How does the chromatic number of a random graph vary?

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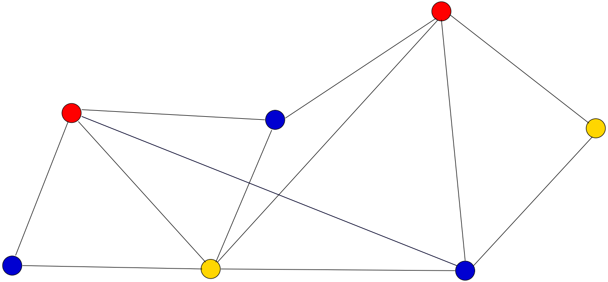
Joint work with Oliver Riordan.

Colouring of G: Colour vertices so that neighbours get different colours



Chromatic number $\chi(G)$: Minimum number of colours we need

Pick an *n*-**vertex graph** uniformly at random. Pick **another one**.

Does it have the same chromatic number?

If not, how different are their chromatic numbers likely to be?

Chromatic number of random graphs

$G_{n,\frac{1}{2}}$ = choose a graph on n labelled vertices uniformly at random

What can we say about $\chi(G_{n,p})$?

Value?

Upper and lower bounds?

Concentration?

How much does $\chi(G_{n,p})$ vary?

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Value?

Upper and lower bounds?

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How much does $\chi(G_{n,p})$ vary?

Bollobás 1987: $$\chi(G_{n,\frac{1}{2}}) \sim \frac{n}{2\log_2 n} \text{ whp.}$$

Improvements: McDiarmid '90, Panagiotou & Steger '09, Fountoulakis, Kang & McDiarmid '10.

H. 2016:

$$\chi\left(G_{n,\frac{1}{2}}\right) = \frac{n}{2\log_2 n - 2\log_2\log_2 n - 2} + o\left(\frac{n}{\log^2 n}\right) \text{ whp.}$$

Explicit interval of length $o\left(\frac{n}{\log^2 n}\right)$ which contains $\chi(G_{n,\frac{1}{2}})$ whp.

Concentration?

Shamir, Spencer 1987: For *any* function $p = p(n)$, $\chi(G_{n,p})$ is whp contained in a sequence of intervals of length about $\sqrt{n}$.

$p = 1 - \frac{1}{10n}$: not concentrated on fewer than $\Theta(\sqrt{n})$ values

$p \leqslant \frac{1}{2}$: slight improvement to $\frac{\sqrt{n}}{\log n}$ (Alon)

$p < n^{-\frac{1}{2}-\varepsilon}$: 2 values ('two-point concentration')
(Alon, Krivelevich 97, Łuczak 91)

$\rightarrow \chi(G_{n,p})$ behaves almost deterministically

The opposite question

Question (Bollobás)

Can we show that $\chi(G_{n,\frac{1}{2}})$ is not concentrated on 100 consecutive values?

Upper bound: $\dfrac{\sqrt{n}}{\log n}$ (Alon)

Theorem (H. 2019; H., Riordan 2021)

Let $\varepsilon > 0$, and let $[s_n, t_n]$ be a sequence of intervals such that $\chi(G_{n,1/2}) \in [s_n, t_n]$ whp. Then there are infinitely many values n such that

$$t_n - s_n \geqslant n^{1/2-\varepsilon}.$$

Proof ingredients

Ingredient 1: A (weak) concentration type result

$$|\chi(G_{n,1/2}) - f(n)| \leqslant \Delta(n) \text{ whp}$$

where $f(n)$ is some function with slope

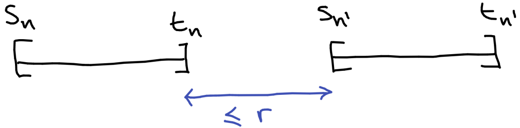
$$\frac{\mathrm{d}}{\mathrm{d}n} f(n) > \frac{1}{\alpha} + \delta.$$

