Infinite-Bin Model and the Longest Increasing Path in an Erdős-Rényi random graph

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Oxford Discrete Mathematics and Probability Seminar
Cooking recipe

Apple pie

1. Preheat the oven.
2. Prepare a dough.
3. Flatten it and place it in a plate.
4. Peal 4 apples.
5. Cut them into thin slices.
6. Put the slices over the pie crust.
7. Put the apple pie in the oven.

Problem

Compute the time necessary for the apple pie to be made depending on the number of cooks.
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Problem

Compute the time necessary for the apple pie to be made depending on the number of cooks.
A cooking recipe
with a single cook

**Apple pie**

1. Preheat the oven.
2. Prepare a dough.
3. Flatten it and place it in a plate.
4. Peal 4 apples.
5. Cut them into thin slices.
6. Put the slices over the pie crust.
7. Put the apple pie in the oven.

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<tbody>
<tr>
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The recipe takes an amount of time equal to its number of steps.
A cooking recipe
with two cooks

Apple pie

1. Preheat the oven.
2. Prepare a dough.
3. Flatten it and place it in a plate.
4. Peal 4 apples.
5. Cut them into thin slices.
6. Put the slices over the pie crust.
7. Put the apple pie in the oven.

Some task can be parallelized, allowing for a reduction of the number of steps needed to realize the recipe.

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A cooking recipe
with three cooks

Apple pie

1. Preheat the oven.
2. Prepare a dough.
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4. Peel 4 apples.
5. Cut them into thin slices.
6. Put the slices over the pie crust.
7. Put the apple pie in the oven.

Increasing the number of cooks allows to decrease the number of step needed.

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A cooking recipe
with three cooks or many more...

Apple pie

1. Preheat the oven.
2. Prepare a dough.
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4. Peal 4 apples.
5. Cut them into thin slices.
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Increasing the number of cooks allows to decrease the number of step needed... up to a point.
Formalizing the problem

The dependencies of the tasks of the recipe are represented as an oriented graph (without cycles).

- The vertices of the graph represent the different tasks.
- Edges denote dependencies.

Lemma

The minimal number of steps needed to realize the project is equal to the length of the longest path in the oriented graph.
Formalizing the problem

The dependencies of the tasks of the recipe are represented as an oriented graph (without cycles).

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**Lemma**

The minimal number of steps needed to realize the project is equal to the length of the longest path in the oriented graph.
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**Lemma**

The minimal number of steps needed to realize the project is equal to the length of the longest path in the oriented graph.
Formalizing the problem

The dependencies of the tasks of the recipe are represented as an oriented graph (without cycles).

- The vertices of the graph represent the different tasks.
- Edges denote dependencies.

**Lemma**

*The minimal number of steps needed to realize the project is equal to the length of the longest path in the oriented graph.*
Outline

1. Barak-Erdős graph
2. Infinite-bin models
3. Coupling of the IBM and the Barak-Erdős graph
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Outline

1 Barak-Erdős graph

2 Infinite-bin models

3 Coupling of the IBM and the Barak-Erdős graph
The Barak-Erdős graph

Definition

The Barak-Erdős graph is a directed version of the Erdős-Rényi graph in which every edge \(\{i, j\}\) is directed from \(i\) to \(j\) if \(i < j\).

In other words, given \(p \in [0, 1]\) and \(n \in \mathbb{N}\), for any \(1 \leq i < j \leq n\), put an edge from \(i\) to \(j\) with probability \(p\), independently from any other edge.

