geq (1-\gamma)n_{2} \) \& \( e_{M_2}(N_{G}(v)) \geq (1-\gamma)^{\frac{q_{2}^{3}}{2} n_{2}^{2}} \)

\( -\frac{1-\beta}{m_{2}(H)} \ll q \ll n^{-\frac{2}{D+3}} \) \quad \gamma = n^{-\frac{3\beta}{6}}
Maker's strategy for $b \leq cn^{2/3}$

Build a "pseudo-random" graph

(P1) $\delta(M) \geq \frac{np}{2}$ & for all $X, Y, |X| \geq \frac{\log n}{p}$, $|Y| \geq \alpha n$  \exists v \in X \text{ s.t.}  
\[ |N_H(v) \cap Y| \geq \frac{|Y|}{2} \]

(P2) $\forall v \forall X \subseteq N(v) \exists \bar{\omega} \text{ s.t.} |X| \geq \alpha np : K_\bar{\omega} \subseteq M \setminus X$

(P3) $\forall V_1, \ldots, V_D, |V_i| \geq n^{1/3}$  \exists $K_D \subseteq V_i$
Maker's strategy for \( b \leq c n^{\frac{2}{D+3}} \)

use (P3) to find chains:

\[ \text{length } n^{1-\beta} \]

\[ K_{D+1} \]
Maker's strategy for $b \leq cn^{\frac{2}{3D+3}}$

use (P3) to find chains:

$D_4$
Maker's strategy for $b \leq cn^{3/5 + 3}$

use (P3) to find chains:

\[
\begin{align*}
\text{length } n^{1-\beta} & \left\{ \begin{array}{c} n^{\beta} \text{ many} \\
\text{length } n^{1-\beta-\delta} & \left\{ \begin{array}{c} n^{\beta+\delta} \text{ many} \\
\end{array} \\
\end{array} \\
\end{align*}
\]

\[
\vdots
\]

\[
\begin{align*}
\text{Co length 0 (vertices)} & \left\{ \begin{array}{c} n^m \text{ many} \\
\end{array} \\
\end{align*}
\]
Maker's strategy for $b \leq cn^{\frac{2}{3n+3}}$

Use (P3) to find chains:

- Length $n^{1-\beta}$
- No many

For any such that there exist contains $K_{D+1}$-factor.

$C_0$ (vertices)
Maker's strategy for $b \leq cn^{\frac{2}{n}+3}$

Chain

$C_0$

A

Absorbing property

$\forall W \subseteq C_0 \exists K_{d+1}$-factor of $A \setminus W$

Rest of graph

Cover with $K_{d+1}$'s (somewhat greedily)

Using $v$'s from $C_0$

$\rightarrow$ use (P1) & (P2)
Conjecture: for every \( H \) on \( \leq n \) vertices with \( \Delta(H) \leq D \):

\[
b^*(H \text{-game}, n) \geq c \cdot n^{\frac{2D+3}{D+3}}
\]

\( K_{D+1} \)-factor is hardest for Maker

Problem: Find explicit winning strategies for Maker in the \( H \)-game or the Hamiltonicity game.

Problem: What is the threshold bias for the \((2:b)\)-game?
• Conjecture: for every \( H \) on \( \leq n \) vertices with \( \Delta(H) \leq D \):
  \[ b^*(H, n) \geq c \cdot n^{\frac{2D+3}{3D+3}} \]
  \( \rightarrow K_{D+1} \)-factor is hardest for Maker

• Problem: Find explicit winning strategies for Maker in the \( H \)-game or the Hamiltonicity game.

• Problem: What is the threshold bias for the \((2:b)\)-game ?

Thank you!