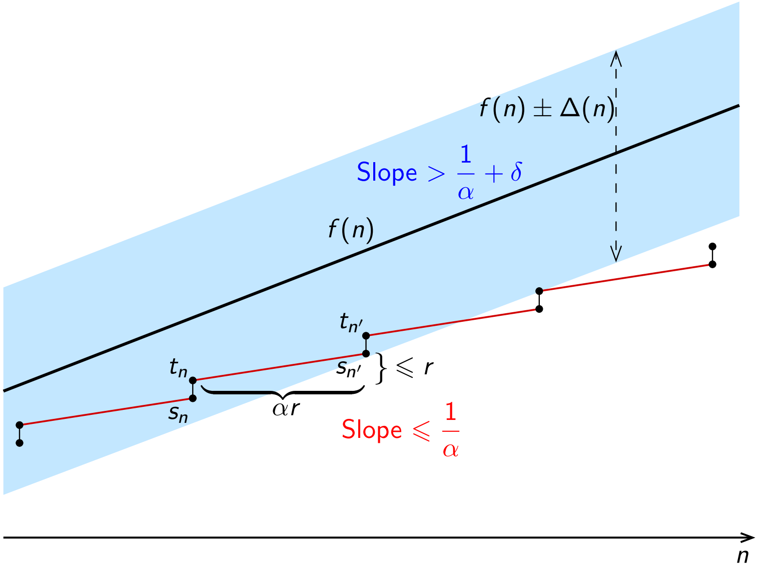
(Will specify $\alpha = \alpha(n)$ later.)

Ingredient 2: A coupling result

Couple $G_{n,1/2}$ and $G_{n',1/2}$ with $n' = n + \alpha r$ (same α as above) so that

$$\mathbb{P}\Big(\chi(G_{n',1/2}) \leqslant \chi(G_{n,1/2}) + r\Big) > \frac{1}{4}.$$



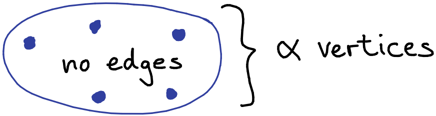


If all intervals **short**: Contradiction!

So there is at least one long interval. (Length $\approx \alpha\delta r$)

What's $\alpha(n)$?

Independence number $\alpha(G)$: Size of the largest independent vertex set (= set without edges).



For most n: $\alpha(G_{n,\frac{1}{2}}) = \alpha(n) \approx 2\log_2 n$ whp.

What does this have to do with colourings?

Every colour class is an independent set, so if there are n vertices,

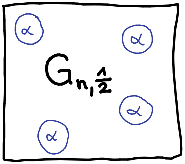
$$\chi(G) \geqslant \frac{n}{\alpha(G)}$$

In fact: $$\chi(G_{n,\frac{1}{2}}) = \frac{n}{\alpha - O(1)}$$

Intuition: An optimal colouring of $G_{n,\frac{1}{2}}$ contains all or almost all independent α-sets as colour classes.

$\chi(G_{n,\frac{1}{2}})$ **should vary at least as much as** X_α**.**

$X_\alpha = \#$ independent α-sets



$X_\alpha \underset{\text{roughly}}{\sim} \mathrm{Poi}_\mu \quad \to$ varies by $\pm\sqrt{\mu}$

where $\mu = n^\rho, \quad 0 \leqslant \rho(n) \leqslant 1.$

Ingredient 1: The (weak) concentration type result

Want:

$$\chi(G_{n,\frac{1}{2}}) = f(n) \pm \Delta(n)$$

$$\frac{\mathrm{d}f}{\mathrm{d}n} \geqslant \frac{1}{\alpha} + \delta$$

H. 2016:

$$\chi\left(G_{n,\frac{1}{2}}\right) = \underbrace{\frac{n}{2\log_2 n - 2\log_2\log_2 n - 2}}_{f(n)} + \underbrace{o\left(\frac{n}{\log^2 n}\right)}_{\Delta(n)} \text{ whp.}$$

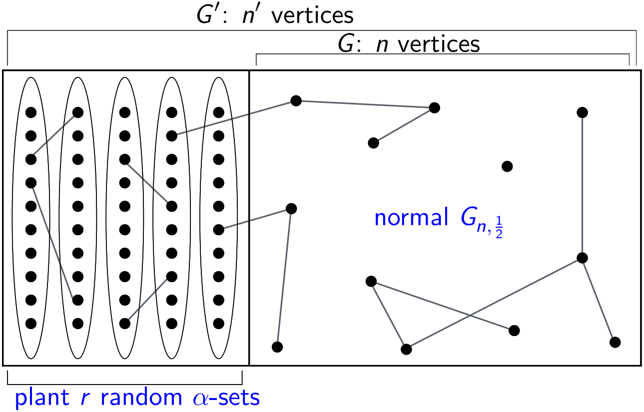
then (unless μ_α is very close to n)