We take interest in the length \(L_n(p)\) of the longest increasing path in this graph.
The Barak-Erdős graph

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Figure: A Barak-Erdős graph

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![Diagram of a Barak-Erdős graph](image)

**Figure:** A Barak-Erdős graph

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\[ \begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7
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![Figure: A Barak-Erdős graph](image-url)

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The Barak-Erdős graph

Some references

- Model introduced by Barak and Erdős in 1984.
- The length of the longest increasing path is one of the most studied features of this model.
- Applications span over a wide array of fields:
  - Performance evaluation of computer systems (Gelenbe-Nelson-Philips-Tantawi ’86, Isopi-Newman ’94);
  - Mathematical ecology (food chains) (Cohen-Newman ’86,’91);
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Existing results
Existence of a limiting function

**Theorem (Newman ’92)**

*There exists a function $C$ such that for any $p \in [0, 1]$,

$$
\lim_{n \to +\infty} \frac{L_n(p)}{n} = C(p) \quad \text{in probability.}
$$

Moreover, $C$ is continuous, increasing and $C'(0) = e$.\*
Existing results
Existence of a limiting function

Figure: Graph of the function $C$
Existing results

Existence of a limiting function

Figure: Graph of the function $p \mapsto C(p)/p$
Existing results

Bounds on the function $C$

**Theorem (Foss-Konstantopoulos ’03)**

There exist two explicit functions $L$ and $U$ such that $L(p) < C(p) < U(p)$ for any $p \in (0, 1)$. This in particular yields, as $p \to 1$,

$$C(1 - p) = 1 - (1 - p) + (1 - p)^2 - 3(1 - p)^3 + 7(1 - p)^4 + O(1 - p)^5.$$
Contribution from infinite-bin models theory I
Improved bounds in a neighbourhood of 1

Theorem (M.-Ramassamy)

There exist sequences of upper bounds \((U_k)\) and lower bounds \((L_k)\) that converge to \(C\) exponentially fast for any \(p > 0\).

In particular, the Taylor expansion of \(C\) can be computed explicitly to any order around \(p = 1\).
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In particular, the Taylor expansion of $C$ can be computed explicitly to any order around $p = 1$. 

$$C(p) \quad L_{12}(p) \quad U_{12}(p)$$

![Graph showing the relationship between $C(p)$, $L_{12}(p)$, and $U_{12}(p)$]
Contribution from infinite-bin models theory II

Analyticity of $C$ in a neighbourhood of 1

**Theorem (M.-Ramassamy)**

The function $C$ is analytic on $(0, 1]$, and there exists an explicit sequence of integers $(a_k)$ such that

$$C(p) = \sum_{k=0}^{+\infty} a_k (1 - p)^k \text{ for all } p \geq 3/4.$$ 

**First coefficients**

$$C(p) = 1 - (1 - p) + (1 - p)^2 - 3(1 - p)^3 + 7(1 - p)^4 - 15(1 - p)^5$$
$$+ 29(1 - p)^6 - 54(1 - p)^7 + 102(1 - p)^8 - 197(1 - p)^9$$
$$+ 375(1 - p)^{10} - 687(1 - p)^{11} + 1226(1 - p)^{12} - 2182(1 - p)^{13}$$
$$+ 3885(1 - p)^{14} - 6828(1 - p)^{15} + 11767(1 - p)^{16} + \cdots$$

(sequence A321309 of OEIS)
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Contribution from infinite-bin models theory III

Asymptotic behaviour of \( C \) as \( p \to 0 \)

**Theorem (M.-Ramassamy)**

\[
C(p) = ep \left( 1 - \frac{\pi^2}{2} \frac{1}{(\log p)^2} \right) + o \left( \frac{p}{(\log p)^2} \right) \text{ as } p \to 0.
\]
Outline

1. Barak-Erdős graph
2. Infinite-bin models
3. Coupling of the IBM and the Barak-Erdős graph
The infinite-bin model

Description

- Infinite number of bins on $\mathbb{Z}$.
- At each time $n$, a new ball is put to the right of the $\xi_n$th ball, with $(\xi_j)$ i.i.d. sequence of random variables on $\mathbb{N}$.
- We take interest in the speed of the front.
### Description

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**Diagram:**

- Several circles representing balls placed at different positions along the integers.
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$x_1 = 5$
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\[ \xi_1 = 5 \]
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$$\xi_2 = 4$$
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\[
\begin{array}{cccccc}
\xi_2 &=& 4 \\
10 & | 9 & 8 & 6 & 4 & 2 & 1 \\
7 & 5 & 3 &
\end{array}
\]
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\[ \xi_3 = 7 \]
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\[ \xi_3 = 7 \]
The infinite-bin model

Description

- Infinite number of bins on $\mathbb{Z}$.
- At each time $n$, a new ball is put to the right of the $\xi_n$th ball, with $(\xi_j)$ i.i.d. sequence of random variables on $\mathbb{N}$.
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$\xi_3 = 7$

\[\begin{array}{cccccccccc}
11 & 10 & 9 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
\end{array}\]
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<table>
<thead>
<tr>
<th>12</th>
<th>11</th>
<th>10</th>
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\[
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& & & & & & \xi_5 = 3 & \\
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$$\xi_6 = 1$$
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On the infinite-bin model

Existing results

- Aldous and Pitman (1993) studied a version of this model when $\xi$ is the uniform distribution on $\{1, \ldots, N\}$.
- This general version introduced by Foss and Konstantopoulos in 2003.
- Studied using the existence of renewal event when $E(\xi) < +\infty$ (Foss, Konstantopoulos, Chernysh, Ramassamy, Zachary).
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Construction of the infinite-bin model

Definition
Given $X$ a configuration and $k \in \mathbb{N}$, we denote by $\Psi_k(X)$ the configuration with a ball added to the right of the $k$th rightmost ball in $X$.

Infinite-bin model
Given $(\xi_n)$ i.i.d. random variables of law $\mu$ and $X_0$ a starting configuration, we call an IBM($\mu$) the process

$$\forall n \in \mathbb{N}, X_n = \Psi_{\xi_n}(X_{n-1}).$$
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A coupling for infinite-bin models

Partial order

Given $X$ and $Y$ two configurations, we say that $X \preceq Y$ if for every $k$, there are more balls to the right of $k$th urn in $Y$ than in $X$.

Lemma

The function $(X, k) \mapsto \Psi_k(X)$ is decreasing with $k$ and increasing with $X$.

Proposition

If $(X_n)$, $(Y_n)$ are two infinite-bin models defined with $(\xi_n)$, $(\zeta_n)$, such that $X_0 \preceq Y_0$ and $\xi_k \geq \zeta_k$ for all $k \in \mathbb{N}$, then

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Theorem (Foss-Konstantopoulos, M.-Ramassamy)

For any probability measure $\mu$ on $\mathbb{N}$, there exists $v_\mu \in [0, 1]$ such that writing $F_n$ for the front at time $n$ of an IBM($\mu$), we have

$$\lim_{n \to +\infty} \frac{F_n}{n} = v_\mu \quad \text{a.s.}$$
Proof of the existence of the speed

Proof.

- If the measure has finite support $K$, then the relative positions of the rightmost $K$ balls form a Markov process.
- Hence the speed exists by ergodicity.
- If $\mu$ has no finite support, setting $\mu_K = \mu 1_{\{\leq K\}}$, we have

$$v_{\mu_K} \leq v_\mu \leq v_{\mu_K} + \mu([K+1, +\infty)).$$

- We conclude that $v_\mu = \lim_{K \to +\infty} v_{\mu_K}$. 
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Infinite-Bin Model

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Proof of the existence of the speed

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Infinite-Bin Model

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Outline

1. Barak-Erdős graph

2. Infinite-bin models

3. Coupling of the IBM and the Barak-Erdős graph
Coupling the IBM and the Barak-Erdős graph

Coupling

One can couple a Barak-Erdős graph with parameter $p$ with an IBM with geometric distribution $\mu_p(k) = p(1 - p)^{k-1}$.

- Start with the empty graph, and the configuration with an infinite number of balls in bin $-1$.
- At each step $n$, add the vertex $n$ and the links with the previous vertices. Add a ball in the bin with index given by the longest path ending at $n$.