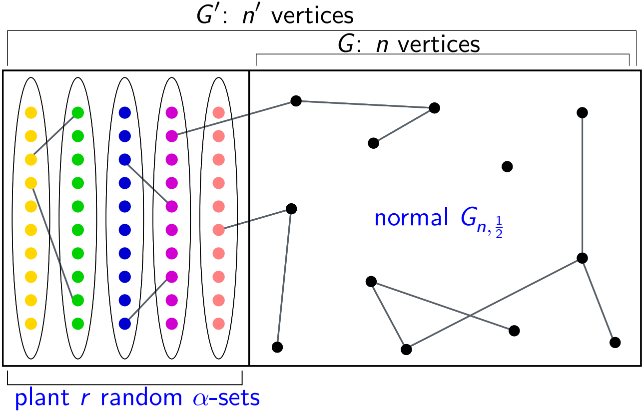
$$\frac{\mathrm{d}f}{\mathrm{d}n} \geqslant \frac{1}{\alpha} + \underbrace{\Theta\left(\frac{1}{\log^2 n}\right)}_{\delta(n)}$$

Ingredient 2: The coupling result

Want: Coupling of $G_{n,1/2}$ and $G_{n',1/2}$ with $n' = n + \alpha r$ so that

$$\mathbb{P}\Big(\chi(G_{n',1/2}) \leqslant \chi(G_{n,1/2}) + r\Big) > \frac{1}{4}.$$





- Inner random graph: $G \sim G_{n,\frac{1}{2}}$
- Also clear:

$$\chi(G') \leqslant \chi(G) + r$$

- Need to prove: G' similar to $G_{n',\frac{1}{2}}$ if r is not too big

Key Lemma

Planted model $G^{\mathrm{pl}}_{n,\frac{1}{2}}$: Plant an independent α-set uniformly at random, and include all other edges independently with probability $\frac{1}{2}$.

d_{TV}: Total variation distance

Key Lemma

$$d_{\mathrm{TV}}\left(G_{n,\frac{1}{2}}, G^{\mathrm{pl}}_{n,\frac{1}{2}}\right) = O\left(\frac{1}{\sqrt{\mu}}\right),$$

where $\mu = \mathbb{E}[X_\alpha]$.

This means: $G_{n,\frac{1}{2}}$ and $G^{\mathrm{pl}}_{n,\frac{1}{2}}$ can be coupled so that they agree with probability

$$1 - O\left(\frac{1}{\sqrt{\mu}}\right).$$

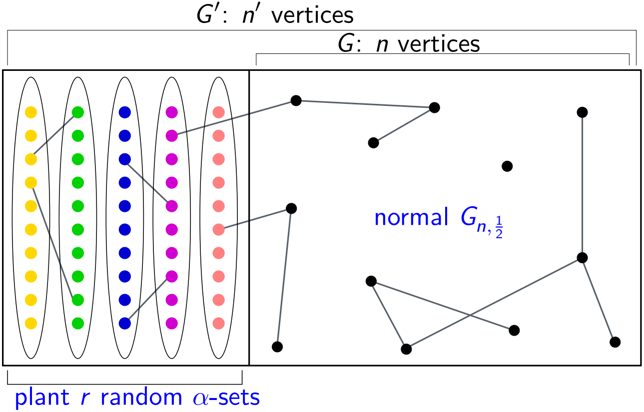
Key Lemma

$$d_{\mathrm{TV}}\left(G_{n,\frac{1}{2}}, G^{\mathrm{pl}}_{n,\frac{1}{2}}\right) = O\left(\frac{1}{\sqrt{\mu}}\right),$$

where $\mu = \mathbb{E}[X_\alpha]$.