Consequence

For any $p \in [0, 1]$, we have $C(p) = v_{\mu_p}$.
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\[ \xi_6 = 7 - 1 \]

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Infinite-Bin Model

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Infinite-Bin Model
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Construction of the coupling

\[ \xi_6 = 7 - 1 0 1 2 3 4 5 6 \]

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Construction of the coupling

\[ \xi_6 = 7 - 10 \]

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Infinite-Bin Model

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Construction of the coupling

\[ \xi_6 = 7 - 10123456 \]

Bastien Mallein (USPN)
Construction of the coupling

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Bastien Mallein (USPN)
Construction of the coupling

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\[ \begin{array}{cccccccc}
-1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \]
Construction of the coupling

\[ \xi_6 = 7 - 10 \]

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Construction of the coupling

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Bastien Mallein (USPN)
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\[ \begin{align*}
\xi_6 &= 7 - 1 \\
\end{align*} \]
Construction of the coupling

\[ ξ_6 = 7 - 10123456 \]

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Infinite-Bin Model

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Bastien Mallein (USPN)

Infinite-Bin Model

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Bastien Mallein (USPN)
Asymptotic behaviour of $C$ as $p \to 1$

Strategy of proof

- We use the $L^1$ convergence of the position of the front at time $nF_n$:

$$\lim_{n \to +\infty} \frac{1}{n} E(F_n) = C(p).$$

- We observe that $E(F_n)$ can be computed for large $p$ as the sum of the contributions of small complex patterns arising in the middle of long sequences of 1.

- We prove the convergence for $p > 1/2$ of the series of the contributions made by these small patterns.
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A perturbative estimate

We assume that $p$ is close to 1. Recall that $\mu_p(k) = p(1 - p)^{k-1}$.

Approximate behaviour

Up to $o(1)$ corrections, $(\xi_n)$ looks like

$$(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots)$$

Therefore $C(p) = 1 + o(1)$.
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Bastien Mallein (USPN)
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Therefore $C(p) = 1 + o(1)$. 
A perturbative estimate

We assume that $p$ is close to 1. Recall that $\mu_p(k) = p(1 - p)^{k-1}$.

Approximate behaviour

Up to $o((1 - p))$ corrections, $(\xi_n)$ looks like

$$(1, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, \ldots)$$

Therefore $C(p) = 1 - p(1 - p) + o((1 - p))$. 
A perturbative estimate

We assume that $p$ is close to 1. Recall that $\mu_p(k) = p(1 - p)^{k-1}$.

Approximate behaviour

Up to $o((1 - p))$ corrections, $(\xi_n)$ looks like

$$(1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, \ldots)$$

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Bastien Mallein (USPN)
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We assume that $p$ is close to 1. Recall that $\mu_p(k) = p(1 - p)^{k-1}$.

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Up to $o((1 - p)^2)$ corrections, $(\xi_n)$ looks like

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Therefore $C(p) = 1 - p(1 - p) + (p(1 - p))^2 - p(1 - p)^2 + o((1 - p)^2)$. 
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Bastien Mallein (USPN)
Finding the asymptotic expansion

Aim

To each finite pattern \( u = (u(1), \ldots, u(n)) \in \bigcup \mathbb{N}^n \), we would like to associate a term \( \varepsilon(u) \in \{-1, 0, 1\} \) such that for any \( N \in \mathbb{N} \),

\[
C(p) = \sum_{n \geq 0} \sum_{u \in \bigcup \mathbb{N}^n} \varepsilon(u) \mathbb{P}(\xi_1 = u(1), \ldots \xi_n = u(n)) + o((1 - p)^N)
\]

Definition

For each finite pattern \( u \), we denote by \( d(u) \) the distance the front travels when applying successively \( \Psi_{u(1)}, \ldots, \Psi_{u(n)} \).