Proof:

$$\begin{aligned}
d_{\mathrm{TV}}\left(G_{n,\frac{1}{2}}, G^{\mathrm{pl}}_{n,\frac{1}{2}}\right) &= \frac{1}{2}\sum_G \left|\mathbb{P}\left(G^{\mathrm{pl}}_{n,\frac{1}{2}} = G\right) - \mathbb{P}\left(G_{n,\frac{1}{2}} = G\right)\right| \\
&= \frac{1}{2}\sum_G \left|\frac{X_\alpha(G)}{\binom{n}{\alpha}}\left(\frac{1}{2}\right)^{\binom{n}{2}-\binom{\alpha}{2}} - \left(\frac{1}{2}\right)^{\binom{n}{2}}\right| \\
&= \frac{1}{2}\sum_G \left(\frac{1}{2}\right)^{\binom{n}{2}} \frac{\left|X_\alpha(G) - \binom{n}{\alpha}\left(\frac{1}{2}\right)^{\binom{\alpha}{2}}\right|}{\binom{n}{\alpha}\left(\frac{1}{2}\right)^{\binom{\alpha}{2}}} \\
&= \frac{1}{2}\mathbb{E}\left[\frac{|X_\alpha - \mu|}{\mu}\right] = O\left(\frac{1}{\sqrt{\mu}}\right)
\end{aligned}$$



If $r = o\left(\sqrt{\mu}\right)$, then

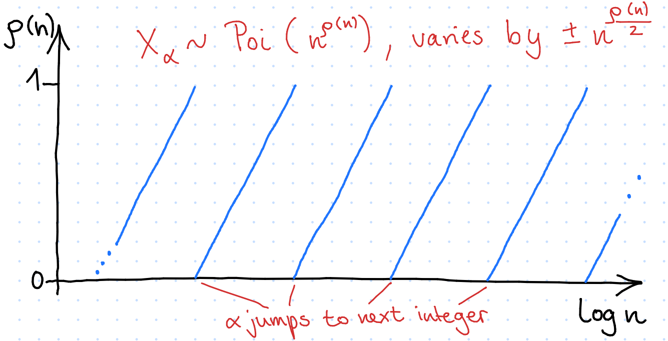
$$d_{\mathrm{TV}}\left(G', G_{n',\frac{1}{2}}\right) = o(1)$$

So can couple $G_{n,\frac{1}{2}}$ and $G_{n',\frac{1}{2}}$ such that, whp,

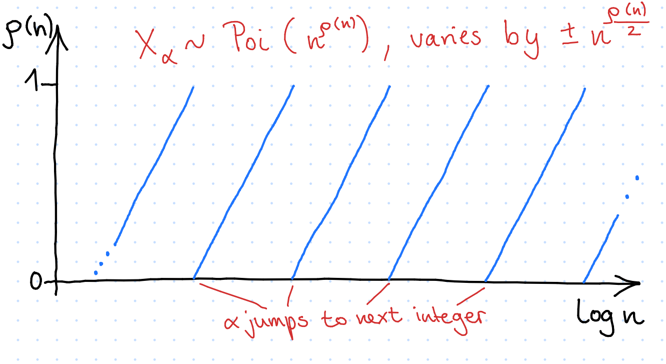
$$\chi(G_{n',\frac{1}{2}}) \leqslant \chi(G_{n,\frac{1}{2}}) + r$$

So what's the truth?

Recall: $X_\alpha \underset{\text{roughly}}{\sim} \mathrm{Poi}_\mu$ where $\mu = n^\rho, \quad 0 \leqslant \rho(n) \leqslant 1.$



Conjecture: $\chi(G_{n,\frac{1}{2}})$ is not concentrated on fewer than $n^{\rho/2}/\log n$ values.



Conjecture: $\chi(G_{n,\frac{1}{2}})$ is not concentrated on fewer than $n^{\rho/2}/\log n$ values.

Theorem(H., Riordan 21)

Let $[s_n, t_n]$ be a sequence of intervals and suppose that $\chi(G_{n,\frac{1}{2}}) \in [s_n, t_n]$ whp. Then for every n with $\rho(n) < 0.99$, there is some $n^* \sim n$ such that

$$t_{n^*} - s_{n^*} \geqslant \frac{(n^*)^{\rho(n^*)/2}}{2000 \log n^*}.$$

Sources of non-concentration?

- Number of α-sets: conjectured lower bound $n^{\frac{\rho}{2}+o(1)}$
- Number of edges?

  Don't seem to matter much. Can couple $G_{n,p}$ and $G_{n,m}$ so that the chromatic numbers only differ by about $\log n$.

- Number of $(\alpha-1)$-sets

  $X_{\alpha-1}$ roughly Poisson with mean about $n^{1+\rho+o(1)}$.