We define \( \varepsilon \) as the solution of the following equation:

\[
d(u) = \sum_{v \text{ subpattern of } u} \varepsilon(v) = \sum_{k=1}^{\left| u \right|} \sum_{j=1}^{\left| u \right| - k} \varepsilon(u(j), u(j+1), \ldots u(j+k-1))
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Finding the asymptotic expansion

**Aim**
To each finite pattern $u = (u(1), \ldots u(n)) \in \cup \mathbb{N}^n$, we would like to associate a term $\varepsilon(u) \in \{-1, 0, 1\}$ such that for any $N \in \mathbb{N}$,

$$C(p) = \sum_{n \geq 0} \sum_{u \in \cup \mathbb{N}^n} \varepsilon(u) P(\xi_1 = u(1), \ldots \xi_n = u(n)) + o((1 - p)^N)$$

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For each finite pattern $u$, we denote by $d(u)$ the distance the front travels when applying successively $\Psi_{u(1)}, \ldots, \Psi_{u(n)}$. We define $\varepsilon$ as the solution of the following equation:

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A direct definition for $\varepsilon$

**Definition**

For $u$ a pattern, we write $\pi u$ the pattern obtained by forgetting the last number and

$$\delta(u) = d(u) - d(\pi u) \in \{0, 1\},$$

**Lemma**

For $u$ a pattern, we write $\omega u$ the pattern obtained by forgetting the first number, we have

$$\varepsilon(u) = \delta(u) - \delta(\omega u).$$
A direct definition for $\varepsilon$

**Definition**

For $u$ a pattern, we write $\pi u$ the pattern obtained by forgetting the last number and

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**Lemma**

For $u$ a pattern, we write $\varpi u$ the pattern obtained by forgetting the first number, we have

$$\varepsilon(u) = \delta(u) - \delta(\varpi u).$$
Analyticity of $C$

**Theorem**

For any $p \in (1/2, 1]$, we have

$$C(p) = \sum_u \varepsilon(u) p^{|u|} (1 - p)^{\sum(u(j) - 1)}.$$

**Proof.**

$$C(p) = \lim_{n \to +\infty} \frac{1}{n} \mathbb{E}(d(\xi_1, \ldots, \xi_n))$$

$$= \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{n-k} \mathbb{E}(\varepsilon(\xi_j, \xi_{j+1}, \ldots, \xi_{j+k-1}))$$

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For any $p \in (1/2, 1]$, we have $C(p) = \sum_u \varepsilon(u)p^{|u|}(1 - p)^{(u(j)-1)}$.

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\[C(p) = \lim_{n \to +\infty} \frac{1}{n} E(d(\xi_1, \ldots \xi_n)) = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{n-k} E(\varepsilon(\xi_j, \xi_{j+1}, \ldots \xi_{j+k-1})) = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} (n - k)E(\varepsilon(\xi_1, \xi_2, \ldots \xi_k)) = \sum_{k=1}^{+\infty} E(\varepsilon(\xi_1, \xi_2, \ldots \xi_k)). \]
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Conclusion

We were able to study the function $C$ by coupling Barak-Erdős graphs with Infinite-Bin models. This function:

- Is analytic on $(0, 1]$;
- Behaves as $ep(1 - \pi^2/2(\log p)^2)$ at $p = 0$;
- Its series expansion can be computed as a perturbation expansion.

Some open questions:

- Is $p \mapsto C(p)/p$ convex?
- Can similar computations be made with $C_k(p)$ the time taken to undertake a series of tasks with $k$ servers.
Thank you for your attention!
Asymptotic behaviour of $C$ as $p \to 0$

### Strategy of proof

- Using the increasing coupling, we have
  \[ C(p) \approx \text{speed of an IBM with uniform distribution on } \{1, \ldots, \left\lfloor \frac{1}{p} \right\rfloor \}. \]

- The speed of an IBM with uniform distribution is coupled with a branching random walk with selection.

- The speed of a branching random walk with selection is computed using Bérard and Gouéré’s result.
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Bound with an infinite-bin model with uniform distribution

Notation
We write $w_N$ the speed of an infinite-bin model with uniform distribution on $\{1, \ldots, N\}$.

Upper bound
For any $p \in \left[\frac{1}{N+1}, \frac{1}{N}\right]$, we have $C(p) \leq w_N$.
Indeed, we have $\sum_{j=1}^{k} p(1-p)^{j-1} \leq (pk)^{\wedge} 1$, thus we can couple a geometric random variable $G$ and a uniform random variable $U$ such that $G \geq U$ a.s.

Lower bound
For any $p \in [0, 1]$, we have $C(p) \geq Np(1-p)^N w_N$.

Conclusion
$$C(1/N) \approx w_N \quad \text{as } N \to +\infty.$$
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$$C(1/N) \approx w_N \quad \text{as } N \to +\infty.$$
A $N$-branching random walk in continuous-time

**Behaviour of the rightmost $N$ balls**

We consider the process $(X_{P_t}, t \geq 0)$, where $P$ is an independent Poisson process of intensity $N$.

- At rate $N$ an event occurs.
- With probability $1/N$, one of the $N$ rightmost ball makes an offspring to its right.
- The leftmost ball is removed from consideration.

**Alternative description**

- A clock on each of the $N$ rightmost balls will ring at rate 1 independently.
- The selected ball makes an offspring to its right.
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- The selected ball makes an offspring to its right.
- The leftmost ball is erased.
A $N$-branching random walk in continuous-time

**Behaviour of the rightmost $N$ balls**

We consider the process $(X_{P_t}, t \geq 0)$, where $P$ is an independent Poisson process of intensity $N$.

- At rate $N$ an event occurs.
- With probability $1/N$, one of the $N$ rightmost ball makes an offspring to its right.
- The leftmost ball is removed from consideration.

**Alternative description**

- A clock on each of the $N$ rightmost balls will ring at rate 1 independently.
- The selected ball makes an offspring to its right.
- The leftmost ball is erased.
Brunet-Derrida behaviour of branching random walks with selection

**Theorem (Bérard-Gouéré 2010)**

*Under some assumptions, if we denote by $v_N$ the speed of a branching random walk with selection, there exist explicit $v_\infty$ and $\chi > 0$ such that*

$$v_N - v_\infty \sim N \to +\infty - \frac{\chi}{(\log N)^2}.$$

**Notation**

More precisely, setting $\kappa(\theta) = \log \mathbb{E}(\sum_{|u|=1} e^{\theta V(u)})$, we have

$$v_\infty = \inf_{\theta > 0} \frac{\kappa(\theta)}{\theta}, \quad \theta^* \text{ solution of } \theta \kappa'(\theta) - \kappa(\theta) = 0,$$

$$\sigma^2 = \kappa''(\theta^*), \quad \chi = -\frac{\pi^2 \sigma^2}{2 \theta^*}.$$
Brunet-Derrida behaviour of branching random walks with selection

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Under some assumptions, if we denote by $v_N$ the speed of a branching random walk with selection, there exist explicit $v_\infty$ and $\chi > 0$ such that

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$$v_\infty = \inf_{\theta > 0} \frac{\kappa(\theta)}{\theta} \quad \text{\theta_* solution of } \theta \kappa'(\theta) - \kappa(\theta) = 0$$

$$\sigma^2 = \kappa''(\theta^*)$$

$$\chi = -\frac{\pi^2 \sigma^2}{2\theta^*}.$$
Conclusion

Theorem

We have $C(p) = p \left( e - \frac{\pi^2 e}{2(\log p)^2} \right)$.

Proof.

Recall that $C(1/N) \approx \frac{1}{N} \nu_N$. We have $\kappa(\theta) = e^\theta$, thus:

- $\nu_\infty = e$;
- $\theta^* = 1$;
- $\sigma^2 = e$.

This concludes the proof.