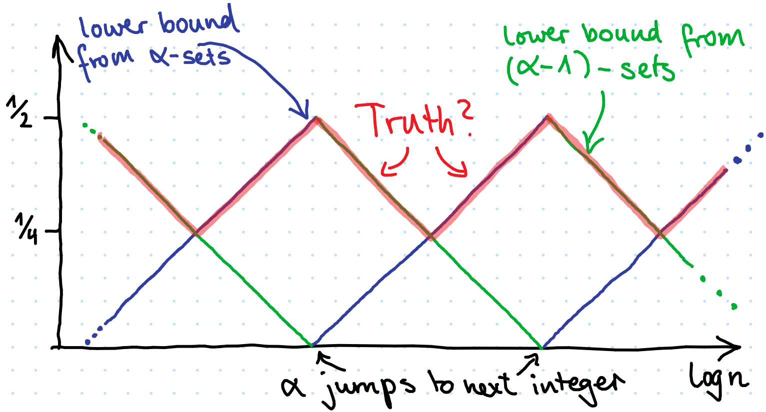
  If $X_{\alpha-1}$ decreases by $n^{\frac{1+\rho}{2}+o(1)}$, we need about $n^{\frac{1-\rho}{2}+o(1)}$ more colours to make the expected number of colourings 1.

Is that all?

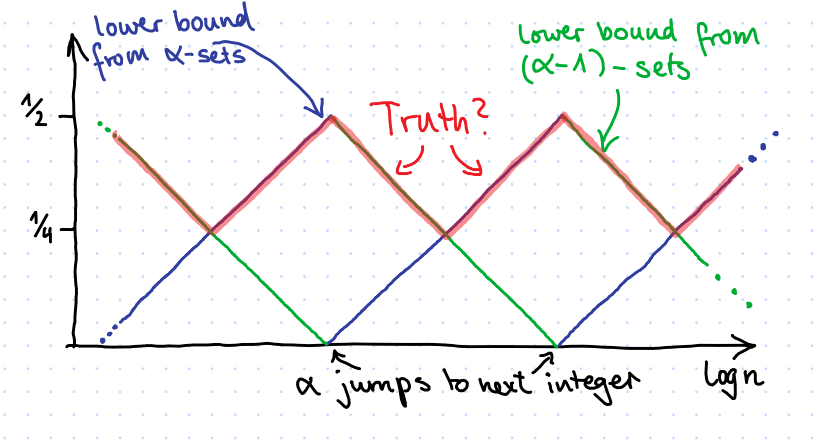
H., Panagiotou 21+: The $(\alpha-2)$-bounded chromatic number of $G_{n,m}$, $m=\frac{1}{2}N$, takes one of at most 2 consecutive values whp.

Zig-zag conjecture

(Bollobás, H., Morris, Panagiotou, Riordan, Smith)



Finer conjectures



- Gaussian limiting distribution
- Best case: $\dfrac{n^{1/4}}{\log^{7/4} n}$
- Worst case: $\dfrac{\sqrt{n}\log\log n}{\log^3 n}$, at $\mu_\alpha = \Theta\left(\dfrac{n}{\log^2 n}\right)$

Best lower concentration bound?

At the moment: $n^{1/2-o(1)}$ for infinitely many n.

Bottleneck: Error term $o\left(\frac{n}{\log^2 n}\right)$ in the explicit estimate for $\chi(G_{n,1/2})$.

H., Panagiotou 21+: Sharper explicit estimate for $\chi(G_{n,1/2})$.

H., Riordan 21: Assuming this estimate, the lower bound on the interval length can be improved to

$$c\frac{\sqrt{n}\log\log n}{\log^3 n}$$

for infinitely many n.

Open questions

- Does the correct concentration interval length zigzag between $n^{1/4+o(1)}$ and $n^{1/2+o(1)}$?
- The proof only finds some n^* near n where the chromatic number is not too concentrated. Can we prove something for every n?
- Alon's upper bound: $\frac{\sqrt{n}}{\log n}$. Our lower bound: $n^{\frac{1}{2}-o(1)}$.

  Conditional lower bound: $\frac{\sqrt{n}\log\log n}{\log^3 n}$. Show that this is optimal?
- Other ranges of p?

  $p < n^{-\frac{1}{2}-\varepsilon}$: two-point concentration. How "far down" does non-concentration go?

Thank you!